Topological structure of the final motions and a search for periodic orbits in the free-fall three-body problem

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1 Introduction and Motivation

The equations of motion are

$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = \frac{\partial U}{\partial \mathbf{r}_i}, i = 1, 2, 3 \tag{1}$$

where $m_i, \mathbf{r}_i, i = 1, 2, 3$ are the masses and position vectors of the three bodies. U is the potential with

$$U = G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_2 m_3}{r_{23}} + \frac{m_3 m_1}{r_{31}}\right). \tag{2}$$

There is an old question.

Question. How various orbits occupy the phase space of the general three-body problem? Which orbits occupy positive areas?

This question was asked after the proofs of non-integrability of the three-body problem by Bruns and Poincaré in the end of the 19th century.

If we cannot follow analytically individual solutions, then we follow qualitatively the groups of solutions.

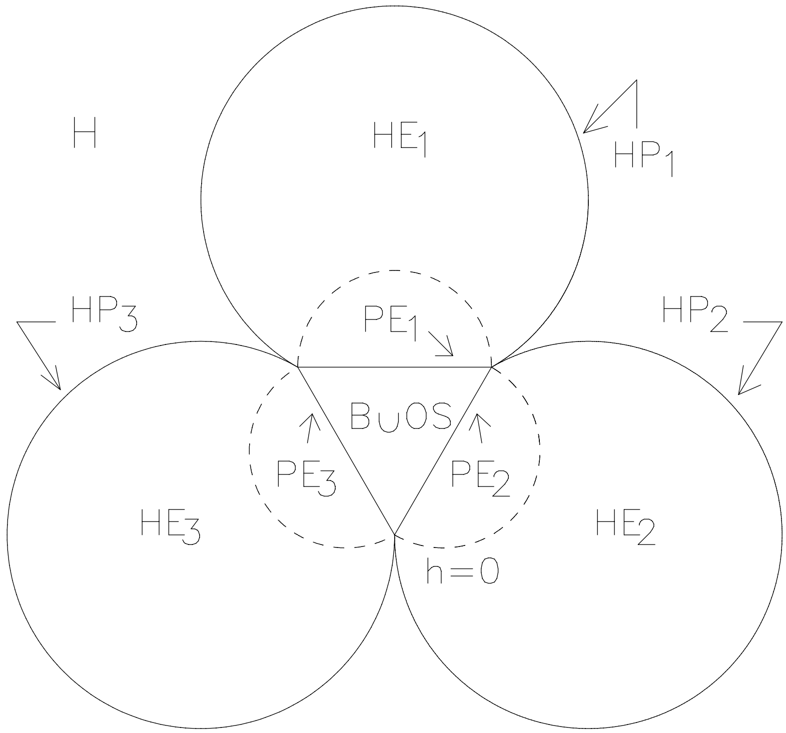

Figure 1: Division of the phase space by various final motions. Chazy (1922,1929), Alekseev(1976,1981). **H: Hyperbolic motions; HP: Hyperbolic-Parabolic motions; HE: Hyperbolic-Elliptic motions; PE: Parabolic-Elliptic motions; OS: Oscillatory motions; and B: Bounded motions.**

Then, the Russian school (Hil'mi, Merman and others) after Chazy connected the initial motions and final motions.

Settings of the problem

Planar problem

Planar problem with zero angular momentum

Stable periodic orbits: Figure-8 orbit.

$\boxed{\text{Planar problem with zero initial velocities}}$
(The free-fall problem)

Collinear problem

We consider the free-fall problem. The system is considered most unstable among others. In particular, the equal mass case is the most unstable among other combinations of masses.

Question. What is the structure of $B \cup OS$ in the free-fall problem?

2 Assertions

Assertion 1. If there are no stable periodic orbits, $B \cup OS$ is a Cantor-like set in the f-f problem.

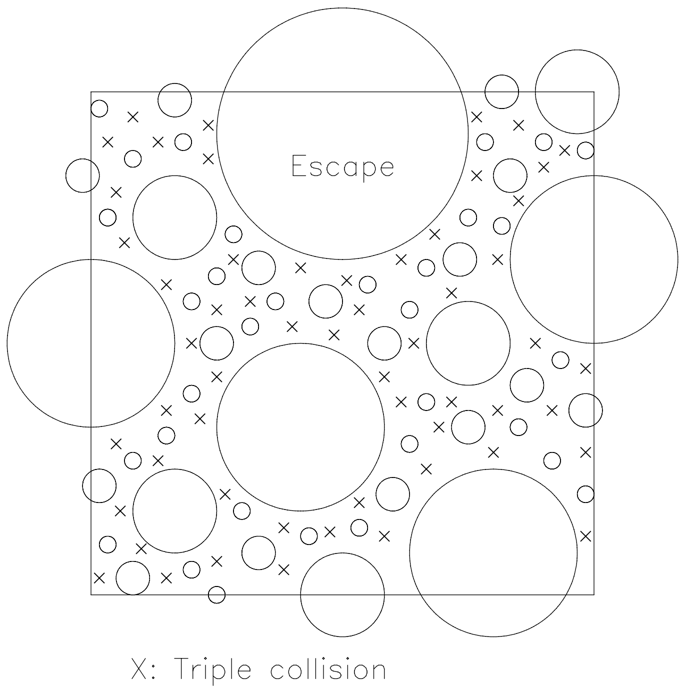

Figure 2: Topology of the final motions of the f-f problem. In case no stable periodic orbits exist.

Let S be the collection of the stable regions of stable periodic orbits.

Assertion 1′. If there are stable periodic orbits, $B \cup OS \setminus S$ is a Cantor-like set in the f-f problem.

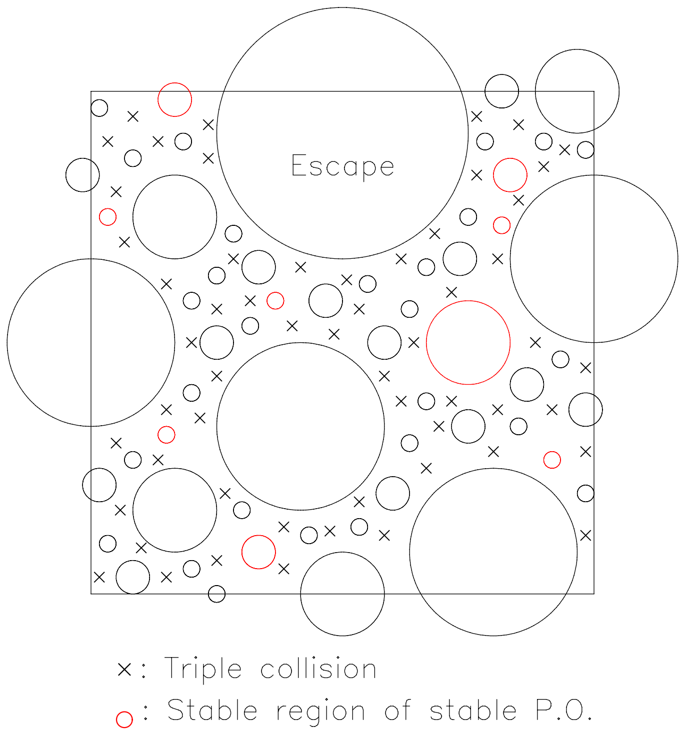

Figure 3: Topology of the final motions of the f-f problem. In case stable periodic orbits exist.

In both cases, crucial is the abundance of triple collisions (Section 3.4 in this report).

Assertion 2. Triple collisions are dense in $B \cup OS$ (resp. $B \cup OS \backslash S$).

Assertion 3. Binary collisions are dense in $B \cup OS$ (resp. $B \cup OS \backslash S$).

Assertion 4. OS are dense in $B \cup OS$ (resp. $B \cup OS \backslash S$).

We do not give proofs in this talk. Later we give evidence for Assertion 2.

We ask:

Are there stable periodic orbits?

The is the motivation of a search for periodic orbits

3 The free-fall problem

3.1 Definition of the problem (cf. Agekyan & Anosova 1967; Tanikawa et al. 1995)

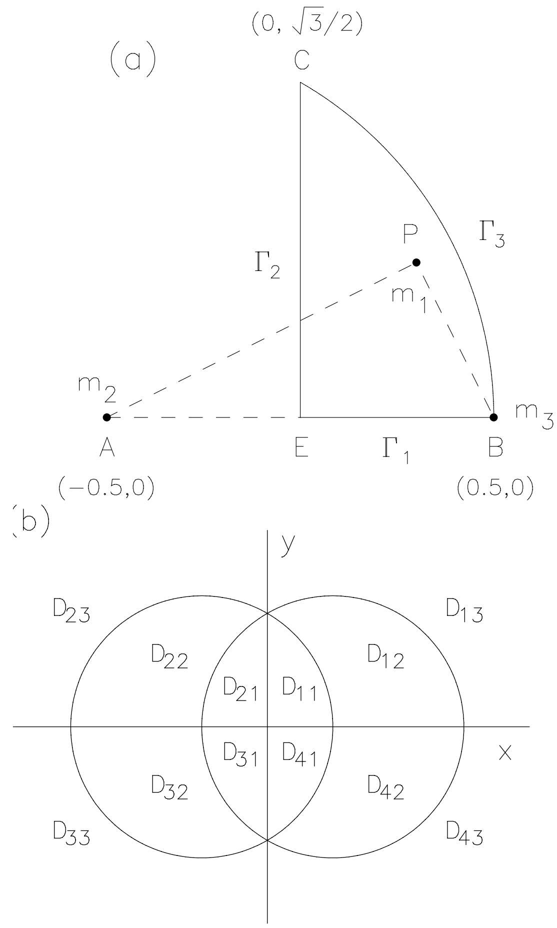

Figure 4: The geometry of the free-fall problem. (a) The initial condition plane. (b) The shape space.

3.2 Symbols and symbol sequences

(Tanikawa & Mikkola 2008, 2015; Montgomery 1998)

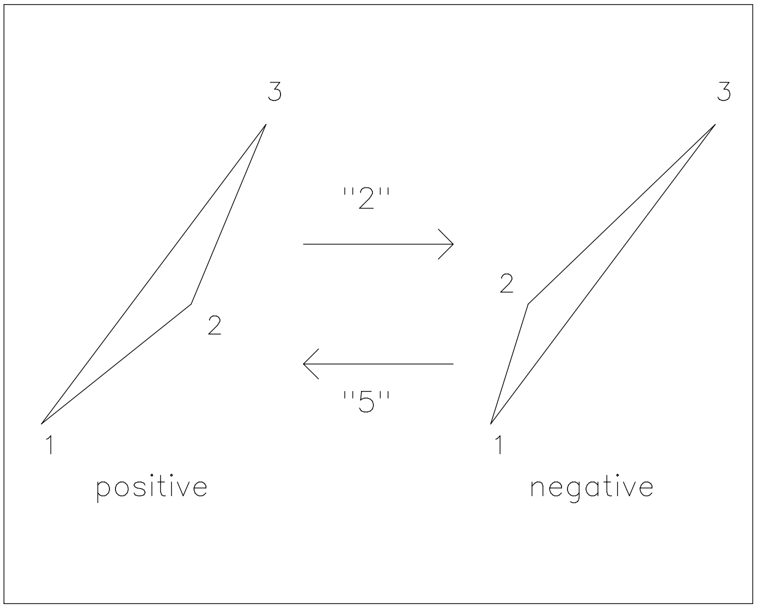

Figure 5: Definition of symbols

We denote a symbol sequence s as

$$s = \dots s_{-3}s_{-2}s_{-1\bullet}s_1s_2s_3\dots \tag{3}$$

where s_i is either 1, 2, 3, 4, 5, or 6.

$$s =_\bullet s_1s_2s_3\dots \tag{4}$$

k-cylinder

$$s =_\bullet s_1s_2s_3\dots s_k *** \quad \text{with arbitrary } *** \tag{5}$$

3.3 Division of the initial condition plane

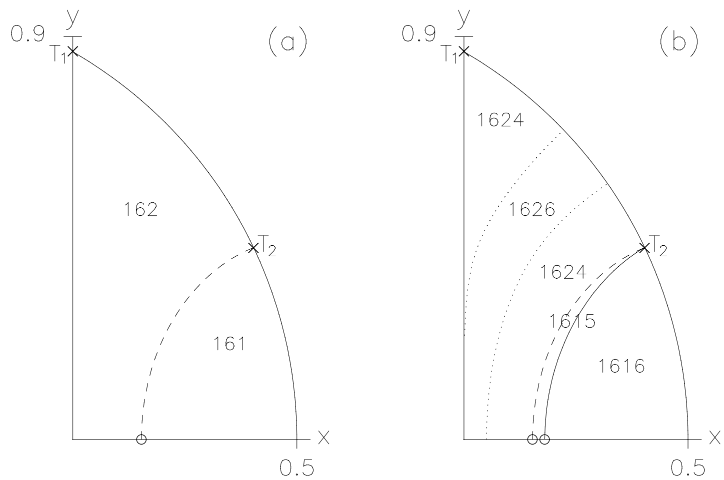

Figure 6: The structure of the initial condition plane. (a) **3-cylinders** $_\bullet$162... and $_\bullet$161..., (b) Division by **4-cylinders**.

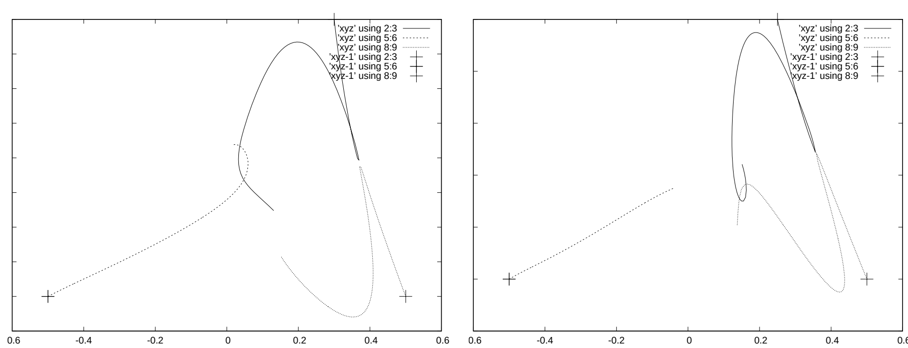

Figure 7: An orbit and symbol sequence. Left:162, right:161.

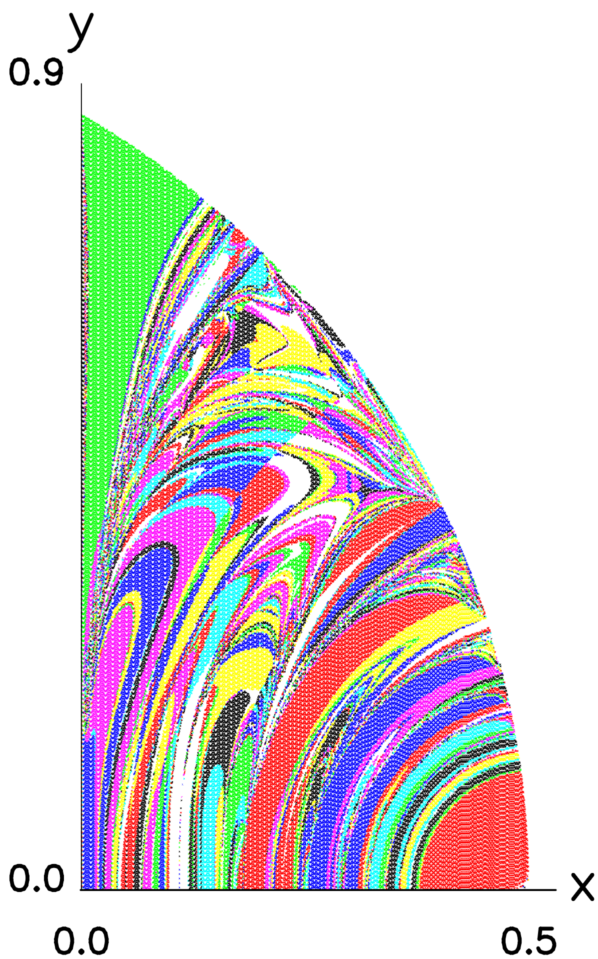

Figure 8: 18 digits, i.e. 18-cylinders.

3.4 Some properties of orbits

3.4.1 Collision curve

A collision curve is a curve of initial conditions of orbits which experience binary collision. Collision curves are the stable (resp. unstable) manifolds of the binary collision manifold (Llibre 1982).

Property 1. Boundaries of cylinders are formed with collision curves. (Tanikawa et al. 1995; Tanikawa & Mikkola 2008, 2015).

Proof. Suppose that cylinders $A =_{\bullet} \cdots 1 * **$ and $B =_{\bullet} \cdots 2 * **$ have a common boundary. In A, body 1 passes through between bodies 2 and 3, while in B, body 2 passes through between bodies 1 and 3. Then at the boundary, bodies 1 and 2 necessarily collide.

A				B		
*————	*———	*		+———	+———	+
3	1	2		1	2	3

A B

3.4.2 Code

A code is a shortest word (finite sequcence of symbols) of a periodic sequence.

Property 2. Any code has the following form:

$$
\begin{aligned}
&s_1 s_2 \dots s_{2k-1} s_{2k} \\
&\text{with} \;\; s_{2k} = s_1 - 3, \\
&\qquad\quad s_{2k-1} = s_2 + 3, \\
&\qquad\qquad \dots .
\end{aligned}
$$

Proof. A periodic sequence can be written as

$$
\begin{array}{c}
\dots s_1 s_2 \dots s_{2k-1} s_{2k} \bullet s_1 s_2 \dots s_{2k-1} s_{2k} \dots \\
\text{past} \quad \Longleftarrow \quad \Longrightarrow \quad \text{future}
\end{array}
$$

The triangles are the same for s_1 and s_{2k}. However, the directions of syzygy crossings are opposite.

Example. 1615341 4162434 (a periodic orbit due to Orlov & Iasko 2015)

Property 3.
(1) If half the period ends at the positive side, then the length of the code is $4k$ for positive k.

$$\begin{array}{ll} \text{(p)}\ s_1\ \text{(n)}\ s_2\ \ \text{(p)}\ \ s_3\ \ \ \text{(n)}\ldots\text{(p)}s_{2k-1}\text{(n)}s_{2k} & \\ & \text{(p)} \\ \text{(p)}s_{4k}\text{(n)}s_{4k-1}\text{(p)}s_{4k-2}\text{(n)}\ldots\text{(p)}s_{2k+2}\text{(n)}s_{2k+1} & \end{array}$$

(2) if half the period ends at the negative side, then the length of bthe code is $2k$ for positive k.

$$\begin{array}{ll} \text{(p)}\ \ s_1\ \ \ \text{(n)}\ s_2\ \ \text{(p)}\ \ s_3\ \ \ \text{(n)}\ldots\text{(p)}s_{2k-1} & \\ & \text{(n)} \\ \text{(p)}s_{4k-2}\text{(n)}s_{4k-3}\text{(p)}s_{4k-4}\text{(n)}\ldots\text{(p)}\ s_{2k} & \end{array}$$

The shortest candidate is '1634'. However, this does not exist. The next candidates are '161434', '162534', and '163634'. It seems that these are not existent. The period is too short.

3.4.3 Triple collision point

A triple collision point is a point whose orbit ends in triple collision.

Property 4. Triple collision points are obtained as intersections of different types of collision curves.

Proof. Suppose that a collision curve in which bodies i and j collide and a collision curve in which bodies i and k collide intersect. Then at intersections, bodies i, j, and k collide.

(Almost all) cylinders have triple collision points on their boundaries.

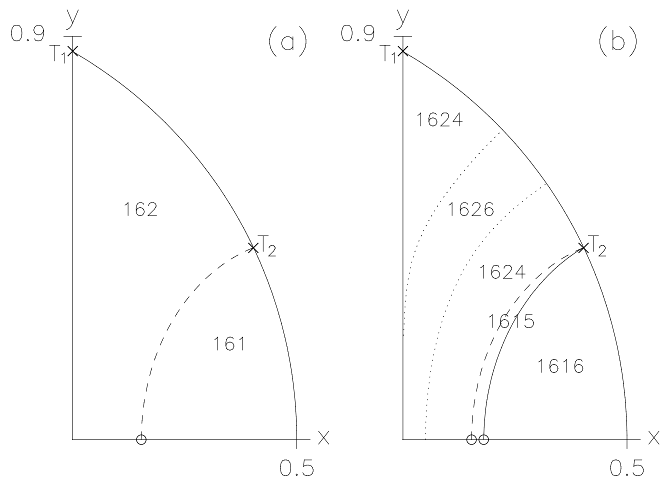

Figure 9: The structure of the initial condition plane. (a) **3-cylinders** $\bullet$162... and $\bullet$161..., (b) Division by **4-cylinders**.

4 Search for periodic orbits

4.1 Procedures of search for periodic orbits

We propose a one-dimensional search along certain curves in the plane.

1) Orbits which experience binary collisions.
2) Orbits which do not experience collisions.

The preceding works adopt two dimensional searches:

Szebehely & Peters (1967): masses 3:4:5;
Orlov & Iasko (2015): the free-fall problem;
Suvakov & Dmitrasinovic (2011,2013,2015);
Dmitrasinovic & Suvakov (2015):
the case of zero-angular-momentum;
Rose (2015): the case of zero-angular-momentum.

4.2 A perodic orbit by Szebehely & Peters (1967)

The Pythagorean problem:

Put the three masses on the vertices of the Pythagorean triangle 3:4:5, and start integrations with zero initial velocities.

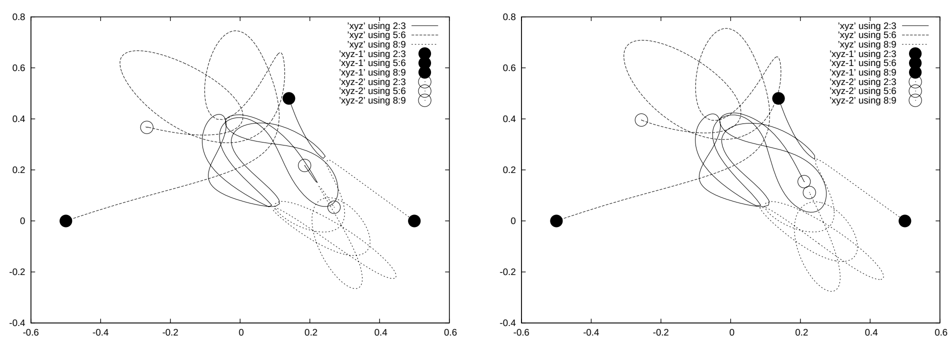

Figure 10: The free-fall problem with mass 3:4:5. Left panel: the so-called Pythagorean orbit. Right panel: A periodic orbit with collision close to the Pythagorean orbit.

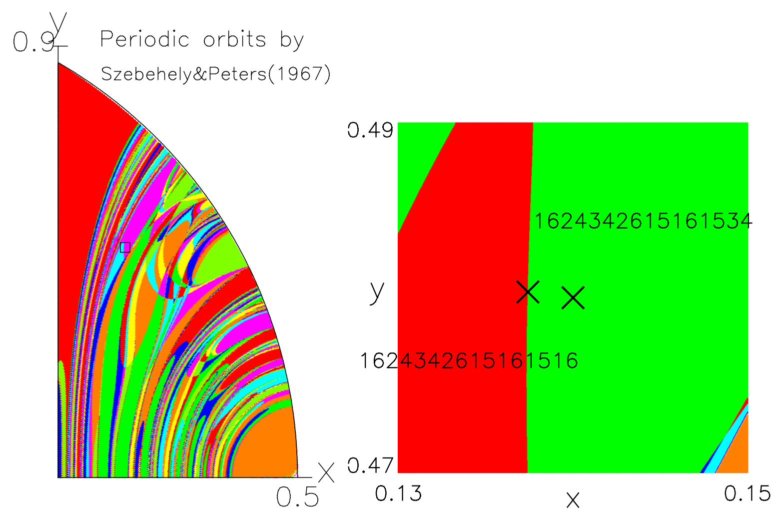

Figure 11: The free-fall problem with mass 3:4:5. Left panel: The Pythagorean orbit is in a small box. Right panel: Enlargement of the box. The rightcross corresponds to the Pythagorean orbit, and the left cross corresponds to the periodic orbit dixcovered by SP(1967). Colored regions are cylinders up to and including 16 digits.

4.3 Perodic orbits by Orlov & Iasko (2015)

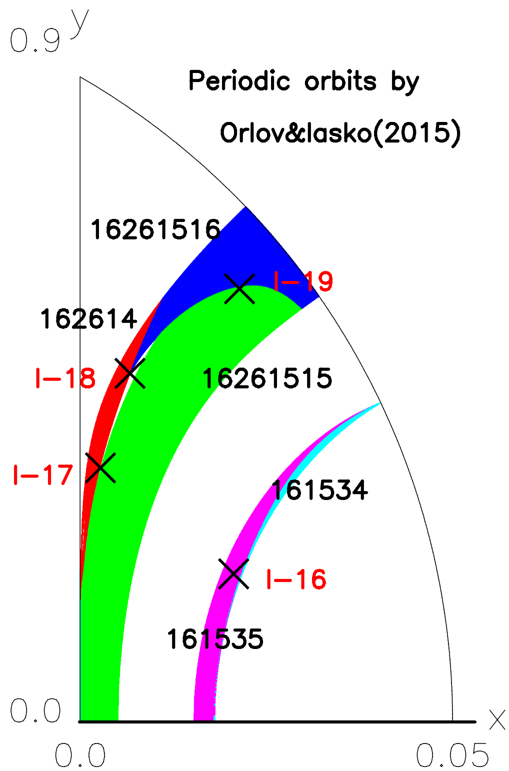

Figure 12: The equal-mass free-fall problem. Crosses indicate the positions of periodic orbits inside D: I-16, I-17, I-18, and I-19. Colored regions are cylinders.

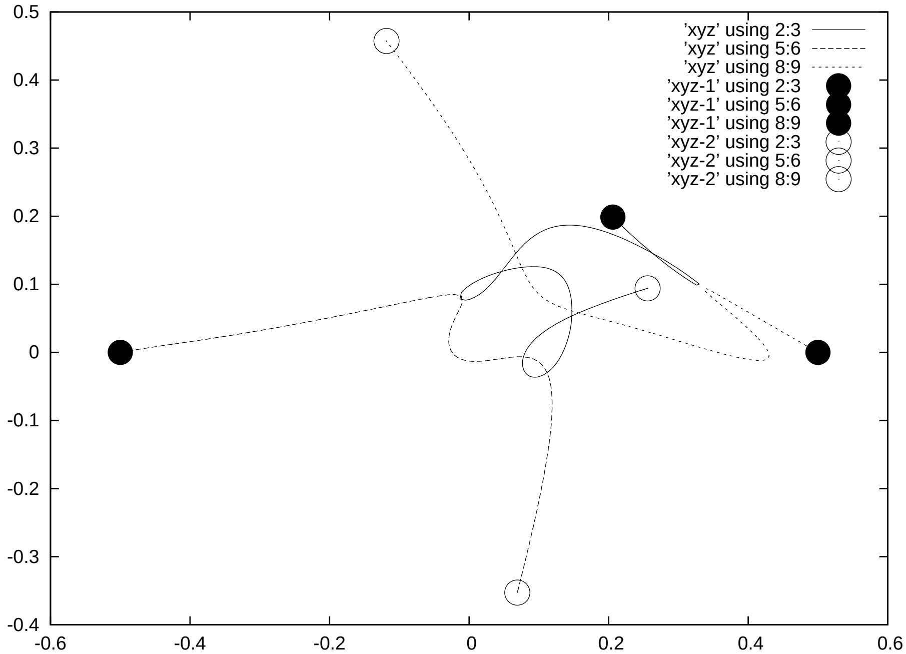

Figure 13: The equal-mass free-fall problem. A periodic orbit inside D: I-16 (Orlov & Iasko 2015).

The code is

$$1615341\ 4162434$$

of length 14.

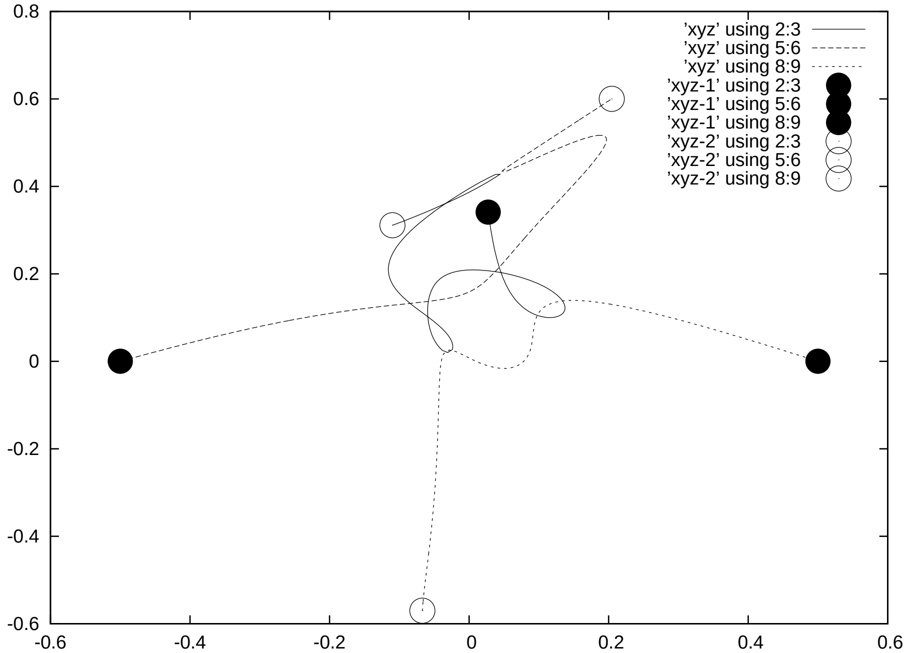

Figure 14: The equal-mass free-fall problem. A periodic orbit inside D: I-17 (Orlov & Iasko 2015).

The code is

$$1626151\ 4243534$$

of length 14.

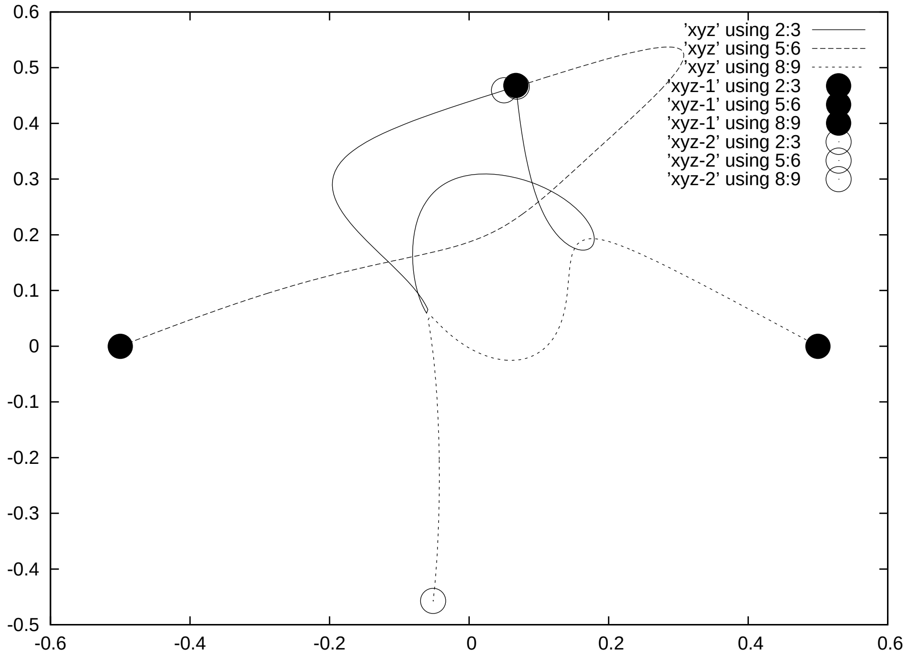

Figure 15: The equal-mass free-fall problem. A periodic orbit inside D: I-18 (Orlov & Iasko 2015).

The code is

$$16261^{51}_{42}43534$$

of length 12.

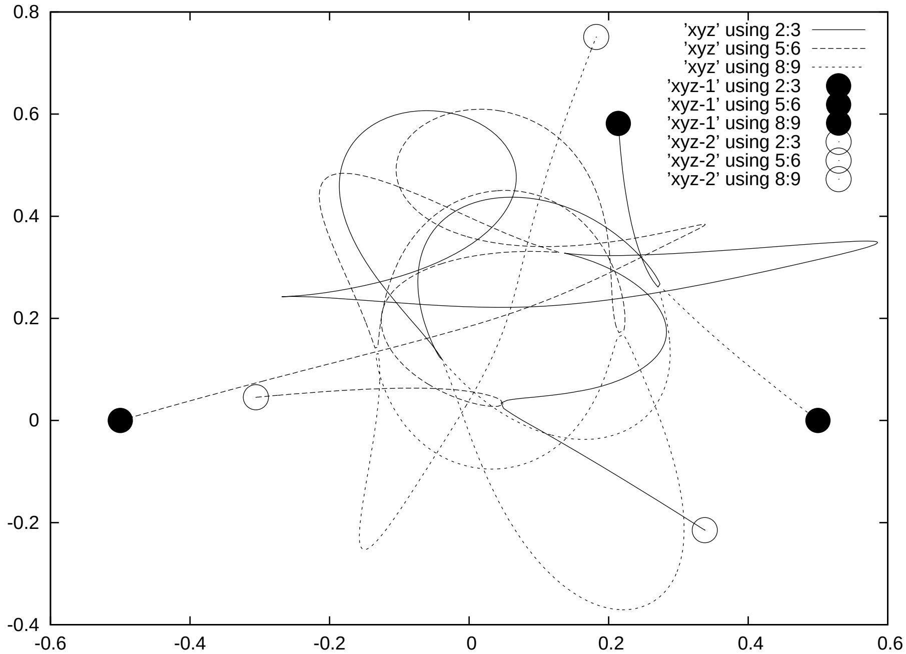

Figure 16: The equal-mass free-fall problem. A periodic orbit with collision inside D: I-19 (Orlov & Iasko 2015).

The code is

$$1626151534 26^{15}_{24} 342\ 516^{24}_{15} 351624243534$$

of length 34.

4.4 One-dimensional search along curves

Procedure 1(orbits with collision).
Periodic orbits which experience binary collision are on the collision curves. So, look for them on **the arc of collision curves between two triple collisions**.

Procedure 2(orbits without collision).

Partial Free-Fall point (PFF point): A point whose orbit experiences in some future a standstill of one of the bodies, and which is the closest to the initial condition plane along its orbit.

Partial Free-Fall curve (PFF curve): A curve formed with PFF points in the phase spece of the f-f problem.

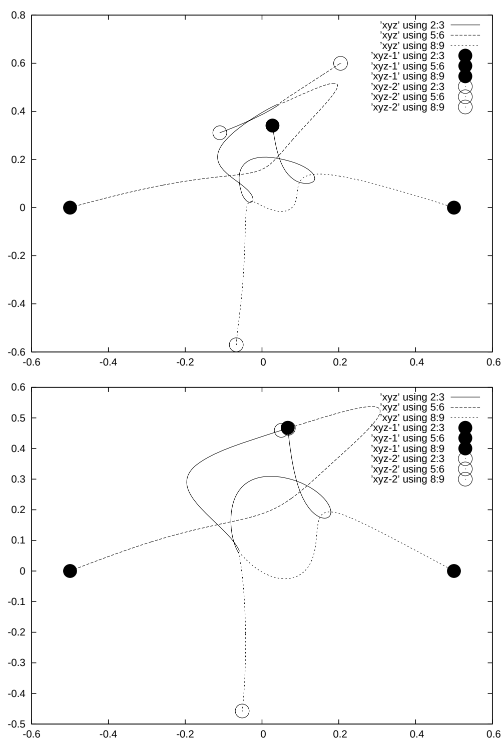

Figure 17: The equal-mass free-fall problem. Periodic orbitis I-17 and I-18 (Orlov & Iasko 2015).

Conjecture. There are PFF curves in the full phase space of the f-f problem. These passes through periodic points in the initial condition plane. So, there are approximate PFF curves as shadows of PFF (SPFF) curves.

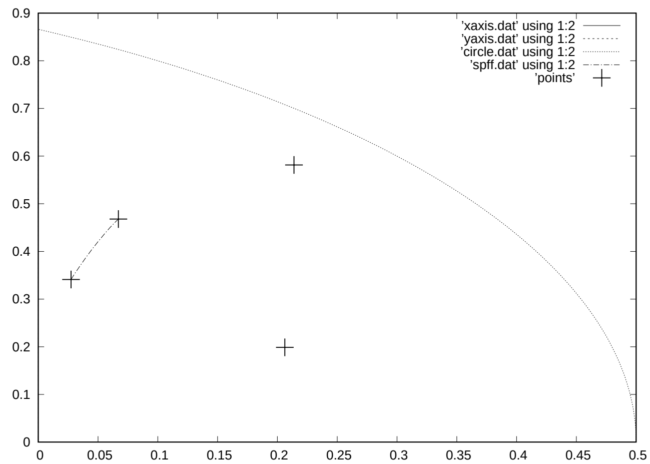

Figure 18: Evidence for the Shadow of the PFF curves which passes I-17 and I-18.

4.5 Stability of periodic orbits

Periodic orbits with collision are expected to be unstable because the future of the orbits are different in the different sides of the collision curves.

5 Concluding remarks

- We have shown that $B \cup OS$ (resp. $B \cup OS \backslash S$) is a Cantor-like set. The structure of the non-escape final motions depends on the stability of periodic orbits.

- The search for periodic orbits can be done one-dimensionally along curves.

- The free-fall problem has variety of initial conditions depending on the combinations of masses such as $m_i \to 0$ or $m_i, m_j \to 0$. The discussions on the stability of periodic orbits may be very difficult.

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