

# Shadowing Lemma and chaotic orbit determination

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**Dynamics and chaos in astronomy and physics**

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What happens if we need to have an accurate quantitative knowledge of chaotic orbits?

*“In fact because of the exponential variety of trajectories which exists, the rotation state at the midpoint of the interval covered by the observations, and the principal moments of inertia, are determined with exponential accuracy.*

***Thus the knowledge gained from measurements on a chaotic dynamical system grows exponentially with the time span covered by the observations.”***

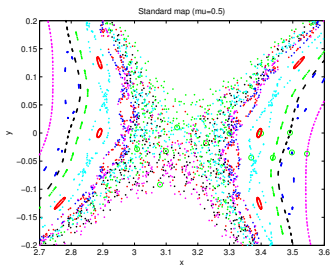
Wisdom, J.

**Urey Prize Lecture:  
Chaotic Dynamics in the Solar System**  
*Icarus* **72**, 241-275 (1987)

## Standard Map: Definition

The **standard map** of the pendulum is a conservative discrete dynamical system, defined on a 2-dimensional torus, which has both ordered and chaotic orbits.

$$S_{\mu}(x_0, y_0) = \begin{cases} x_{k+1} & = x_k + y_{k+1} \\ y_{k+1} & = y_k - \mu \sin(x_k). \end{cases}$$



The system has more **regular orbits** for small  $\mu$ , and more **chaotic orbits** for large  $\mu$ .

We choose an **intermediate value**  $\mu = 0.5$ , in such a way that both ordered and chaotic orbits are present.

## Standard map linearization

The least square parameter estimation process can be performed by an **explicit formula**.

- **Linearized map:**

$$DS = \begin{pmatrix} \frac{\partial x_{k+1}}{\partial x_k} & \frac{\partial x_{k+1}}{\partial y_k} \\ \frac{\partial y_{k+1}}{\partial x_k} & \frac{\partial y_{k+1}}{\partial y_k} \end{pmatrix} = \begin{pmatrix} 1 - \mu \cos(x_k) & 1 \\ -\mu \cos(x_k) & 1 \end{pmatrix}$$

- **Linearized state transition matrix:**

$$A_k = \frac{\partial(x_k, y_k)}{\partial(x_0, y_0)}; \quad A_{k+1} = DS A_k; \quad A_0 = I$$

- **Variational equation:**

$$\begin{aligned} \frac{\partial(x_{k+1}, y_{k+1})}{\partial \mu} &= DS \frac{\partial(x_k, y_k)}{\partial \mu} + \frac{\partial S}{\partial \mu} \\ &= DS \frac{\partial(x_k, y_k)}{\partial \mu} + \begin{pmatrix} -\sin(x_k) \\ -\sin(x_k) \end{pmatrix} \end{aligned}$$

## Observations process

- Both coordinates  $x$  and  $y$  are observed at each iteration, and the observations are **Gaussian random variables** with mean  $x_k$  ( $y_k$ , respectively) and standard deviation  $\sigma$ .

## Residuals

- The residuals contain two components: a **random** one for the observation error, and a **systematic** one because the true value  $\mu_0$  is not the same as the current guess.

$$\begin{cases} \xi_k &= x_k(\mu_0, \sigma) - x_k(\mu_1) \\ \bar{\xi}_k &= y_k(\mu_0, \sigma) - y_k(\mu_1), \end{cases}$$

with  $k = -n, \dots, n$ .

## Standard map orbit determination II

- The **least squares fit** is obtained from the normal equations:

$$C = \sum_{k=-n}^n B_k^T B_k; \quad D = - \sum_{k=-n}^n B_k^T \begin{pmatrix} \xi_k \\ \bar{\xi}_k \end{pmatrix}$$

$$B_k = \frac{\partial(\xi_k, \bar{\xi}_k)}{\partial(x_0, y_0, \mu)} = - \left( A_k \middle| \frac{\partial(x_k, y_k)}{\partial \mu} \right)$$

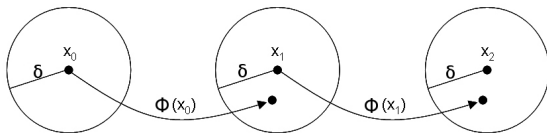
- An iteration of **differential corrections** is a correction  $\Gamma D$  obtained from the **covariance matrix**  $\Gamma = C^{-1}$ .
- At convergence of the iterations to the **least squares solution**  $(x^*, y^*, \mu^*)$ , weights should be assigned to the residuals consistently with the probabilistic model.

## Shadowing Lemma: $\delta$ -pseudotrajectory

### $\delta$ -pseudotrajectory

A  $\delta$ -pseudotrajectory is a sequence of points  $(x_k, y_k)$  connected by an approximation of the map  $\Phi$ , with error  $< \delta$  at each step:

$$|\Phi(x_k, y_k) - (x_{k+1}, y_{k+1})| < \delta$$



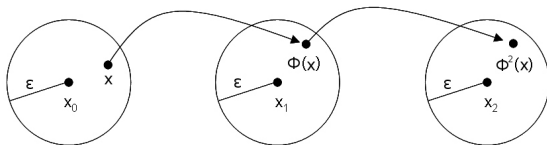
## Shadowing Lemma: $\varepsilon$ -shadowing

### $\varepsilon$ -shadowing

The orbit with initial conditions  $(x, y)$  ( $\varepsilon, \Phi$ )-shadows a  $\delta$ -pseudotrajectory  $(x_k, y_k)$  if:

$$|\Phi^k(x, y) - (x_k, y_k)| < \varepsilon$$

for every  $k$ .





## Shadowing Lemma

### Shadowing Lemma

If  $\Lambda$  is an hyperbolic set for a diffeomorphism  $\Phi$ , then there exists a neighborhood  $W$  of  $\Lambda$  such that

**for every**  $\varepsilon > 0$

**there exists**  $\delta > 0$

such that

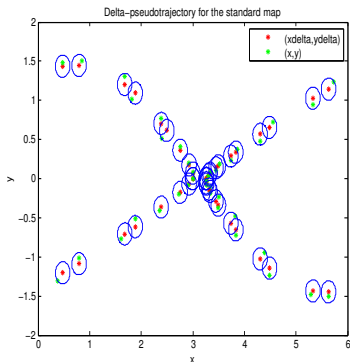
**for every**  $\delta$ -pseudotrajectory in  $W$  **there exists** a point in  $W$  that  $\varepsilon$ -shadows the  $\delta$ -pseudotrajectory.

- An **hyperbolic set** is (roughly) an invariant set with every orbit having a positive and a negative Lyapounov exponent.
- There is an  $L > 0$ , function of the Lyapounov exponents, such that  $\delta < \varepsilon/L$ .

# Shadowing Lemma and least squares solution I

To connect orbit determination and the Shadowing Lemma, we need to show first that the **observations**  $x_k(\mu_0, \sigma)$ ,  $y_k(\mu_0, \sigma)$  are a  $\delta$ -**pseudotrajectory** for the dynamical system  $S_{\mu^*}$ :

- $\mu^*$  is the value of the dynamical parameter found from the least squares solution
- $\delta = \sqrt{2}|\mu_0 - \mu^*| + K\sigma$



Example of a  $\delta$ -pseudotrajectory.

**Initial conditions:**

$$x_0 = 3, y_0 = 0, \mu_0 = 0.5.$$

**Options:**

$$\delta\mu = 10^{-1}, \sigma = 10^{-3}.$$

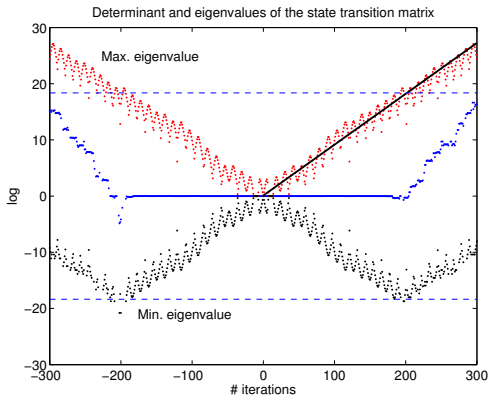
## Shadowing Lemma and least squares solution II

- We select  $\varepsilon > K\sigma$ :  $K$  is a number such that, at convergence, no larger norm  $|(\xi_k, \bar{\xi}_k)|$  is found among the residuals for  $-n \leq k \leq n$
- The orbit with initial conditions  $(\mathbf{x}^*, \mathbf{y}^*)$   **$\varepsilon$ -shadows** the  **$\delta$ -pseudotrajectory** formed by the observations  $(x_k, y_k)$ .

### Summary

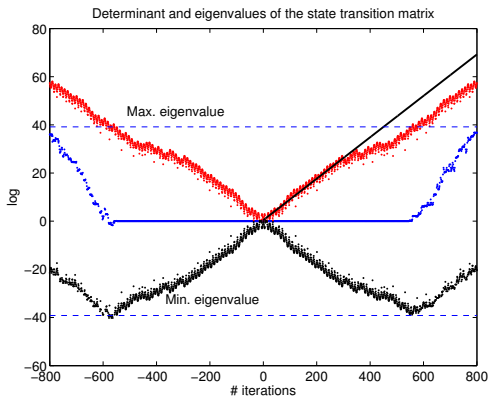
- The **observations** are a **pseudotrajectory** because of errors and systematics due to imperfect knowledge of the dynamics.
- The **least squares solution** is the **shadowing** of the observations.
- The **Shadowing Lemma** is a **minimization of the infinite dimensional space of all orbits**, while the **orbit determination** is a **minimization of the norm of a finite number of residuals**.

# Computability Horizon



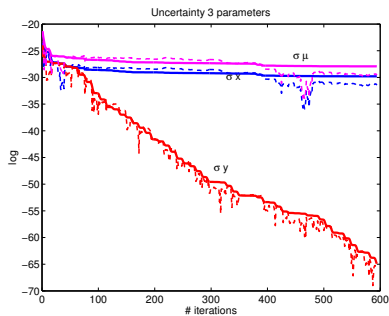
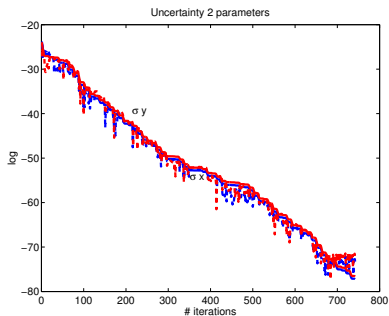
- Initial conditions:  $x_0 = 3$ ,  $y_0 = 0$ ,  $\mu_0 = 0.5$
- Positive Lyapounov exponent is  $\chi \simeq 0.091$ , and the Lyapounov time is  $t_L = 1/\chi \simeq 11$ .
- The numerical instability at  $k \simeq 200$  occurs because  $\exp(200/t_L) \sim 10^8$ .
- The inversion of the normal matrix C fails.

# Meaning of the computability horizon



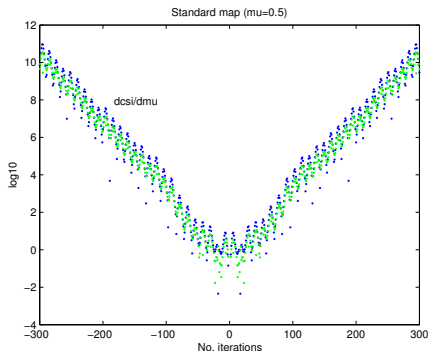
- The computability horizon is  $\simeq 600$  iterations of  $S$ .
- The computability horizon is an **absolute barrier to the determination of a least squares orbit**.
- We need to admit that we can only solve for a  $\delta$ -pseudotrajectory.

## Results: with and without dynamical parameter



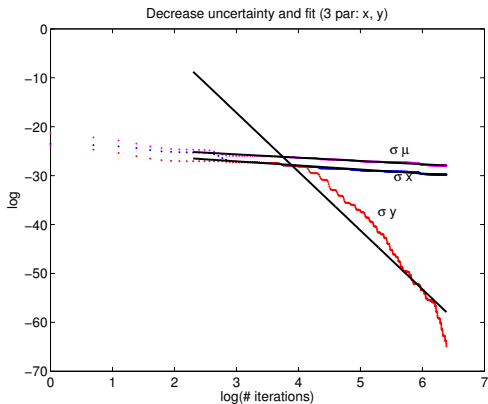
- Least squares solution for up to 600/700 iterates in quadruple precision, by using a progressive method.
- If the solutions has a **fixed**  $\mu = \mu_0$ , the improvement in accuracy is **exponential** in  $k$ .
- If we solve for **2 initial conditions** and  $\mu$ , the improvements is **not exponential** in at least two variables (including  $\mu$ ).

# Loss of accuracy when adding a dynamical parameter



- The much lower accuracy in the determination of  $\mu$  and at least one initial condition is **not due to lack of sensitivity**.
- Correlations grow.
- Orbit determination is degraded by aliasing.
- This is a **finite-time analog of the shadowing lemma**.

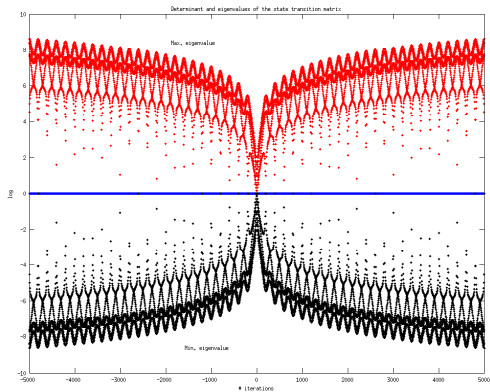
## Results: chaotic orbit, power law improvement



- Power law accuracy improvement with the number  $n$  of iterations like  $n^a$  with  $a = -0.675$  for  $\mu$ ,  $a = -0.833$  for  $x$ , while for  $y$  a power law is not appropriate.
- The slopes are sensitive to the initial conditions.

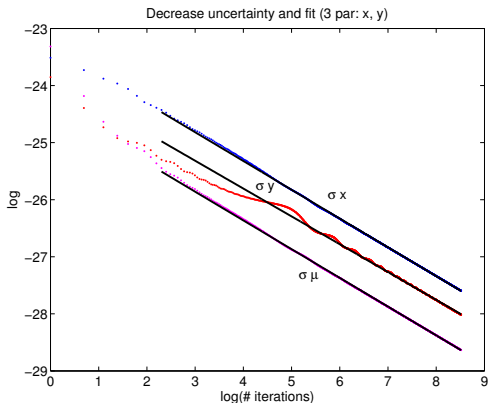


# Results: ordered case, power law improvement



- Initial conditions:  $x_0 = 2$ ,  $y_0 = 0$ ,  $\mu_0 = 0.5$
- The computability horizon **disappears**.
- The Lyapounov exponent could be zero.

## Results: ordered case, gaussian statistics



- The difference between the case with 2 parameters and the one with 3 **disappears**.
- In a log-log plot we find an **accuracy improvement  $n^a$**  with  $a = -0.504, -0.504, -0.488$  for  $\mu, x, y$  respectively.
- We conjectured that the exponent is actually  $-0.5$ .

**Orbit determination is possible only within the computability horizon.**

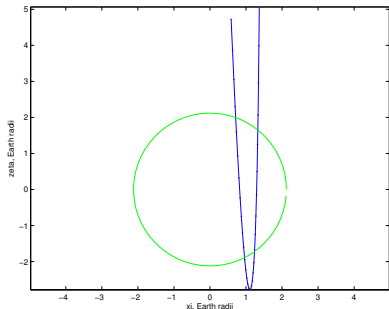
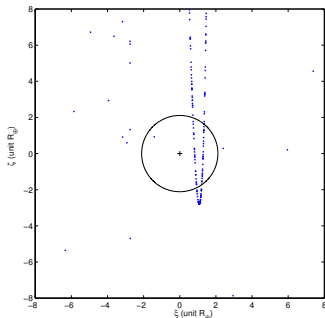
**If a parameter is estimated, the least squares solution is a finite time span shadowing of the observations**

**Chaotic orbit:** the precision of the solution can grow only as a power law  $n^a$  with  $-1 < a < 0$ .

**Ordered orbit:** the computability horizon is not a problem and the precision improvement is  $n^a$  with  $a \simeq -1/2$ .

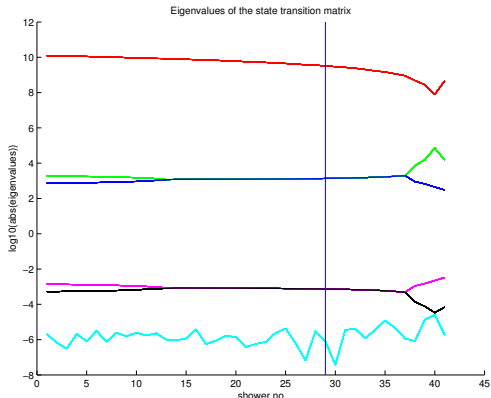
**How soon will we find practical problems of dynamical astronomy in which we need to use these concepts and preliminary results?**

## Example from impact monitoring



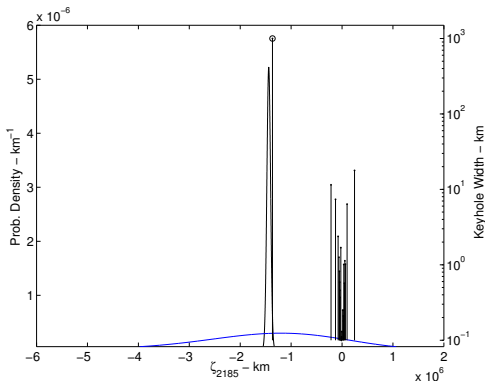
- The **twin Virtual Impactors (VI)** (connected patches of initial conditions colliding with Earth) for asteroid (101955) Benu in year 2182.
- Orbits compatible with the observations (up to 2011) and a Yarkovsky model.
- Vertical axis is along the projection of Earth's velocity onto the TP.

## Example from impact monitoring



- State transition matrix for the integral flow of the equations of motion for (101955) Bennu, after the ca of 2182.
- The eigenvalues are 6, 2 real and 4 complex, always in couples  $\lambda, 1/\lambda$ .

## Determination of Yarkovsky effect



- Prediction on the Target Plane of an encounter in the 2185 with a **Yarkovsky dynamical parameter estimated**, and **without it**.
- The keyholes for impact in the year 2185 to 2196 are **all in the possible range of values with Yarkovsky**.