Universal regime of fidelity decay in realistic quantum computations

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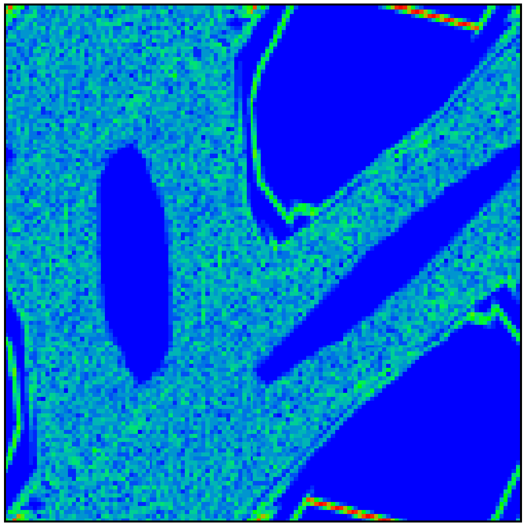

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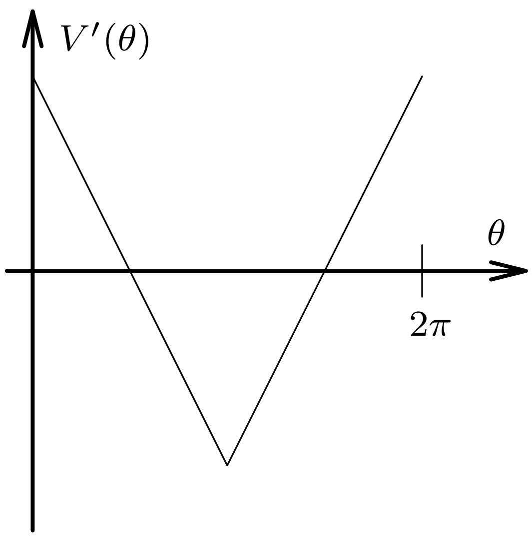
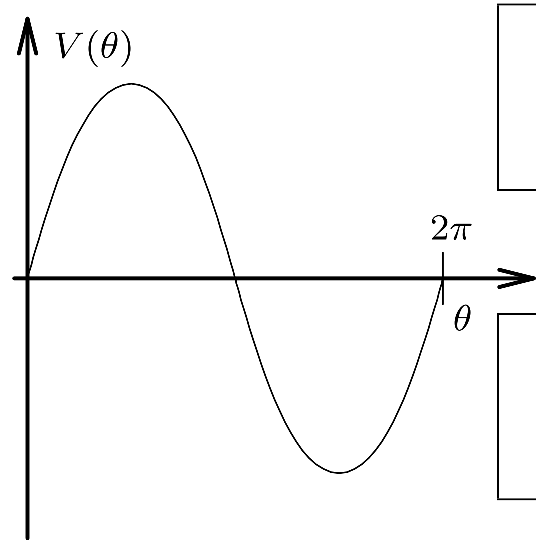
Model

$$H(t) = \frac{T\,p^2}{2} + V(\theta) \sum_{n=-\infty}^{\infty} \delta(t-n)$$





Classical map :

$$p_{n+1} = p_n - V'(\theta_n)$$
$$\theta_{n+1} = \theta_n + T\,p_{n+1}$$

Quantum map :

$$|\psi(t+1)> = U\,|\psi(t)>$$
$$U = e^{-iTp^2/2}\,e^{-iV(\theta)}$$

$$V(\theta) = \begin{cases} -\frac{k}{2}\theta(\theta-\pi) \\ \frac{k}{2}(\theta-\pi)(\theta-2\pi) \end{cases},\quad V'(\theta) = \begin{cases} k(\frac{\pi}{2}-\theta) & \text{if} \quad 0 \le \theta \le \pi \\ k(-\frac{3\pi}{2}+\theta) & \text{if} \quad \pi \le \theta \le 2\pi \end{cases}$$

Quantum algorithm

$$p = \sum_{j=0}^{n_q-1} \alpha_j \, 2^j \;\in\; \{0, \ldots, N-1\}$$

$N = 2^{n_q}$ = dimension of Hilbert space; n_q = number of qubits; $\alpha_j \in \{0,1\}$.

Identification: $|p\rangle \equiv |\alpha_0\rangle_0 \, |\alpha_1\rangle_1 \, \ldots \, |\alpha_{n_q-1}\rangle_{n_q-1}$.

$$\Rightarrow \quad e^{-iT\hat{p}^2/2} \, |p\rangle = \prod_{j<k} \underbrace{e^{i(\cdots)\alpha_j \alpha_k}}_{B^{(2)}_{jk}(\cdots)} \prod_{j} \underbrace{e^{i(\cdots)\alpha_j}}_{B^{(1)}_{j}(\cdots)} |p\rangle$$

with simple and controlled phase-shift:

$$B_j^{(1)}(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \quad , \quad B_{jk}^{(2)}(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{pmatrix} .$$

Similar expression for $e^{-iV(\hat{\theta})}|\theta>$ requires double controlled phase-shift:

$$B^{(3)}_{jkl}(\phi) = B^{(2)}_{jl}\left(\frac{\phi}{2}\right) B^{(2)}_{jk}\left(\frac{\phi}{2}\right) C^{(N)}_{kl}\, B^{(2)}_{jk}\left(-\frac{\phi}{2}\right) C^{(N)}_{kl} \ .$$

Quantum Fourier transform (QFT) is used for: $|p> \ \leftrightarrow \ |\theta>$.

$\Rightarrow$ The quantum map :

$$|\psi(t+1)>= U\,|\psi(t)> \quad , \quad U = e^{-iTp^2/2}\, e^{-iV(\theta)}$$

can be effectively simulated on a quantum computer with n_q qubits.

For this the total number of 2-qubit quantum gates for one iteration is:

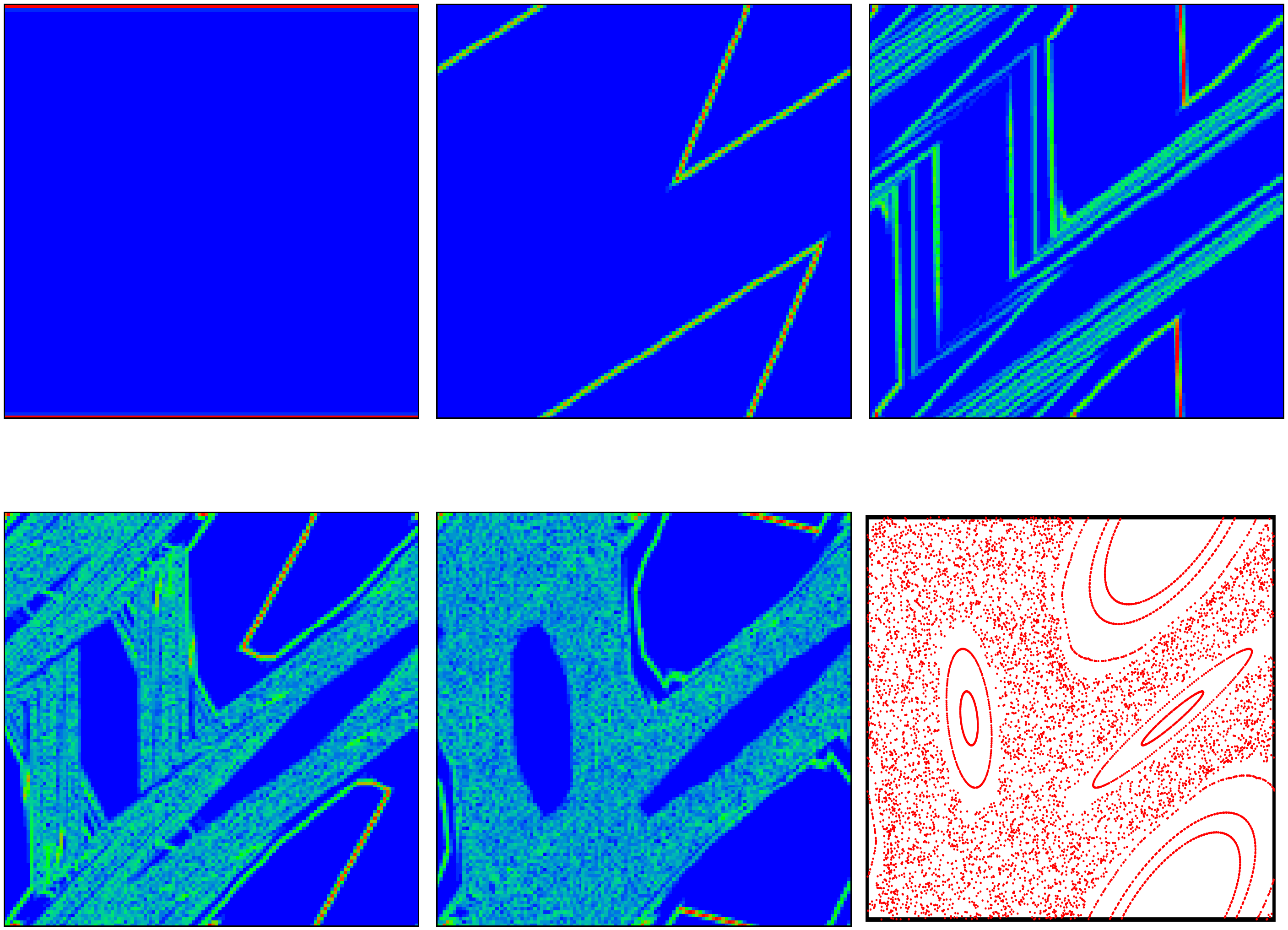
$$N_g = \frac{9}{2}n_q^2 - \frac{11}{2}n_q + 4 \ .$$

A classical computer needs $\mathcal{O}(2^{n_q}\, n_q^2)$ operations.

Semi-classical limit : $T = \frac{2\pi}{N} = \frac{2\pi}{2^{n_q}} \quad , \quad k = \frac{k_{cl.}}{T}$.

First Illustration

$k_{cl.} = 1.7, \quad n_q = 14, \quad t = 0, 1, 5, 10, 20, \quad |\psi(0)> = |0>$



Fidelity decay due to errors

$$|\psi(t)> = U^t\,|\psi(0)> \quad , \quad U = \underbrace{U_{N_g} \cdot \ldots \cdot U_1}_{\text{elementary gates}} \quad .$$

Errors: $$\boxed{U_j \;\rightarrow\; U_j\, e^{i\delta H} \quad , \quad \delta H \sim \varepsilon \;.}$$

(i) Residual couplings of quantum computer to external bath:

$\Rightarrow$ decoherence

$\Rightarrow$ δH random and different at each j and t.

(ii) Static imperfections in the quantum computer itself:

$\Rightarrow$ δH (random but) constant at each j and t.

A measure of the error is the fidelity:

$$\boxed{f(t) = |<\psi(t)|\psi_\varepsilon(t)>|^2 \;.}$$

Random and Static Errors

$$f(t) = |<\psi(t)|\psi_\varepsilon(t)>|^2 \approx \exp\left(-\frac{t}{t_r}\right)$$

$$t_r = (0.095\varepsilon^2 n_q^2)^{-1} \approx 47/(\varepsilon^2 N_g)$$

(d): Random errors

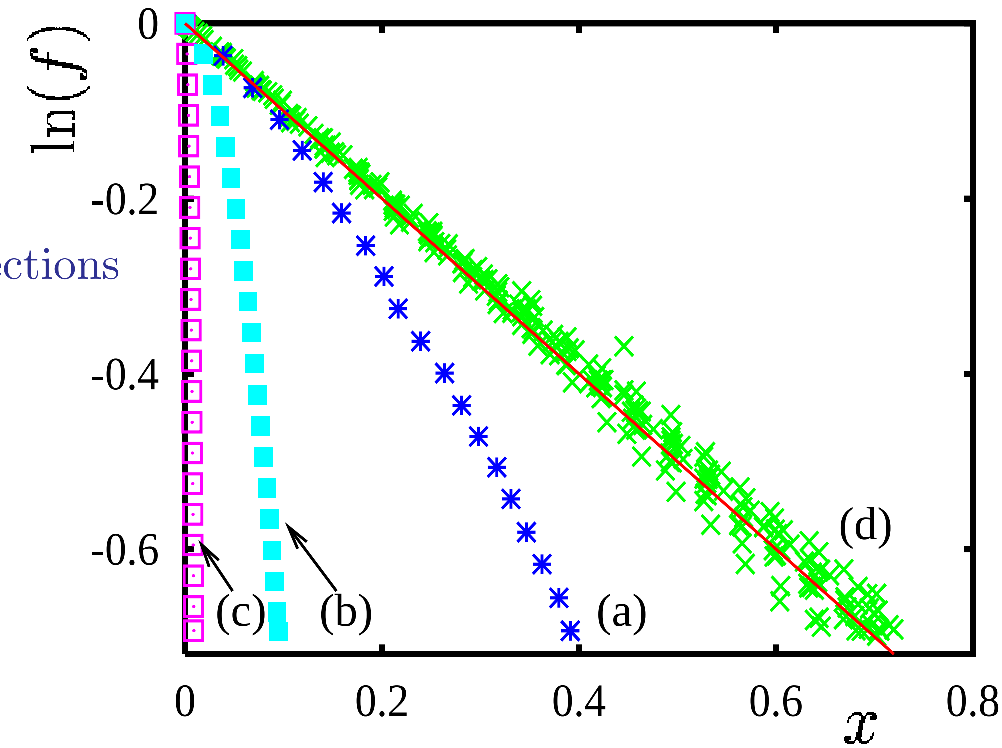
($x = t/t_r$).

(a), (b), (c): Static imperfections

($x = t/t_c$ such that:

$f(t) \approx e^{-t/t_c}$ for $t \to 0$).

—— $\exp(-x)$



Random matrix theory for fidelity decay

$$f(t) = \left| \frac{1}{N} \texttt{tr} \left(U^{-t} \left(U e^{i\delta H_{\rm eff}} \right)^t \right) \right|^2$$

$$(1-f) \ll 1 \quad : \quad f(t) \approx 1 - \frac{t}{t_c} - \frac{2}{t_c} \sum_{\tau=1}^{t-1} (t-\tau)\, C(\tau)$$

with:

$$\frac{1}{t_c} = \frac{1}{N} \texttt{tr} \left(\delta {H_{\rm eff}}^2 \right) \quad , \quad C(\tau) = \frac{t_c}{N} \texttt{tr} \Big(\underbrace{U^{-\tau}\, \delta H_{\rm eff}\, U^{\tau}}_{\delta H_{\rm eff}(\tau)}\, \delta H_{\rm eff} \Big)$$

(General relation between fidelity and correlation function: *Prosen, 2001)*

$U \in$ *COE (CUE)* $\Rightarrow$ *Scaling law:*

$$-\langle \ln f(t) \rangle_U \approx \frac{N}{t_c} \chi\left(\frac{t}{N} \right) \quad , \qquad \chi(s) = s + \frac{2}{\beta} s^2 - 2 \int_0^s d\tilde{\tau}\, (s - \tilde{\tau})\, b_2(\tilde{\tau}) \ .$$

with the "two-level form factor": $b_2(\tilde{\tau})$.

Static imperfections

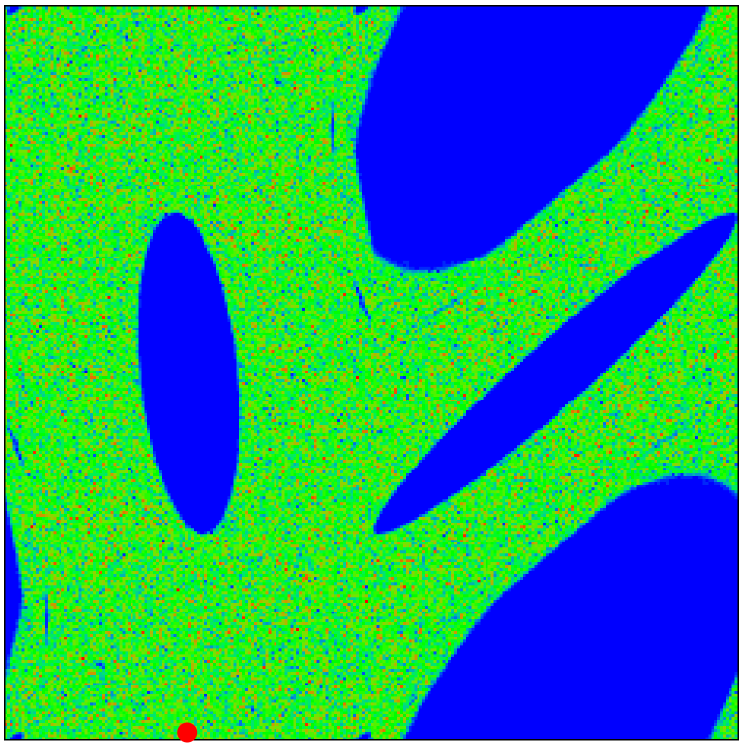
$t = 5625$

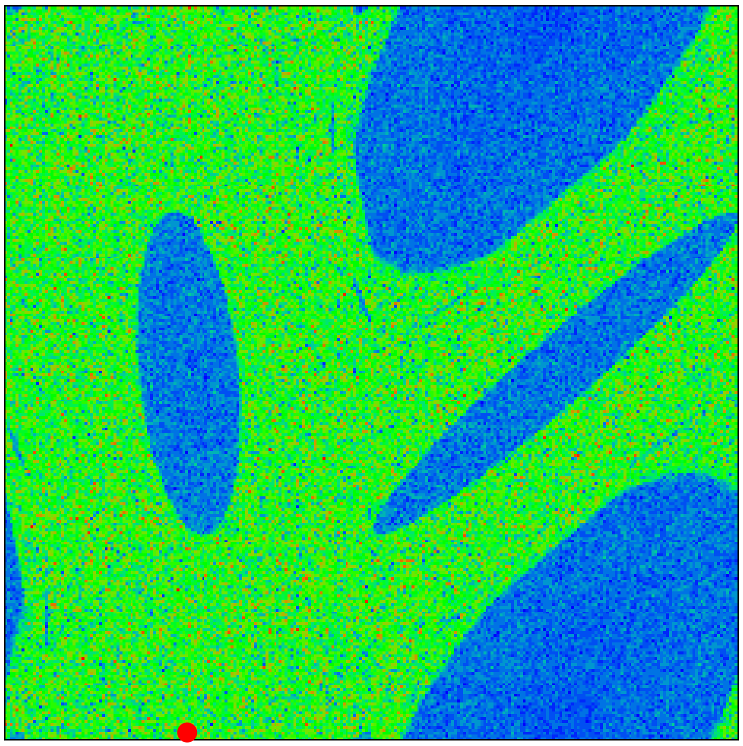
$n_q = 16$

$$\delta H = \sum_{j=0}^{n_q-1} \delta_j \, \sigma_j^{(z)} + 2 \sum_{j=0}^{n_q-2} J_j \, \sigma_j^{(x)} \, \sigma_{j+1}^{(x)}$$

$\delta_j \sim \varepsilon$

$J_j \sim \varepsilon$





$\varepsilon = 0$ $\quad$ $\varepsilon = 7 \cdot 10^{-7}, \quad f = 0.9388$

Position of initial gaussian wave packet

Verification of scaling law

$\beta = 1$, $N = \sigma t_{\mathrm{H}}$ where $\sigma = 0.65 \approx$ fraction of chaotic region of phase-space:

$$-\langle \ln f(t) \rangle_U \approx \frac{\sigma\, t_{\mathrm{H}}}{t_c} \chi\left(\frac{t}{\sigma\, t_{\mathrm{H}}}\right) \approx \frac{t}{t_c} + \frac{2}{\sigma}\,\frac{t^2}{t_c t_{\mathrm{H}}}$$

$$t_c = \frac{1}{\varepsilon^2 n_q N_g^2} \quad , \quad t_{\mathrm{H}} = 2^{n_q}$$

with $N_g = \frac{9}{2} n_q^2 - \frac{11}{2} n_q + 4 =$ number of elementary quantum gates.

Static imperfections:

(a) $\varepsilon = 3 \cdot 10^{-5}$

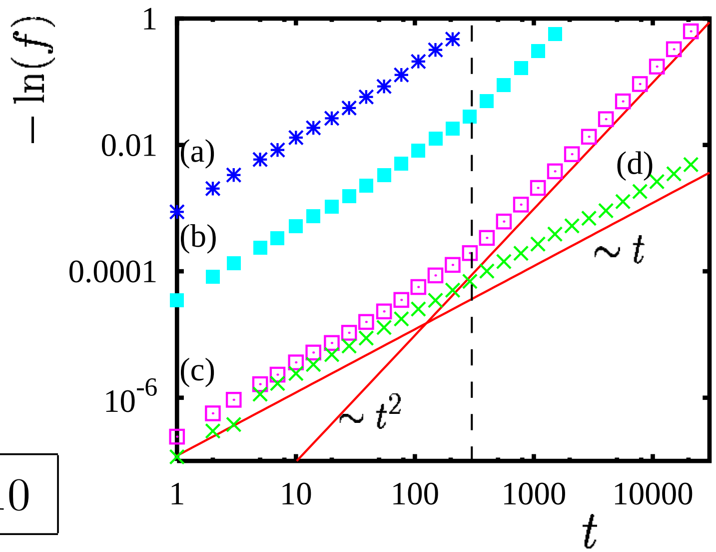
(b) $\varepsilon = 6 \cdot 10^{-6}$

(c) $\varepsilon = 5 \cdot 10^{-7}$

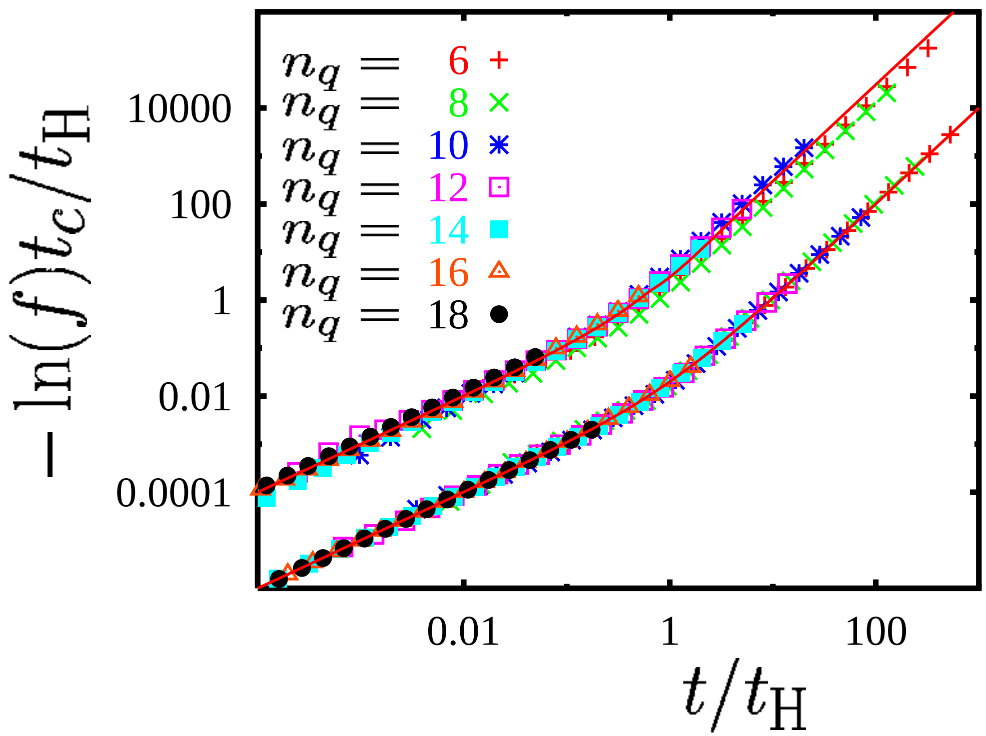
Random errors:

(c) $\varepsilon = 1.59 \cdot 10^{-4}$

$n_q = 10$



Scaling analysis for chaotic dynamics

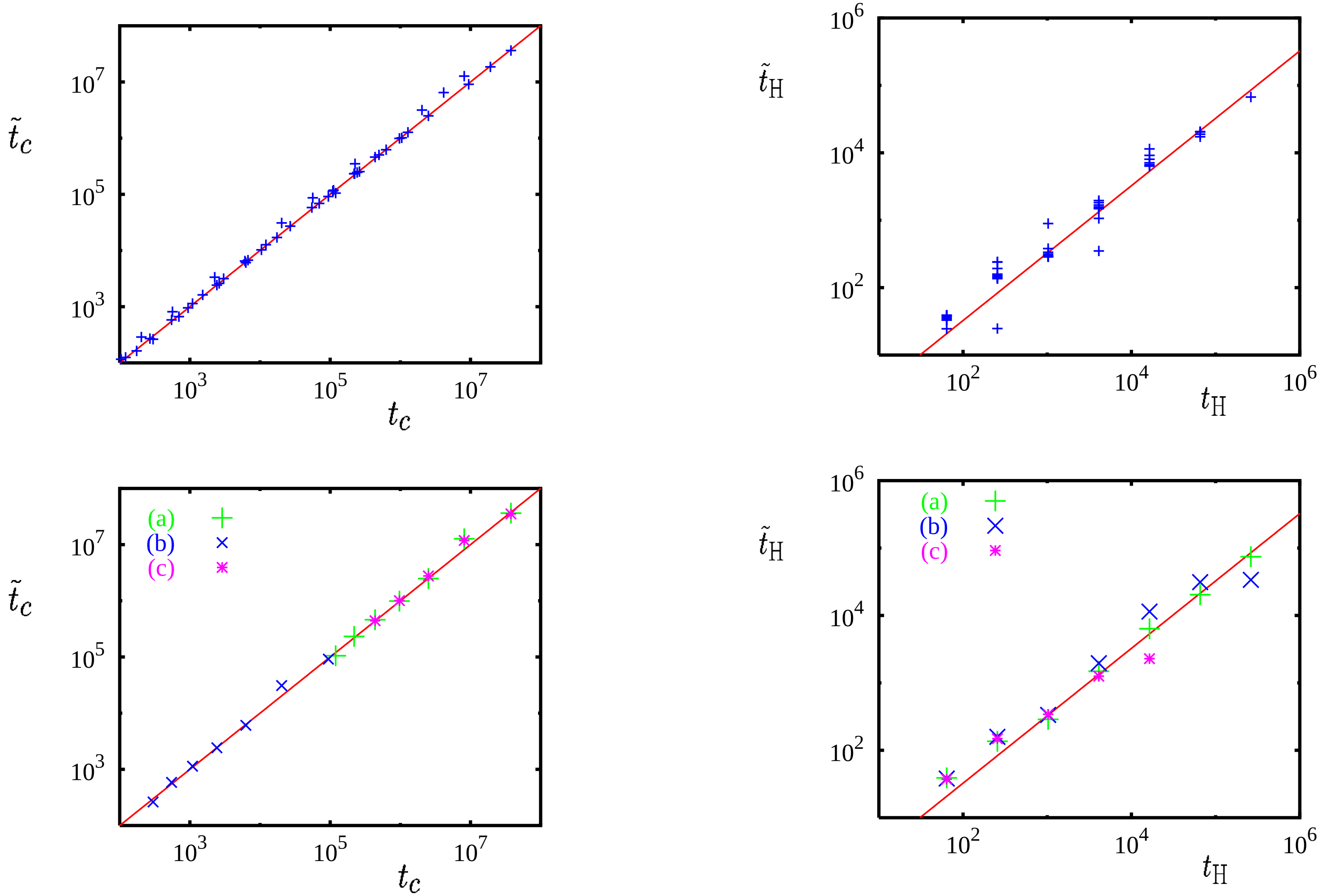


Upper curve: with theoretical values of t_c and $t_{\rm H}$.

Lower curve: with fit values $\tilde{t}_c$ and $\tilde{t}_{\rm H}$ from

$$-\ln(f(t)) = \frac{t}{\tilde{t}_c} + \frac{t^2}{\tilde{t}_c \tilde{t}_{\rm H}} \; .$$

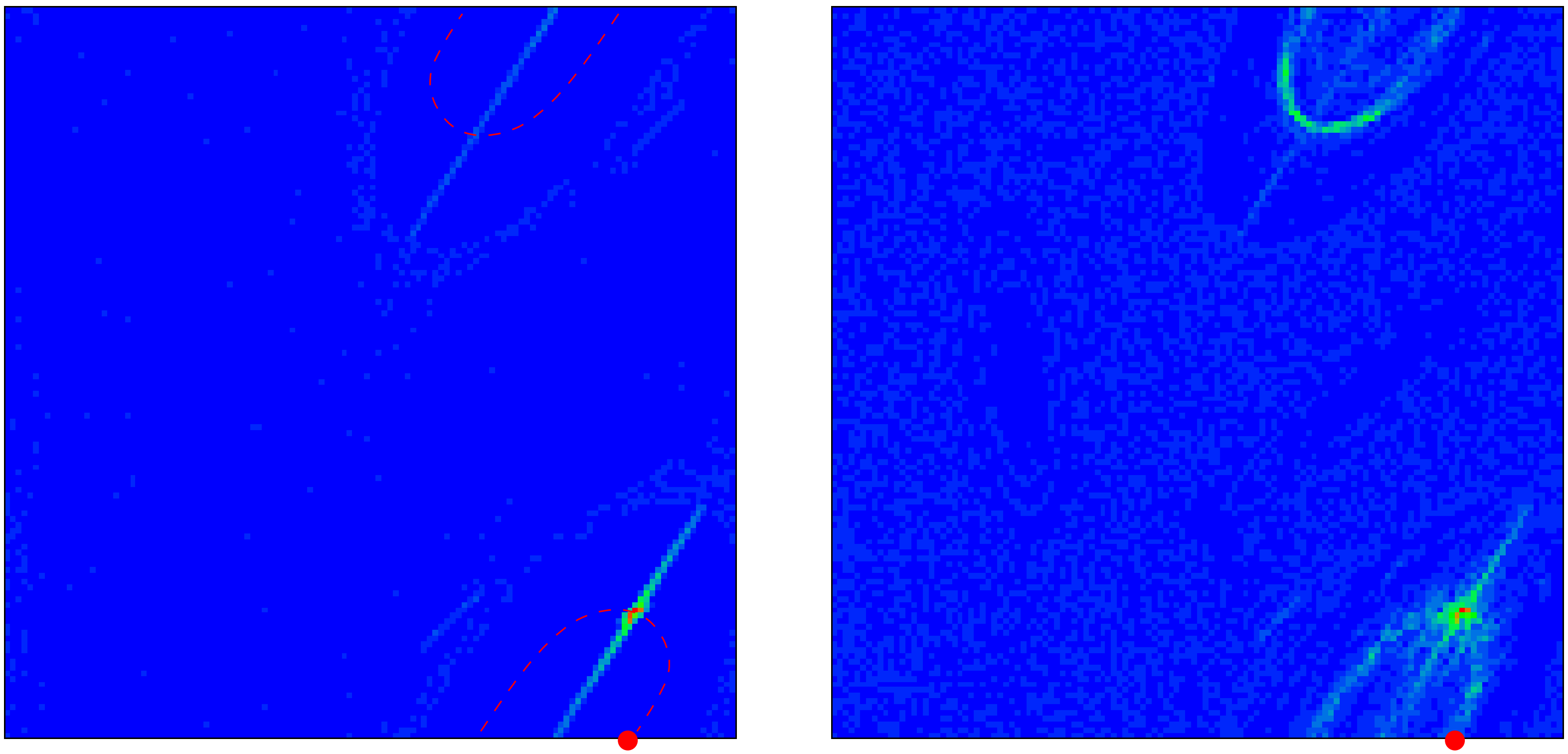
$\tilde{t}_c$ and $\tilde{t}_{\mathrm{H}}$ for chaotic dynamics



Integrable dynamics

$t = 22783$

$n_q = 14$



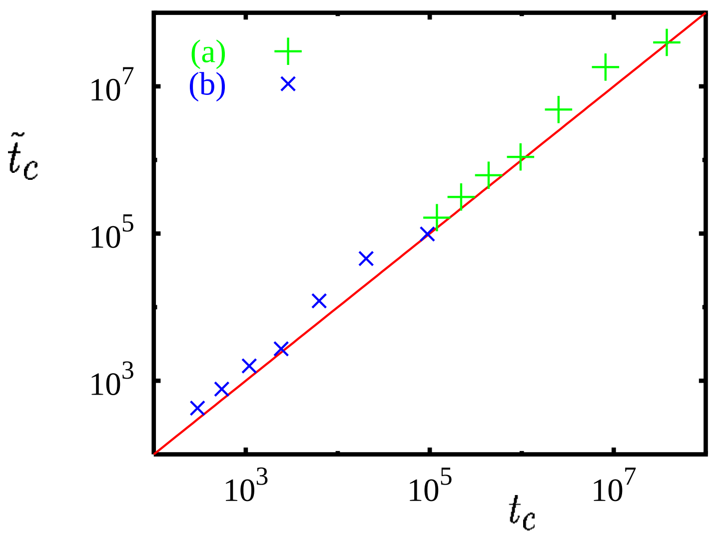
$\varepsilon = 0$ $\varepsilon = 5 \cdot 10^{-7}, \quad f = 0.5$

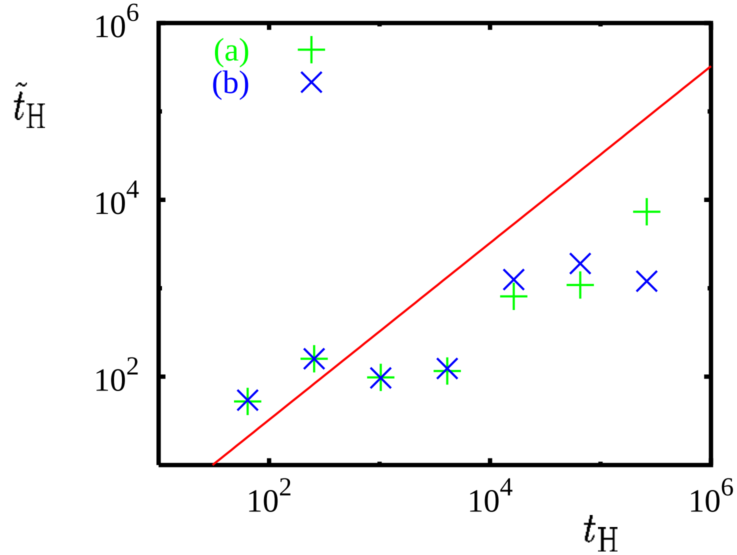
Position of initial gaussian wave packet

Scaling analysis for integrable dynamics

Fit values $\tilde{t}_c$ and $\tilde{t}_{\mathrm{H}}$ from

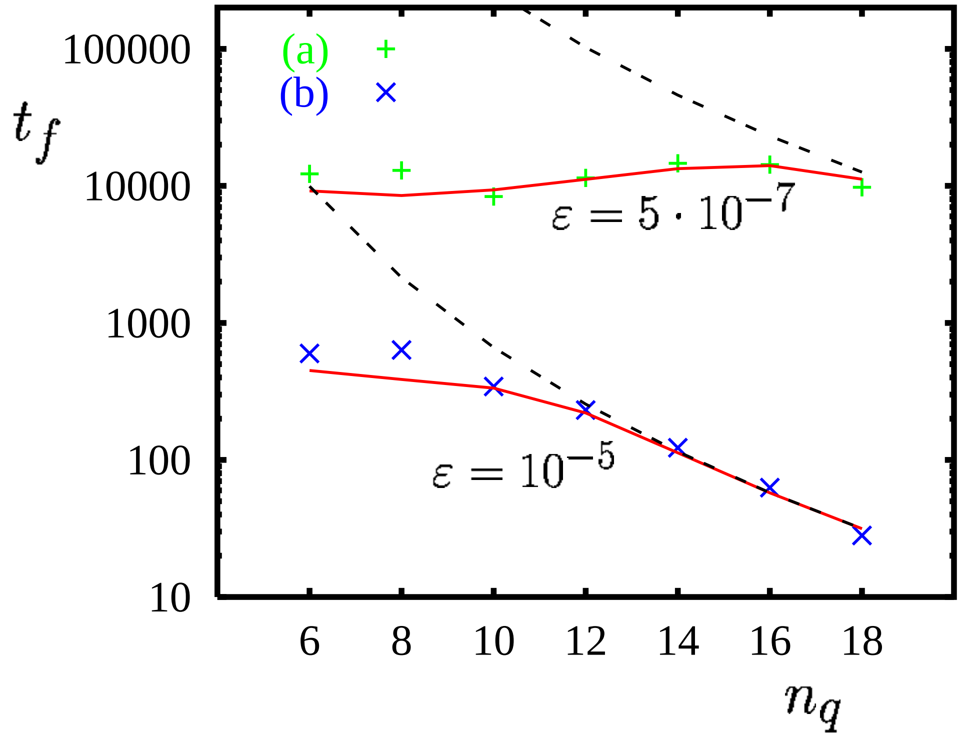
$$-\ln(f(t)) = \frac{t}{\tilde{t}_c} + \frac{t^2}{\tilde{t}_c \tilde{t}_{\mathrm{H}}} \; .$$





Time scale t_f with $f(t_f) = 0.9$

(for chaotic dynamics)



Theory:

$$t_f \approx \begin{cases} 0.105\, t_c & \text{if} \quad \varepsilon \gg (2^{n_q} N_g^2 n_q)^{-1/2} \\ 0.185\, \sqrt{t_c t_{\mathrm{H}}} & \text{if} \quad \varepsilon \ll (2^{n_q} N_g^2 n_q)^{-1/2} \end{cases}$$

Box-Husimi functions

Circle state:

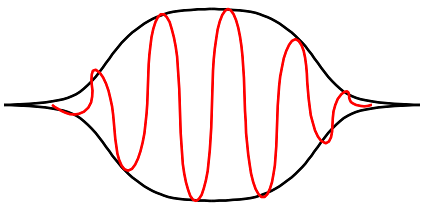
$$|\psi > \sim \sum_{(p,\theta)\in\bigcirc} |\varphi(p,\theta)>$$

Husimi function:

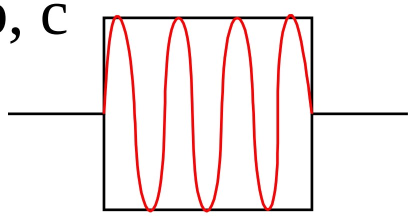
$$\rho_H(p,\theta) = |<\varphi(p,\theta)\,|\,\psi>|^2$$

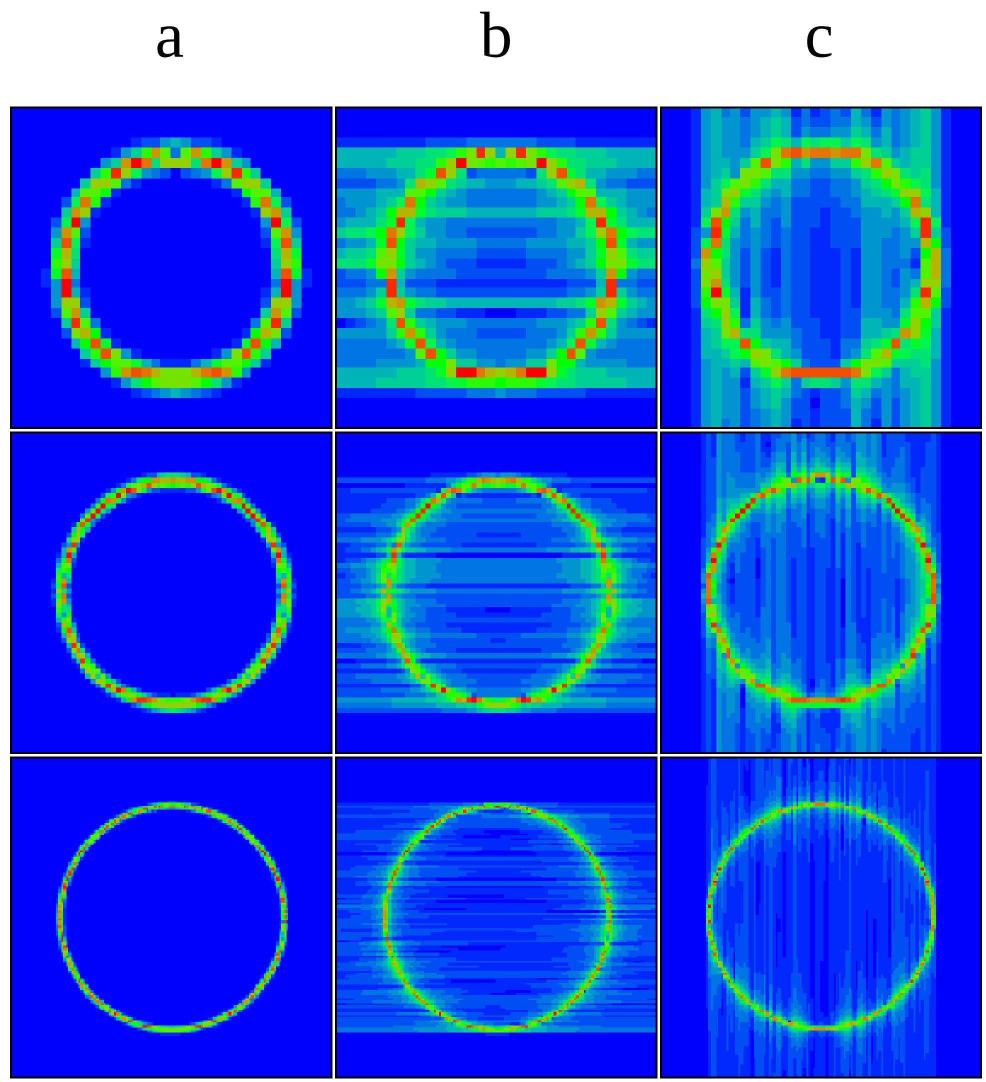
Coherent state: $|\varphi(p,\theta)>$

a



b, c





Conclusion

- Interesting model of quantum chaos accessible to quantum computation with a moderate number of qubits (10-20).
- Relation between two level form factor from RMT and fidelity.
- Universal scaling regime for fidelity decay.
- Modified Husimi function (nearly) accessible to quantum computation.
- Stability of quantum computation for classically chaotic maps.

- Implications for other quantum algorithms ?
- Diffusive regime ?

$$f(t) \approx f_{\mathrm{RM}}(t) \exp\left(-\mathrm{const.}\frac{t^{3-d/2}\, t_{\mathrm{Th}}^{d/2}}{t_c\, t_{\mathrm{H}}^2}\right)$$

- Localized regime ? Modified Heisenberg time ?