Ulam Method, Fractal Weyl Law, and Complex Networks

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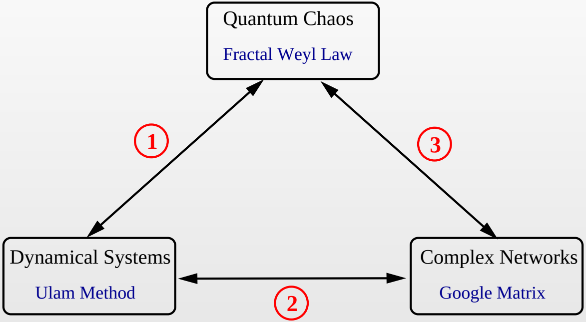
Outline

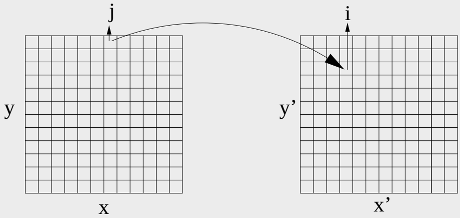


1. Fractal Weyl law in Ulam approximant of Perron-Frobenius operators. Dissipation and scattering cases.
2. Ulam method for 1d intermitency maps: modeling Google matrix.
3. Fractal Weyl law in complex networks: Linux Kernel networks.

Ulam approximant of Perron-Frobenius operator

Discretized phase-space:
Adjacency matrix $\mathbf{A} = P(j \rightarrow i)$

$N = N_x \times N_y$ cells.



N_c: traj. from cell j
N_i: traj. to call i

$$\begin{cases} \mathbf{A}_{i,j} = N_i / N_c \\ \sum_i \mathbf{A}_{i,j} = 1 \quad \text{(closed systems)} \end{cases}$$

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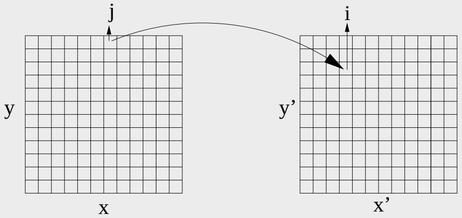

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Spectrum and PF theorem

$$\begin{aligned} \mathbf{A}\psi_i^R &= \lambda_i \psi_i^R \\ \psi_i^L \mathbf{A} &= \psi_i^L \lambda_i \end{aligned}$$

- Unique largest real eigenvalue
- Corresponding eigenvector positive

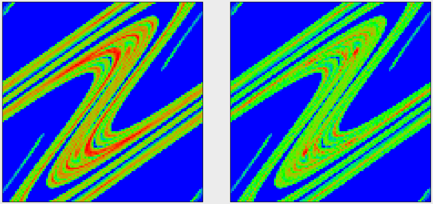
Fractal Wey law for open quantum systems

$$N_\gamma \propto N^\nu, \qquad N = V/\hbar$$
$$\nu = d - 1, \qquad d : \text{FTS}$$

Two models

Scattering

$$\begin{cases} \bar{y} = y + K\sin(x + y/2) \\ \bar{x} = x + (y + \bar{y})/2 \pmod{2\pi} \end{cases}$$



$N = 110 \times 110,\ K = 7, a = 2$

$\lambda_1 = 0.756 \quad \lambda_3 = -0.01 + i0.513$

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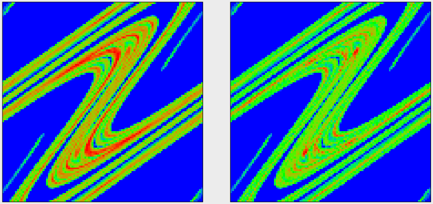

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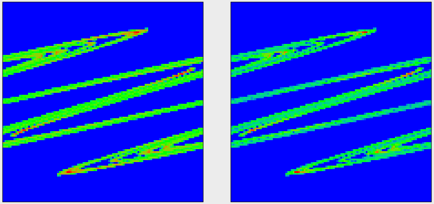
Dissipation

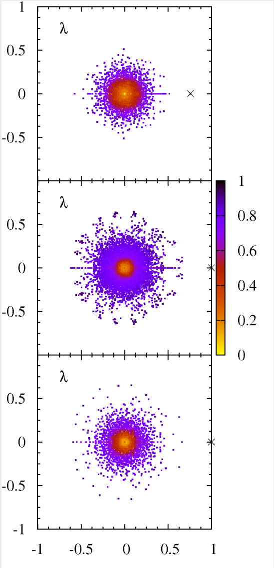
$$\begin{cases} \bar{y} = \eta y + K\sin x \\ \bar{x} = x + \bar{y} \pmod{2\pi} \end{cases}$$



$N = 110 \times 110,\ K = 7, \eta = 0.3$
$\lambda_1 = 1 \quad \lambda_3 = -0.258 + i0.445$

Spectral properties

Spectrum ($\mu_i = \sum_l \psi_1(l)|\psi_i(l)|$)



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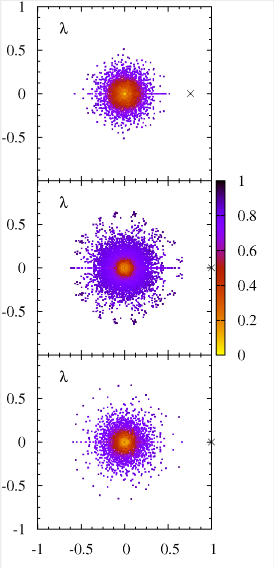
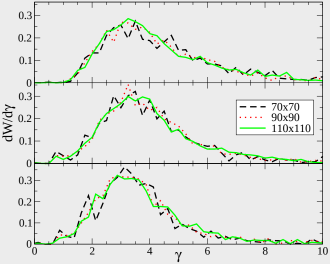
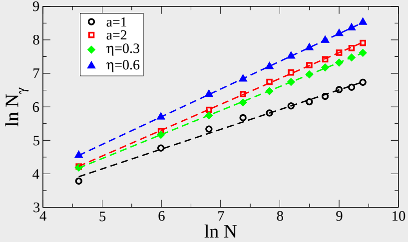


Distribution of γ

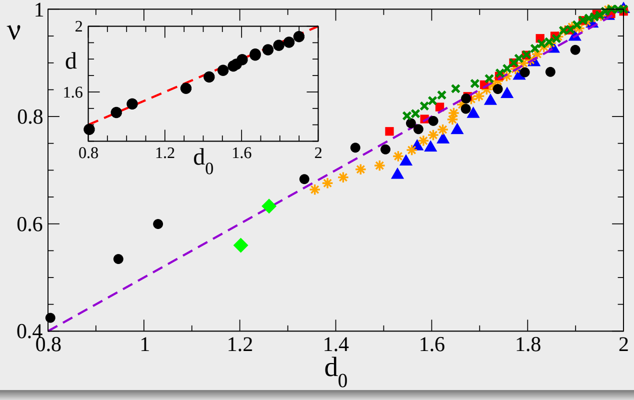


N_λ vs. N



Fractal Weyl law

ν vs. d_0 (repeller-attractor) $\longrightarrow \nu = d_0/2$ (d: FTS)



$d, d_0 \rightarrow$ box counting

- $\bullet \rightarrow$ m1: $K = 7$ $a \in [0.8, 6]$
- $\times \rightarrow$ m2: $K = 15$ $\eta \in [0.3, 1]$
- $* \rightarrow$ m2: $K = 10$
- ■ $\rightarrow$ m2: $K = 12$
- ▲ $\rightarrow$ m2: $K = 7$
- ◆ $\rightarrow$ Henon $a = 1.2, 1.4; b = 0.3$

Google matrix and PageRank

Google matrix

$$\mathbf{G} = \alpha \mathbf{S} + (1-\alpha)\mathbf{E}/N$$

- $\mathbf{S}$ is constructed from the adjacency matrix $\mathbf{A}$ of directed network links between N nodes.
  1. $S_{ij} = A_{ij} / \sum_k A_{kj}$
  2. columns with only zero elements are replaced by $1/N$
- The second term describes a finite probability $1-\alpha$ for WWW surfer to jump at random to any node so that the matrix elements $E_{ij} = 1$.

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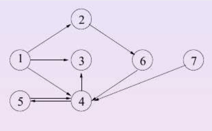
example



$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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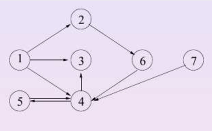

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PageRank: p

- **G** follows PFT (with $\lambda_1 = 1$)
- $\alpha = 0.85$ (random after 6 clicks)
- $\mathbf{G}p = p$

Real networks

Characteristic properties

- **Small world:** average distance between 2 nodes $\sim \log N$
- **Scale–free:** distribution of in/out-coming links $P(k) \sim k^{-\nu}$ ($\nu_i \simeq 2.1$, $\nu_o \simeq 2.7$)

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Explanation

- Constant growth: new nodes appear regularly, and are attached to the network.
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PageRank of WWW

- $p \sim 1/j^{\beta}$ and $\beta \simeq 0.9$ (where j is the order index)
- Conjecture: β and ν are correlated.

Intermittency maps: 2 models

A.S. Pikovsky, Phys. Rev. A **43**, 3146 (1991).

$$f_1(x) = \begin{cases} x + (2x)^{z_1}/2 & \text{for } 0 \leq x < 1/2 \\ (2x - 1 - (1-x)^{z_2} + 1/2^{z_2})/(1 + 1/2^{z_2}) & \text{for } 1/2 \leq x \leq 1 \end{cases}$$

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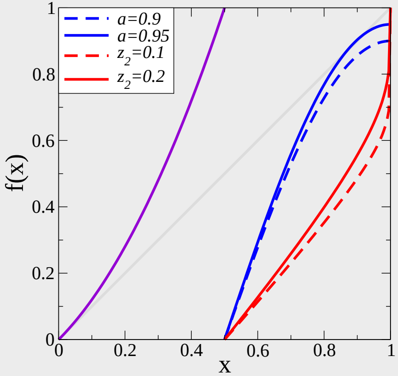

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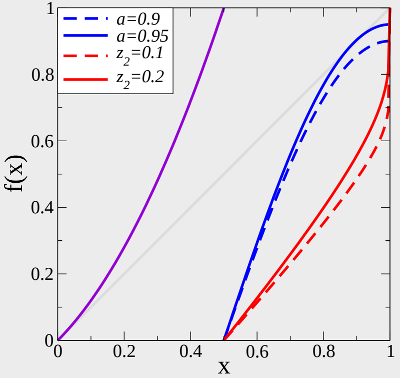

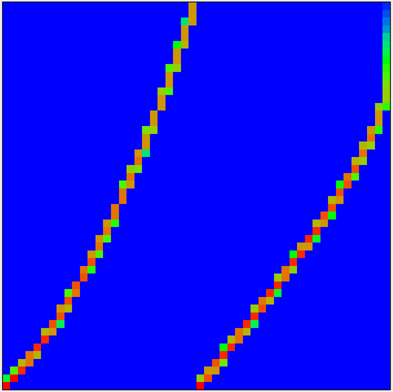

Intermittency maps motivation

steady state invariant distribution

$g(x) \propto t(x)$ by a trajectory ($t \sim \frac{1}{x^{1-z_1}} \propto g(x))^*$: power law distrib. (for small values of x)

- f_1-map: fully chaotic while
- f_2-map: a fixed point attractor appears for $a > 0.945$ (when $f_2(x) = x$).

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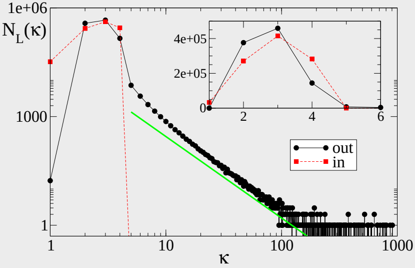
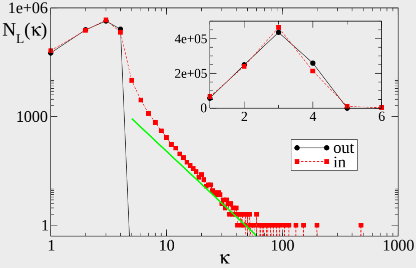
f_1-map link distribution



f_2-map link distribution



Link distribution

- sharp drop of ingoing links
- power law decay of <u>outgoing</u> links

$$\kappa = \frac{d\overline{x}}{dx} \text{ (div. near } x = 1) \longrightarrow \kappa \sim \frac{1}{(1-x)^{(1-z_2)}}$$

The number of nodes with κ links is

$$N_n \sim (1-x) \sim \frac{1}{\kappa^{1/(1-z_2)}}$$

and the differential distribution of nodes

$$N_L^{out} \sim \frac{dN_n}{d\kappa} \sim \frac{1}{\kappa^\mu} \text{ with } \mu = \frac{2-z_2}{1-z_2}:$$

For this case with $z_2 = 0.2 \rightarrow \mu = 9/4$

- sharp drop of outgoing links
- power law decay of <u>ingoing</u> links

$\kappa = \frac{dx}{d\overline{x}} \sim \frac{1}{\overline{x}^{1-1/2\nu}}$ since $\overline{x} \sim (1-x)^{2\nu}$ near $x = 1$ ($\nu = 1$ in our case)

The number of nodes with κ links is

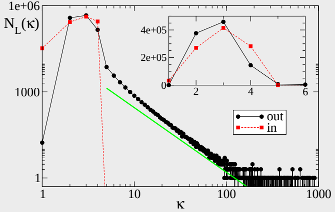
$$N_n \sim \overline{x} \sim \frac{1}{\kappa^{2\nu/(2\nu-1)}}$$

and the differential distribution of nodes

$$N_L^{in} \sim \frac{dN_n}{d\kappa} \sim \frac{1}{\kappa^\mu} \text{ with } \mu = \frac{4\nu-1}{2\nu-1}$$

For our case with $\nu = 1 \rightarrow \mu = 3$

f_1-map link distribution



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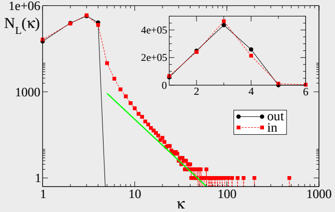
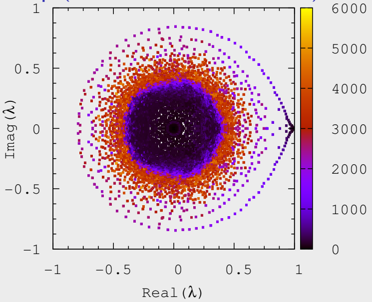
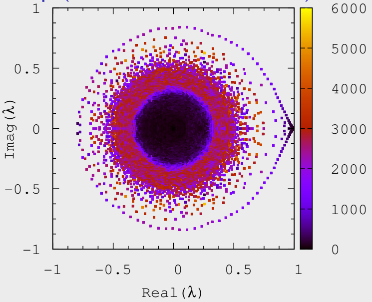


Spectrum

f_1-map ($N = 12000$, $z_1 = 2$, $z_2 = 0.2$)



f_2-map ($N = 12000$, $z_1 = 2$, $a = 0.9$)



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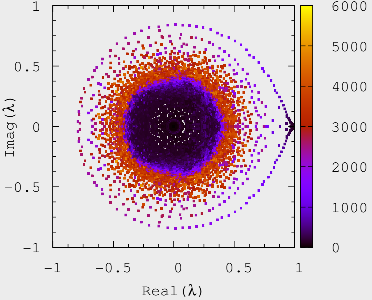

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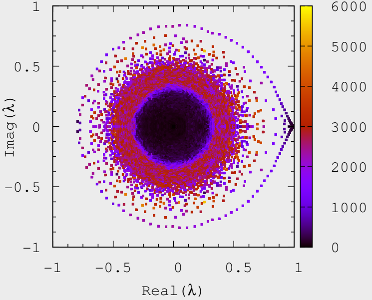
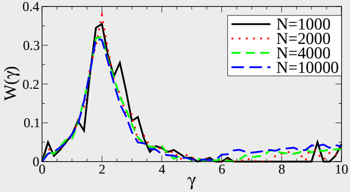
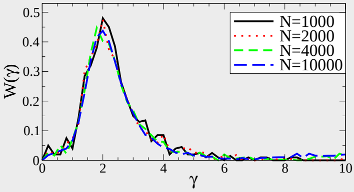


Density of states on $W(\gamma)$ ($\gamma = -2\ln|\lambda|$)





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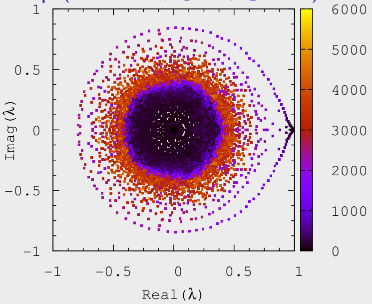

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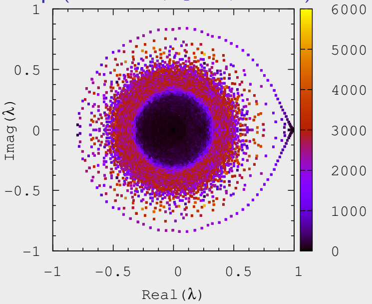
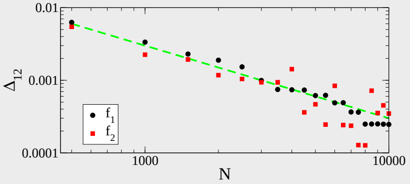
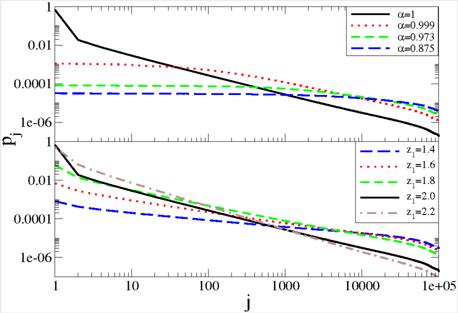


Eigenvalue gap: $\Delta_{12} \simeq \frac{3}{N}$ $\left(\propto \frac{1}{N^{z_1-1}}\right)$



PageRank

f_1-map, $N = 10^5$ ($\mathbf{t}$:$z_1 = 2, z_2 = 0.2$; $\mathbf{b}$:$z_2 = 0.2, \alpha = 1$)



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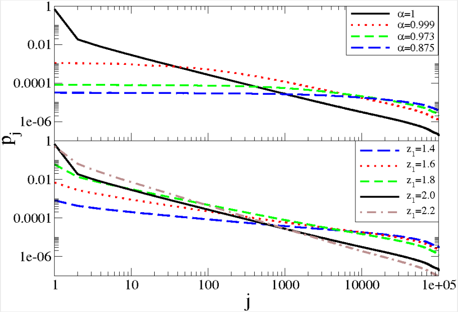
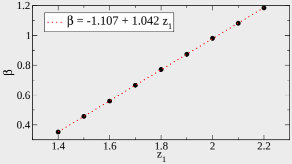
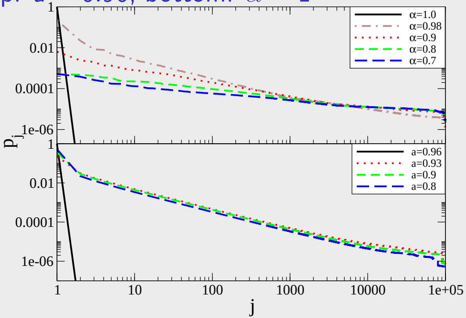


β ($p_j \sim 1/j^\beta$) $N = 10^5$, $z_2 = 0.2$, $\alpha = 1$



PageRank transition for f_2-map ($z_1 = 2$ and $N = 10^5$)

top: $a = 0.96$; bottom: $\alpha = 1$



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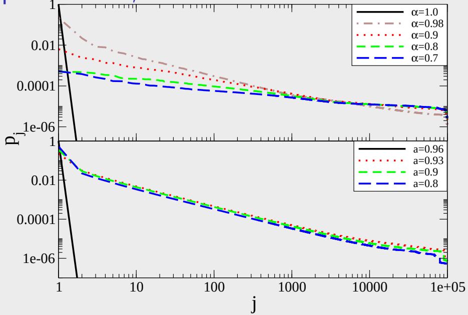
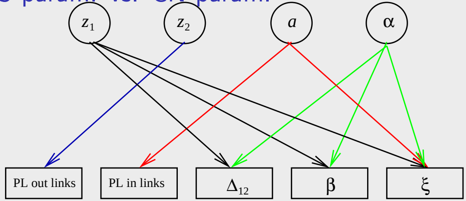
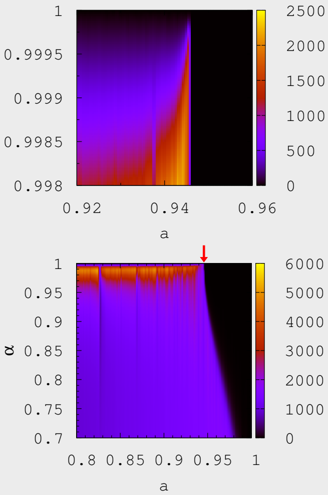


DS param. vs. CN param.



Phase diagram for PageRank



Procedure Call Network of Linux Kernel

10 versions from $V1.0$ to $V2.6$ (A. Chepelianskii arXiv:1003.5455 (2010))

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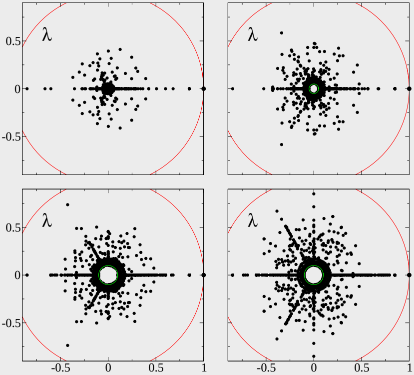
spectrum



TL:$V2.0.40$ $N = 14079$; TR:$V2.4.37.6$ $N = 85756$;

BL:$V2.6.32$ $N = 285509$; BR: (inverted links)

Procedure Call Network of Linux Kernel

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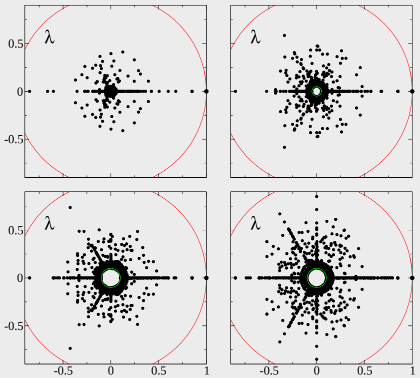
spectrum



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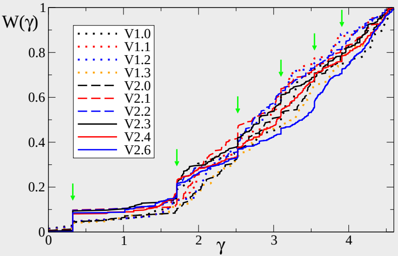
integrated density of states

$$\gamma = -2 \ln |\lambda|$$



from $V1.0$, $N = 14079$;
to $V2.6.32$, $N = 285509$

$\gamma_m = -2 \ln (\alpha/m)$ for $m = 1, 2, 3, 4, 5, 6$

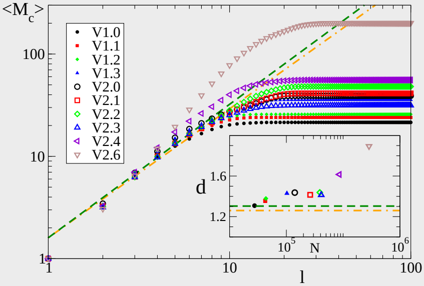
Fractal Weyl law in Linux Kernel

Cluster growing fractal dimension

The average mass $\langle M_c \rangle$,

with uniformly distributed seed.

$\langle M_c \rangle \propto l^d$.



$d \simeq 1.4$

Fractal Weyl law in Linux Kernel

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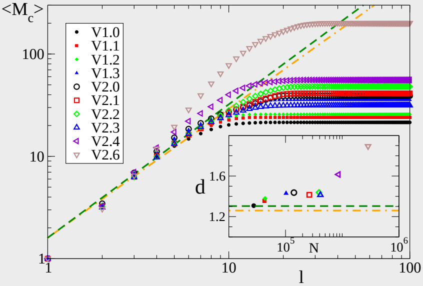
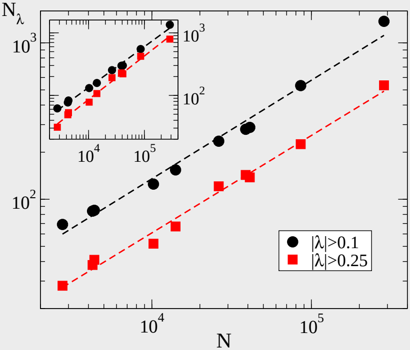


$d \simeq 1.4$

Number of states

$$\gamma = -2 \ln |\lambda|$$



$N_\gamma \propto N^\nu$

directed links:

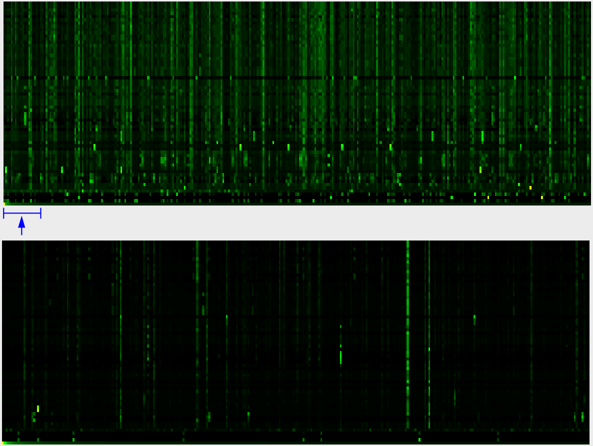
$\nu_{|\lambda|>0.25} \simeq 0.62$; $\nu_{|\lambda|>0.1} \simeq 0.63$

inverted links:

$\nu_{|\lambda|>0.25} \simeq 0.69$; $\nu_{|\lambda|>0.1} \simeq 0.65$

Fractal Weyl law in Linux Kernel

Coarse grained eigenstates $V2.6.32$, $N = 285509$



64 from $|\lambda| = 1$ (B) to $|\lambda| \simeq 0.4$ (T)

t-p: (complete) 307 cells of 930 sites; b-p: (first part) 300 cells, of 62 sites.

Concluding remarks

1. FWL on UA of PFO

   L. Ermann and D. Shepelyansky, EPJB **75**, 299 (2010).

2. Dynamical systems

   ↓

   Ulam network construction for PFO

   ↓

   Complex directed networks

   1-D Intemittency maps: control and tune parameters
   (counterexample of PageRank dependence on in(out)-going distribution)

   L. Ermann and D. Shepelyansky, PRE **81** 036221 (2010).

3. FWL on real complex networks

   ↓

   Linux Kernel Archituecture

   L. Ermann, A. Chepelianskii and D. Shepelyansky, arXiv 1005.1395, submitted PRE.

Thank you