Entanglement versus relaxation and decoherence in a quantum algorithm for quantum chaos

Stefano Bettelli
Laboratoire de Physique Théorique, IRSAMC, Université Paul Sabatier, Toulouse.

Abstract

Since the first quantum computers will likely suffer severe memory constraints, it is interesting to study quantum algorithms for the simulation of chaotic maps, which investigate non-trivial physics with only 10-15 qubits. This poster details a recent study about the entanglement properties of the quantum memory under the effect of such an algorithm, and the sensibility of these properties to noise in the implementation of the quantum computer. (*ref. PRA* **67***, 054303 (2003), available as quant-ph/0301086*).

A need for practical algorithms

Although there is already a number of theoretical results for ideal (noiseless) quantum computers which prove the superiority of the quantum computational model over the classical one, no scalable quantum computer has been built so far, and the technological and (maybe) fundamental problems concerning their design indicate that in the foreseeable future the quantum computer community will not have access to devices with a memory larger than 10-15 qubits.

It is therefore interesting, for the time being, to design and study quantum algorithms with very limited memory requirements, which could however be used as a test-ground for error correction strategies. In this context, the quantum algorithms for the simulation of chaotic maps are ideal candidates, because they satisfy the previous requirement and allow for the investigation of non-trivial physical phenomena.

Classical and quantum (chaotic) maps

We study time-dependent Hamiltonian systems with periodic conditions on q and on p, which can show the phenomenon of dynamical chaos. The discretised dynamics, determined by the parameters K and L, can be understood as a "kick" followed by a free evolution:

$$\begin{cases} \bar{q} &= q + \bar{p} \mod 2\pi \\ \bar{p} &= p - K\frac{dV}{dq} \mod 2\pi L \end{cases} \qquad H(t) = \frac{p^2}{2} - KV(q)\sum_n \delta(t-n)$$

In the quantum case one must consider also the number of levels N (with $\hbar = 2\pi L/N$). The Floquet operator (evolution operator corresponding to one iteration of the map) is a product of two terms, which are diagonal respectively in $\hat{p}$ and $\hat{q}$.

$$\boxed{U_F = e^{-i\hat{p}^2/2\hbar}e^{-iKV(\hat{q})/\hbar}} = \exp\left[-i\frac{\pi L}{N}\hat{n}_p^2\right]\cdot\exp\left[-i\frac{K}{L}\left(\frac{2\pi}{N}\right)^{\alpha-1}\mathcal{P}_\alpha(\hat{n}_q)\right]$$

Since the Floquet operator is a composition of operator which are diagonal in the position and momentum space, it is easily implemented using the QFT ($\mathcal{F}$) to move from one representation to the other:

$$U_F = \mathcal{F}^\dagger \circ e^{-i\alpha x^2} \circ \mathcal{F} \circ e^{-i\beta\hat{V}(x)}$$

Differently from the classical case, this allows for an efficient simulation of the evolution of the system for one time step on a quantum computer, for those potentials which are polynomials, by considering $x = \sum_{j=0}^{n-1} a_j 2^j$:

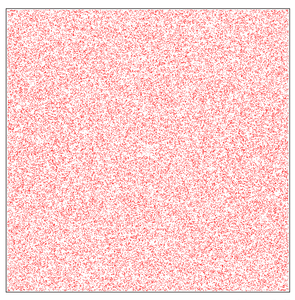
$$e^{-2\pi i\gamma x^\alpha} = \prod_{j_1\cdots j_\alpha} e^{-2\pi i\gamma a_{j_1}\cdots a_{j_\alpha}2^{j_1+\ldots+j_\alpha}} = \prod_{j_1\cdots j_\alpha} \mathtt{RPC}_{j_1\ldots j_\alpha}\left(\gamma 2^{j_1+\ldots+j_\alpha}\right)$$

Complexity: it is easily seen that, if n_q is the number of qubits the wave function is expanded on, the exponentiation circuit requires n_q^α gates, and that it is the most expensive subroutine of the quantum double-well algorithm. Thus this algorithm can calculate one time step in the quantum map in polynomial time (with respect to n_q), while classical algorithms cannot do better than $n_q 2^{n_q}$ (since they are limited by fast Fourier transforms).

All multi-controlled phase shifts are broken down into one or two qubit primitives (Hadamard gates, phase shifts and controlled phase shifts), but this does not change the complexity. For $\alpha = 4$ one additional working qubit is required, while for $\alpha = 2$ all the qubits are used for storing the representation of the quantum state. Further optimisations can reduce the circuit size by a factor $\frac{1}{\alpha!}$

The algorithm in this poster has been implemented using a quantum language developed by one of the authors (S.Bettelli, "Toward an architecture for quantum programming", cs.PL/0103009, accepted on EPJD), and the numerical experiments have been performed up to $n_q \sim 20$ qubits.

The classical sawtooth map

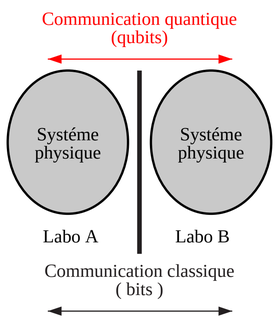


$K = 0.5$

The chaotic map which we have chosen for this study is the sawtooth map, determined by the potential $V(q) = -\frac{q^2}{2}$.

It is easy to understand that for positive K the potential behaves repulsively and the classical map become globally chaotic. Due to the discontinuity of $V'(q)$ the KAM theorem does not apply. The map shows normal diffusion, with the diffusion coefficient, defined by $D_0(K) = \langle p^2\rangle/t$, which grows quadratically for large K ($D_0(K) \simeq D_{ql} = (\pi^2/3)K^2$) but is depressed for small K ($D_0(K) \sim 3.3K^{2.5}$), and shows oscillations around the asymptotic value for $K > 1$.

Entanglement and its characterisation



The possibility for quantum systems to be entangled is more than a simple correlation: the state which describes the whole system cannot be factorised. One of the main problems of the theory of quantum information is to find a measure for the phenomenon of entanglement. The *entanglement of formation* can be understood simply as the number of qubits which are to be exchanged on average in order to build up a given quantum state for a bi-partite system:

$$E(\rho) = \min_{\rho\to\{p_i,\psi_i\}}\sum_i p_i S(\mathrm{tr}_B|\,\psi_i\,\rangle\langle\,\psi_i\,|)$$

For the more general case of multi-partite systems, no satisfactory definition has been found up to now. The additivity properties are not yet clear too.

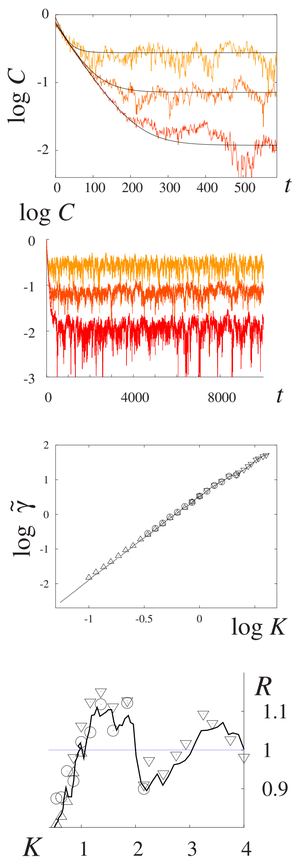
Many researchers think that the power of quantum computation must be linked to the possibility of exploiting entangled states. Without this resource, one should resort either to an exponentially large memory or to measurement devices with exponential precision. That notwithstanding, the simple presence of entanglement is not sufficient to beat classical computation (→ Gottesmann and Knill's theorem).

There exist also a universal computational model (see Raussendorf, "One-way quantum computer", Phys. Rev. Lett. **86**, 5188 (2001)) where a particular initially entangled state is used as the only resource, renouncing to multi-qubit gates: the implementation of the computation exploits only one-qubit gates and one-qubit measures, which destroy the entangled state in order to produce the final result.

We have chosen to study the evolution of the entanglement of formation of the two most significant qubits in the quantum computer memory while the quantum sawtooth map algorithm is running, using the "concurrence" C. The concurrence depends on the reduced density matrix ρ of the two qubits. If one defines $\widetilde{\rho} = (\sigma_y\otimes\sigma_y)\rho^*(\sigma_y\otimes\sigma_y)$, then C is $\max\{0, \lambda_1-\lambda_2-\lambda_3-\lambda_4\}$ where the λ_i are the square roots of the eigenvalues, in decreasing value order, of $\sqrt{\widetilde{\rho}}\rho\sqrt{\widetilde{\rho}}$ [see Wootters, Phys. Rev. Lett. **80**, 2245 (1998)]. The initial state we use is $|\,\psi\,\rangle = \frac{|\,00\,\rangle+|\,11\,\rangle}{\sqrt{2}}|\,\phi\,\rangle$, for which $C = 1$, and L is always a multiple of 4.

This quantity does not account for the overall entanglement of the quantum memory, but it has been proven to be linked to interesting physical properties (like quantum phase transitions), and its degradation due to "errors" in the computation should be correlated to the powerfulness of the computation.

Behaviour of the concurrence on a perfect computer



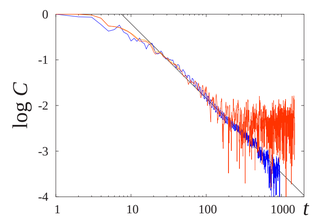
Initial evolution of the concurrence for the sawtooth map at $K = 0.5, L = 4$ and $n_q = 8, 12, 16$ (curves from top to bottom respectively). The smooth curves show the fit $C(t) = A\exp(-\gamma t)+\overline{C}$ of the relaxation to the asymptotic value $\overline{C}$.

Behaviour of $C(t)$ on a larger time scale, showing the asymptotic regime. The initial state is $(|\,00\,\rangle+|\,11\,\rangle)|\,\phi\,\rangle/\sqrt{2}$ where $|\,\phi\,\rangle$ is the uniform superposition of all but the two most significant qubits.

Dependence of the rescaled rate of the concurrence decay, $\tilde{\gamma} = 2\gamma L^2 = (\gamma/\gamma_c)D_0(K)$, on the chaos parameter K for $n_q = 19, L = 16$ (triangles down); $n_q = 18, L = 8$ (circles) and $n_q = 17, L = 4$ (triangles up). The solid curve gives the values of the diffusion rate $D_0(K)$.

This picture shows the data on a larger scale with $R = \tilde{\gamma}/D_{ql}$ (symbols) and $R = D_0(K)/D_{ql}$. It is evident that $\tilde{\gamma}$ follows not only the general trend of D_0 but also its oscillations, showing that γ is almost exactly the classical relaxation rate $\gamma_c = D_0(K)/(2L^2)$.

The scaling of the residual value of the concurrence



The decrease of the concurrence seems to be linked to the classical relaxation of the probability of being located in the initial region. It is therefore natural to write the generic state $|\,\psi\,\rangle$ as $\sum_a |\,a\,\rangle|\,\phi_a\,\rangle$, where $a = 00, 01, 10$ or 1, and to conjecture that C is approximated by

$$\delta\mathcal{P}_{quant} = |W_{00} + W_{11} - W_{01} - W_{10}|$$

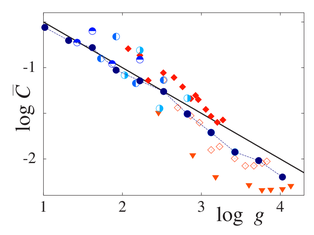
($W_a = \langle\,\phi_a\,|\,\phi_a\,\rangle$ is the a-region probability).

We have checked the validity of this conjecture also in the more complicated case (mixed phase space) of the double-well map, where the decrease of $\delta\mathcal{P}_{class}$ (black, $5\cdot10^6$ points) is polynomial, dominated by the *Poincaré recurrences*. It is possible to see that C (red, $n_q = 16, L = 4$) follows $\delta\mathcal{P}_{class}$ up to $t \sim -\frac{1}{\gamma_c}\ln\overline{C}$.

If one makes the hypothesis that $\delta\mathcal{P}_{quant}$ is a good approximation for C also in the asymptotic regime, then $\overline{C}$ is determined by its fluctuations. Due to the discretisation of the momentum, the symmetry is broken and, in each of the W_a's, the terms have random signs; for this reason one finds $\overline{C} \propto 1/\sqrt{N}$, where $N = 2^{n_q}$ is the number of levels in the discretisation.

It is however known in solid state physics that only the states in the Thouless energy domain Δ_T have a significant scalar superposition, and one can further conjecture that N must be replaced by $N_{\mathrm{eff}} \sim \Delta_T/\Delta \sim g$ (where $g = DN/L^2$ is the *Thouless conductance*), giving $\overline{C} \sim 1/\sqrt{g}$.

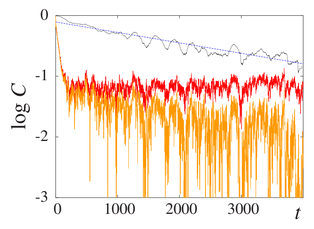
It is possible to estimate Δ_T by using the diffusion relation $\hbar^2\langle n^2\rangle = \langle p^2\rangle = D_0 t$, which determines a characteristic time for "travelling through" the momentum spectrum: $\sqrt{\langle n^2\rangle} \sim N \Rightarrow t_D \sim (N\hbar)^2/D_0 \sim L^2/D_0$. With this characteristic time and the indetermination $\Delta_T t_D \sim \hbar$, one finds $\Delta_T \sim D_0/(NL)$. On the other hand, Δ, the average separation of the quasi-energy levels, is $\Delta \sim \hbar/N \sim L/N^2$, since $\omega = E/\hbar$ and $\Delta\omega \sim 1/N$.



The previous conjecture has been tested numerically. This picture shows the dependence of the residual value of the concurrence $\bar{C}$ versus the system conductance $g = ND_0(K)/L^2$ for a broad range of parameters: half filled circles show the dependence on $L = 4, 8, 12, 16, 20$ for $K = 0.5$ and $n_q = 14, 15, 16$; diamonds and triangles show the variation with K for $n_q = 14, L = 16$; $n_q = 15, L = 8$ and $n_q = 16, L = 4$. The filled circles connected by dashed lines show the dependence on N for $K = 0.5, L = 4$.

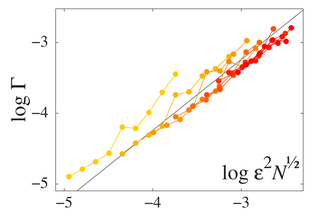
The solid line marks the slope $1/\sqrt{g}$. We attribute the presence of strong fluctuations to the fact that the value $\bar{C}$ is averaged only over the time but there is no averaging over the parameters. Thus, from the point of view of disordered systems $\bar{C}$ represents only one value for one realisation of disorder.

The decay induced by noisy gates



Here $n_q = 12, K = 0.5, L = 4$.

The last part of this study concerns the effect of noisy gates on the residual concurrence. This picture illustrates the procedure for extracting numerically the induced decay rate. The two lower curves show the dependence $C(t)$ for the perfect algorithm (top) and the algorithm with noisy gates at noise amplitude $\epsilon = 0.003$ (bottom). In the latter case the average is done over 20 noise realizations. The curve in the upper part shows the ratio of $C(t)$ at $\epsilon = 0.003$ to its value in a perfect algorithm; this ratio is averaged over a 100 kick moving window to reduce fluctuations. The dashed straight line shows a fit of the ratio to an exponential decay proportional to $e^{-\Gamma t}$ (the initial classical relaxation drop is excluded from the fit).



This picture illustrates the dependence of the decay rate on the error intensity and the number of qubits. The straight line shows the averaged behaviour $\Gamma = 0.58\epsilon^2\sqrt{N}$. Quite naturally we find that $\Gamma \propto \epsilon^2$ [see, for instance, • Phys. Rev. A, **66**, 054301 (2002)]. This scaling becomes better and better for large ϵ values where Γ is larger. However, more surprisingly there is an exponential growth of $\Gamma \propto \sqrt{N}$.

This result is very different from those obtained in other papers [see • and Phys. Rev. Lett. **87**, 227901 (2001)], where the time scale for the fidelity and the decoherence rate for tunnelling oscillations varied polynomially with n. A possible explanation is that the eigenstates are exponentially sensitive to imperfections due to the chaotic structure of the wave functions [see e.g. Eur. Phys. J. D **20**, 293 (2002)], but more investigation is required.

Summary and future studies

- The decay of the concurrence in an operating quantum computer is determined by the underlying relaxation rate of the classical dynamics.
- The residual level of entanglement in a perfect algorithm scales as the inverse square root of the conductance of the system.
- This residual entanglement is destroyed by a decoherence whose effective rate grows exponentially with the number of qubits. (This is an unexpected result).
- We are currently studying the behaviour of the concurrence on a chaotic map with mixed phase space structure (Poincaré recurrences).

Quantum Chaos: Theory and Applications
Villa Olmo, Como, June 17-22, 2003