A measurement which preserves localization

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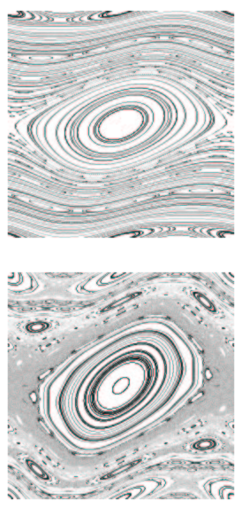
Abstract

It is widely believed that measurements in a quantum system introduce noise and decoherence, thus destroying localization. Indeed, it was shown that, for the kicked rotator in a localized regime, a complete measurement of the momentum induces diffusion. In this poster we show that, contrary to the expectations, if we properly choose our measurement operators, we can still have a localized regime. Such operators can be naturally implemented as single-qubit measurements on a quantum computer simulating a kicked rotator. A transition localization/delocalization, obtained by increasing the kick strength and/or by measuring different qubits is discussed.

Classical kicked rotator - Chirikov Standard Map

$H = \frac{T}{2} I^2 + k\cos\theta \sum_n \delta(\tau - tT)$

$$\begin{cases} I_{t+1} = & I_t + k\sin\theta_t \\ \theta_{t+1} = & \theta_t + T I_{t+1} \end{cases}$$



The classical dynamics depends on $K = kT$

* $K = 0$: integrable system
* $0 < K < K_g \approx 0.97$ mixed phase space - bounded motion
* $K \gg K_g$ diffusion in the momentum space: $\langle (I_t - I_0)^2 \rangle \sim Dt$

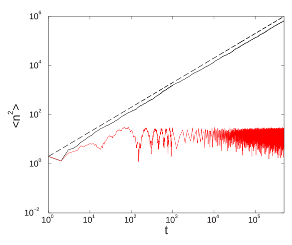
Quantum kicked rotator

Quantization: $\theta \to \hat{\theta}$

$I \to \hat{n} = -i\hbar \frac{\partial}{\partial\theta}$ $(\hbar = 1)$

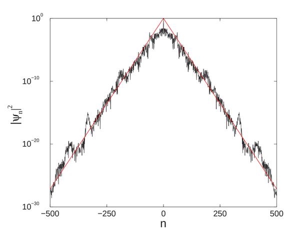
Evolution over a period T: Floquet operator

$$U(T,k) = U_{free}(T) \times U_{kick}(k) = e^{-i\frac{T}{2}\hat{n}^2} e^{ik\cos\hat{\theta}}$$



Dynamical localization:

after a time t_H the diffusion process stops.
The eigenstates of $U(T)$in the momentum representation are localized $\psi_n \sim e^{|n-n_0|/\ell}$ $\ell \sim D/2$ is the *localization length*



Mapping to a solid state model

Measurement

Quantum measurement on a state $|\psi\rangle$ are described by a set of $\{M_m\}$ measurement operators
measurement operators satisfy: completeness equation $\sum_m M_m M_m^\dagger = \mathbb{I}$
Probability of outcome m: $p(m) = \langle\psi|M_m M_m^\dagger|\psi\rangle$.
completeness equation $\iff \sum_m p(m) = 1$
state after the measurement: $|\psi_m\rangle = \frac{M_m|\psi\rangle}{\sqrt{p(m)}}$

POVM measurement

Positive Operator-Valued Measurement $\{E_m\}$
E_m positive operators, $\sum_m E_m = \mathbb{I}$
$p(m) = \langle\psi|E_m|\psi\rangle \geq 0$; $|\psi_m\rangle = \frac{E_m|\psi\rangle}{\sqrt{p(m)}}$

Projective Measurement

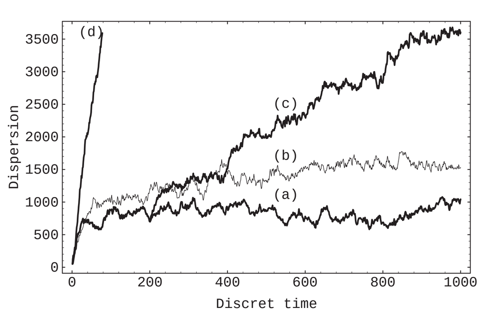
P_m is a projector, $(P_m^\dagger = P_m,\ P_m^2 = P_m)$.
Observable M $M = \sum_m m P_m$
$p(m) = \langle\psi|P_m|\psi\rangle$; $|\psi_m\rangle = \frac{P_m|\psi\rangle}{\sqrt{p(m)}}$

Measurement-induced quantum diffusion

Kaulakys,Gontis, Phys.Rev.A **56**, *1997; Facchi, Pascazio, Scardicchio, Phys.Rev.Lett.***83**, *1999*

Kicked rotator: periodicity in $\theta \to$ discrete momentum states $|n\rangle$; eigenvalues $I_n = n$ $(\hbar = 1)$; $\langle\theta|n\rangle = e^{in\theta}/\sqrt{2\pi}$ Measurements in the momentum basis $P_m = |m\rangle\langle m|$

Kaulakys and Gontis- simulations:
measurement in the momentum basis after each S kicks: Diffusion
$\langle (n_t - n_0)^2 \rangle = \frac{k^2}{2T} t$ (if $S = 1$)
a) no measurements
c) S=200
d) S=1



Facchi et al.: Density Matrix formalism. Measurement at each step, $P_n(t) = Tr(|n\rangle\langle n|\rho_t)$ probability of outcome n for the momentum, after t steps. Analytical calculations for the evolution in time of $\langle n\rangle$ and $\langle n^2\rangle$.

Results: $\langle n\rangle_t = \langle n\rangle_0$ -no drift
$\langle n^2\rangle_t = \langle n^2\rangle_0 + \frac{k^2}{2T}$ - diffusion

The Density Matrix is diagonal in the momentum basis after each measurement (no quantum coherence). $\frac{k^2}{2T}$ is not the *exact* classical diffusion coefficient D
Classical dynamics is not recovered by such a measurement procedure

Classical analog

After a measurement in the momentum basis, for the Heisenberg principle θ is fully undetermined and can be replaced by a random variable. A corresponding classical map (ξ_t is a stochastic process) is:

$$\begin{cases} \theta_t = & \xi_t \\ I_{t+1} = & I_t + k\sin\theta_t \end{cases}$$

This gives rise to a diffusive dynamics and the same scaling $\langle I^2\rangle = \frac{k^2}{2T}$
also in the classically integrable case

Quantum computer

- Quantum bit (qubit): 2 level system, logical states $|0\rangle$, $|1\rangle$; $\alpha|0\rangle + \beta|1\rangle$
- Quantum register: set of n_q qubits, Hilbert space $\mathbb{H}$ of size $N = 2^{n_q}$
- $|j\rangle \in \mathbb{H}$, $|j\rangle = |a_1(j), a_2(j), \ldots, a_{n_q}(j)\rangle$, $j = 0, \ldots, N-1$
- $a_1(j), \ldots, a_{n_q}(j)$ binary representation of j
  $j = 2^{n_q-1}a_1(j) + 2^{n_q-2}a_2(j) + \ldots + a_{n_q}(j)$
- Quantum register described by $|\psi\rangle = \sum_{j\in\mathbb{H}} \psi_j|j\rangle$
- Unitary Evolution - Unitary elementary operations = Quantum Gates

Kicked Rotator simulated on a Quantum computer

Initial state $|\psi_0\rangle = \sum_j \psi_j^0|j\rangle$ in the momentum representation
$j < N/2$: negative momenta; typical initial condition $\psi_j^0 = \delta_{j,N/2}$

Classical Algorithm

- Free rotation $U_{free}(T)$ diagonal in the momentum basis: $\sim N$ operations
- Fast Fourier Transform (FFT) $\sim N\log N$ operations$\to$ position basis
- Kick $U_{kick}(k)$ diagonal in position basis, $\sim N$ operations
- FFT back to momentum representation

Quantum Algorithm

- Free rotation $U_{free}(T)$ decomposed in $\log N^2 = n_q^2$ quantum gates
- Quantum Fourier Transform (QFT)$\sim \log N^2$ operations $\to$ position basis
- Kick $U_{kick}(k)$ realized in $\sim n_q^3$ elementary gates
- QFT back to momentum representation

$\sim N\log N \Longleftarrow$ 1 MAP STEP $\Longrightarrow \sim \log N^3$

Measurements on a quantum computer

Measuring the momentum $\{P_m = |m\rangle\langle m|\} \iff$ Measuring ALL the qubits.

Partial measurements are possible. We measure only a part of the quantum register
The density matrix is not diagonal after such a measurement: Quantum Correlations - Quantum Effects are still present
Localization is a quantum effect due to interference: are there cases in which localization is preserved?
Measuring qubit n_m: 2 projectors $P_0(n_m), P_1(n_m)$, onto the states where qubit n_m is 0 or 1.

$$P_{0,1}(n_m) = \sum_{k\in S_{0,1}(n_m)} |k\rangle\langle k| \qquad S_{0,1}(n_m) : \{k \in [0, N-1]; a_{n_m}(k) = 0, 1\}.$$

Measuring the most significant qubit: selecting positive/negative momenta

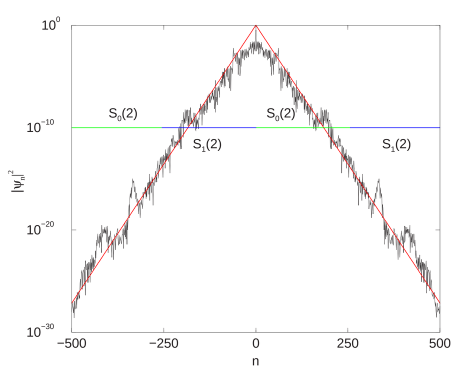
$$P_0(1) = \sum_{k=0,N/2-1} |k\rangle\langle k| \ P_1(1) = \sum_{k=N/2,N} |k\rangle\langle k|$$

Measurement of a qubit: example

$$P_{0,1}(n_m) = \sum_{k\in S_{0,1}(n_m)} |k\rangle\langle k| \qquad S_{0,1}(n_m) : k \in [0, N-1]; a_{n_m}(k) = 0, 1$$

$n_m = n_q$: $P_0(n_q)$ projects onto even momentum states, $P_1(n_q)$ onto odd states.
$n_m = 1$: $P_0(1)$ projects onto negative momentum, $P_1(1)$ on positive momentum states.
The sets $S_{0,1}(n_m)$ are collection of 2^{n_m-1} intervals of size $2^{n_q-n_m}$

Example: $n_q = 10$
$n_m = 2$
sets of size $2^8 = 256$, $S_0(2)$, $S_1(2)$



Kicked Rotator with measurements

We consider a kicked rotator in a localized regime, $T = 2$, $k = 2, \ldots 20$, $N = 2^{n_q}$ $n_q = 9, \ldots 12$. The initial state is concentrated in $n_0 = 0$, $j_0 = N/2$ $j < N/2$ corresponds to negative momenta, $j > N/2$ to positive ones. After each map iteration we perform a measurement of qubit n_m. In the system without measurement, diffusion stops due to localization effects. Depending on the measured qubit n_m, we have different regimes

- Diffusion if n_m is the least significant qubit, $n_m = n_q$
- Localized regimes, depending on the choice of n_m, and on the values of k
- Localization/delocalization transition at fixed n_m, by increasing k

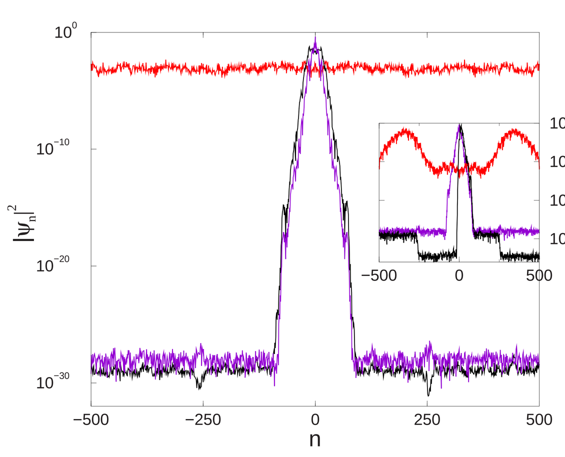
Simulation of the measurement procedure

1 Sampling the measurement procedure: at each step we compute $\langle\psi|P_0(n_m)|\psi\rangle$, $\langle\psi|P_1(n_m)|\psi\rangle$ and according to this probabilities we project onto $P_0(n_m)|\psi\rangle$, $P_1(n_m)|\psi\rangle$. We perform t such steps. Then we average over M realizations of this dynamics. Sampling Method

2 At each step we compute $P_{0,1}(n_m)|\psi\rangle$. Then we add a random phase at each vectors. $|\psi\rangle \to e^{i\phi_0}P_0(n_m)|\psi\rangle + e^{i\phi_1}P_1(n_m)|\psi\rangle$. So $P_0(n_m)|\psi\rangle$ and $P_1(n_m)|\psi\rangle$ are incoherent. Phase Method

Then, we compute $\langle n^2\rangle = \sum_n n^2|\psi_n|^2$ and the Inverse Participation Ratio (IPR) $\xi = 1/\sum_n |\psi_n|^2$ to find localized/extended regimes.
In the case (1) we computed $|\psi_n^{av}|^2 = \frac{1}{M}\sum_{l=1,M} |\psi_n(l)|^2$, where l labels the measurement procedure.

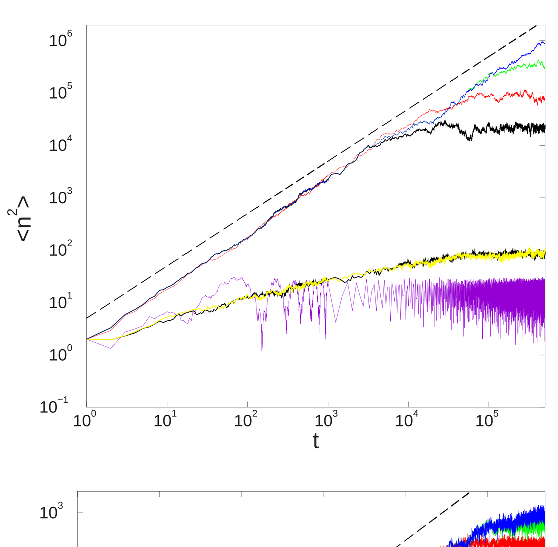
Sampling the measurement procedure: average wave function



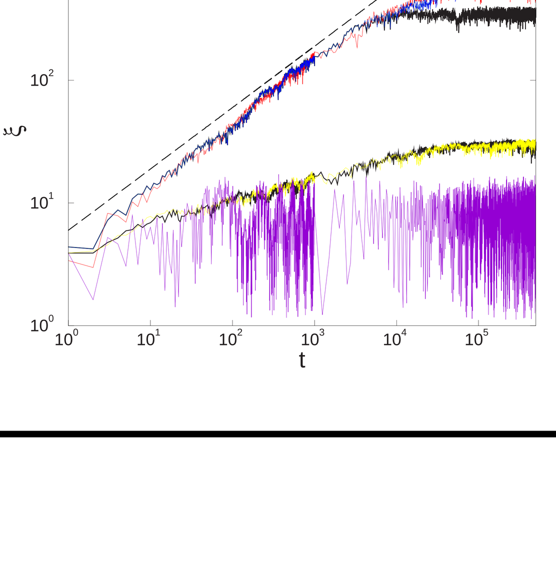
$T = 2$, $k = 2$, $n_q = 10$, $t = 5^5$. Examples of wave function, for the free case, a case where localization is preserved and a case where measuring a single qubits induces diffusion. Violet curve: no measurements, black curve $n_m = n_q - 8$, red curve $n_m = n_q$

Main plot: average over $M = 50$ measurement realizations. Inset: only one measurement realization

Results - Sampling the measurement procedure

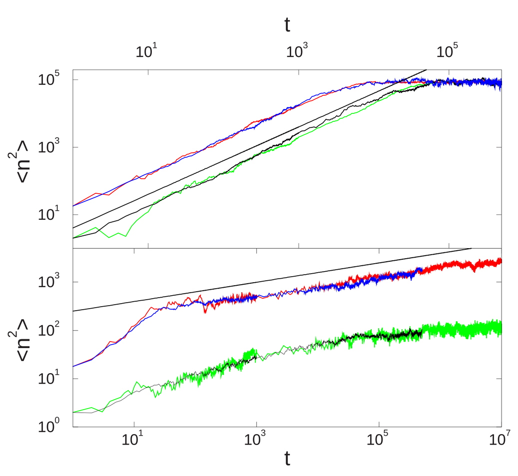


$\langle n^2\rangle$ as a function of time t. Parameters are $T = 2$, $k = 2$. Violet curve: evolution without measurements, the diffusion is frozen by localization effects. Upper curves: we measure the least significant qubit ($n_m = n_q$), thus obtaining a diffusive behavior. Colors are black for $n_q = 9$, red for $n_q = 10$, green for $n_q = 11$, blue for $n_q = 12$. Lower curves (thin black and yellow $n_q = 9, 12$): $n_m = n_q - 8$, The behavior is typical of a localized regime. $\langle n^2\rangle$ does not depend on the size of the system $N = 2^{n_q}$



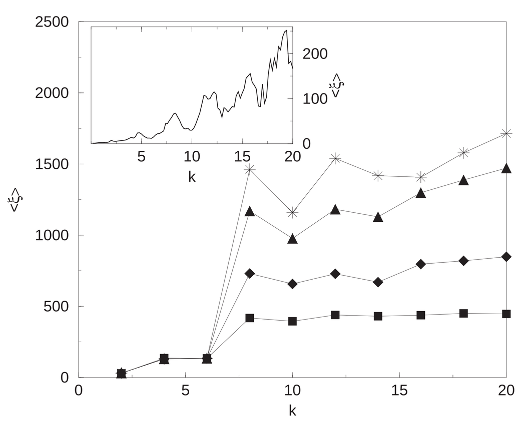
IPR ξ of $|\psi_n^{av}|^2$ as a function of time t. Parameters are $T = 2$, $k = 2$. Violet curve: evolution without measurements, the diffusion is frozen by localization effects. Upper curves: we measure the least significant qubit ($n_m = n_q$), thus obtaining a diffusive behavior $\xi \sim N$. Colors are black for $n_q = 9$, red for $n_q = 10$, green for $n_q = 11$, blue for $n_q = 12$. lower curves (thin black and yellow $n_q = 9, 12$): $n_m = n_q - 8$, The behavior is typical of a localized regime. ξ does not depend on $N = 2^{n_q}$

Comparison - Phase and Sampling methods

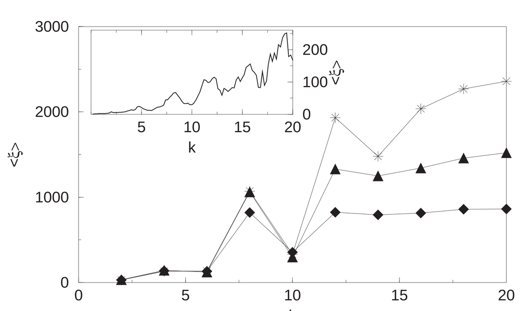


$k = 2, 6$, $T = 2$, $n_q = 10$. Phase method: red ($k = 6$) and green ($k = 2$) curves. Sampling method: black ($k = 2$) and blue ($k = 6$) curves. Upper Plot: comparison for two diffusive cases ($n_m = n_q$). Lower plot: comparison for two localized cases ($n_m = n_q - 8$). We notice that the agreement is very good. Moreover, since the simulations are faster with the phase method, we could reach larger t. For instance, in the lower plot, it is shown that the anomalous diffusion $\langle n^2\rangle \sim t^{0.2}$ stops into a localized state.

k dependence - localization/delocalization transition

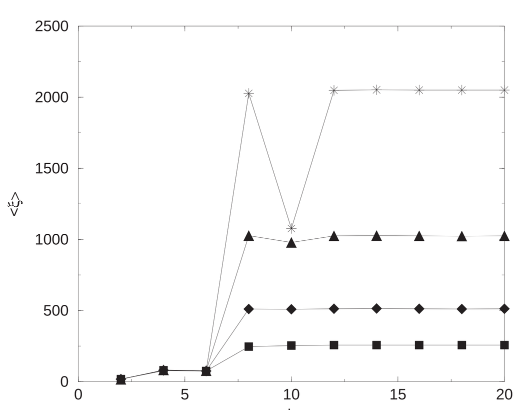


Sampling method: IPR for the average wave function as a function of k, for $n_m = n_q - 8$ and $n_q = 9$ (squares), $n_q = 9$ (diamonds), $n_q = 11$ (triangles), $n_q = 12$ (stars). Up to $k = 6$ ξ does not depends on the size of the system (signature of localization). For larger k, ξ shows a dependence on N, typical of an extended state. In the inset, ξ vs. k for the evolution without measurements

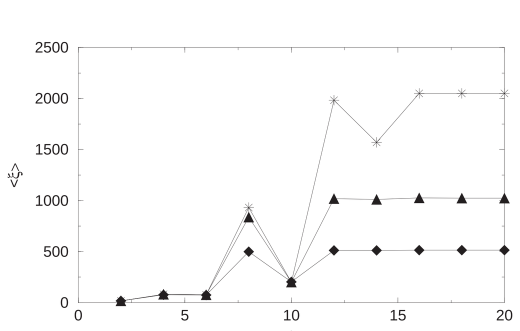


Sampling method: IPR for the average wave function as a function of k, for $n_m = n_q - 9$ and $n_q = 10$ (diamonds), $n_q = 11$ (triangles), $n_q = 12$ (stars). Increasing n_m changes the localization/delocalization border. The transition seems to depend on k via the ratio between the free localization length and $2^{n_q-n_m}$. In the inset, ξ vs. k for the evolution without measurements

k dependence - localization/delocalization transition



Phase Method: IPR as a function of k, for $n_m = n_q - 8$ and $n_q = 9$ (squares), $n_q = 9$ (diamonds), $n_q = 11$ (triangles), $n_q = 12$ (stars). Up to $k = 6$ ξ does not depends on the size of the system (signature of localization). For larger k, ξ shows a dependence on N, typical of an extended state. Here the ξ values are different if compared to the sampling method, but we get the same k values for the localized/extended transition



Phase Method: IPR as a function of k for $n_m = n_q - 9$ and $n_q = 10$ (diamonds), $n_q = 11$ (triangles), $n_q = 12$ (stars). Increasing n_m changes the localization/delocalization border. The transition seems to depend on k via the ratio between the free localization length and $2^{n_q-n_m}$. Here the ξ values are different if compared to the sampling method, but we get the same k values for the localized/extended transition

Summary and perspectives

* Measurement of a single qubit in the Kicked Rotator in a localized regime
  1 $n_m = n_q$ least significant qubit: measurement induces diffusion
  2 $n_m - n_q > 9$ for small k localization is still present
  3 transition localized / extended states in k for fixed $n_m - n_q$
  role of the free localization length in the transition
  4 Random Phase Method and Sampling Method give qualitatively the same results
  5 good agreement for $\langle n^2\rangle$, differences in ξ
* There exist measurements which preserve quantum localization
* Such measurements are naturally implemented on a Quantum Computer as single qubit measurements
* Role of the localization length of the unperturbed system in the transition
- Experimental set-ups
- Detailed study of the measurement process with a density matrix formalism