Simulation, Decoherence

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Overview

- Simulation of a physical system: three stages
- (i) Preparation of the initial state: this talk
- (ii) Simulation of the system dynamics
  - Unitary dynamics for chaotic quantum maps: earlier theoretical work by RHUL, Toulouse, experiments at MIT, etc
  - Nonunitary dynamics: this talk
- (iii) Readout: future work

Overview (cont'd)

- Decoherence functional for iterated quantum maps
- (i) Representation of maps on quantum register (RHUL, Toulouse), quantum symbolic dynamics
- (ii) Coarse graining for chaotic quantum maps
- (iii) Decoherence conditions for fine-grained and coarse-grained evolution
- (iv) Decoherence condition versus initial states
- (v) Approximate decoherence

State preparation

- Our algorithms prepare a $\log_2 N$ qubit register in an arbitrary pure state

$$|\Psi\rangle = \sum_{x=0}^{N-1} \sqrt{p(x)}\, e^{2\pi i \phi(x)} |x\rangle \ ,$$

  with arbitrary fidelity.

- Objection: A typical state of a quantum register cannot be efficiently described, let alone prepared!

- Assumption: We are given classical algorithms to compute $p(x)$ and $\phi(x)$.

Phases

Choose ϵ such that $1/\epsilon \in \mathbb{N}$. Define $U_1, \ldots, U_{1/\epsilon}$ by

$$U_k|x\rangle = \begin{cases} e^{2\pi i \epsilon}|x\rangle & \text{if } \phi(x) > (k-\frac{1}{2})\epsilon \,, \\ |x\rangle & \text{otherwise.} \end{cases}$$

$$|\tilde{\Psi}\rangle = U_1 U_2 \cdots U_{1/\epsilon} \sum_{x=0}^{N-1} \sqrt{p(x)}\,|x\rangle = \sum_{x=0}^{N-1} \sqrt{p(x)}\, e^{2\pi i \tilde{\phi}(x)}|x\rangle \,,$$

where $|\tilde{\phi}(x) - \phi(x)| \leq \epsilon/2$ and hence

$$\left| \langle\tilde{\Psi}| \sum_{x=0}^{N-1} \sqrt{p(x)}\, e^{2\pi i \phi(x)}|x\rangle \right| > 1 - \epsilon^2/8 \,.$$

Efficient for which $p(x)$?

Consider a sequence of probability functions $p_N : \{0,\ldots,N-1\} \longrightarrow [0,1]$, $N = 1,2,\ldots$. For any N, the algorithm prepares the quantum register in a state $|\tilde{\Psi}\rangle$ such that, with probability greater than $1-\nu$, the fidelity obeys the bound

$$|\langle\tilde{\Psi}|\Psi\rangle| > 1-\lambda\ .$$

If there exists $\eta < 1$ such that

$$p_N(x) \leq \frac{1}{\eta N} \quad \text{for all} \quad N \quad \text{and} \quad x\ ,$$

the resources needed by our state preparation algorithm are polynomial in the number of qubits, $\log_2 N$, and the inverse parameters η^{-1}, λ^{-1} and ν^{-1}.

Example 1

Our algorithm scales exponentially with the number of qubits for

$$p_N(x) = \delta_{xy}$$

for some integer $y = y(N)$. It then follows from the optimality of Grover's algorithm that the number of oracle calls needed is proportional to $\sqrt{N}$.

Example 2

Sequences that do satisfy the bound $p_N(x) \leq \frac{1}{\eta N}$ for all N and x arise naturally in the problem of encoding a bounded probability density function $f : [0, 1] \rightarrow [0, f_{\mathrm{max}}]$ in a state of the form

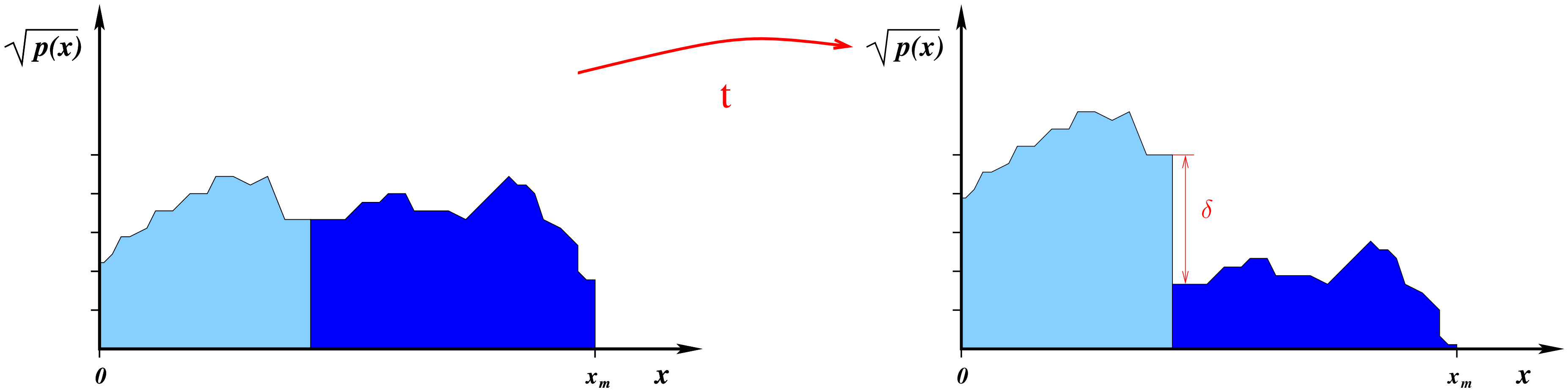
$$|\Psi_f\rangle = \mathcal{N}^{-1} \sum_{x=0}^{N-1} \sqrt{f(x/N)}\, |x\rangle \ ,$$

where $\mathcal{N}$ is a normalization factor.

Our algorithm is efficient for this class of problems.

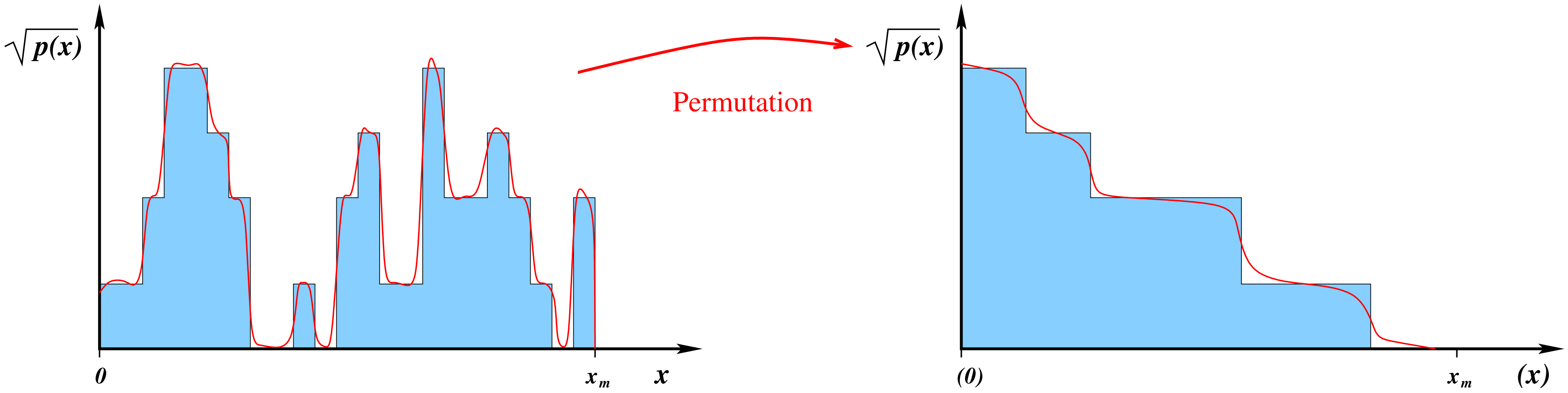
Generalized Grover's search

Grover's algorithm has the following property [Biham, et. al., quant-ph/9807027]:



Grover iterations increase the average amplitude of "good" states while decreasing the average amplitude of the "bad" states. The rest of the amplitude profile remains intact.

Monotone representation



Similarly to the original algorithm by Grover, our algorithm accesses individual values of x via quantum oracles. This allows us to introduce a permutation Π of the x values so that $p(x)$ appears as a monotonically decreasing function of $\Pi(x)$. This helps visualizing the action of our algorithm, although no explicit knowledge of this permutation is required.

The algorithm (1)

- Choose $\epsilon < \lambda\eta/3$ and define oracles $O_k(x) = \begin{cases} 1 & \text{if } \sqrt{p(x)} > \frac{1-k\epsilon}{\sqrt{\eta N}}, \\ 0 & \text{otherwise.} \end{cases}$

- For each oracle, O_k, estimate the number of solutions using the quantum counting algorithm and infer the number of iterations, t_k.

  For $k = 1, \ldots, T = \epsilon^{-1}$, define the Grover operator $\hat{G}(O_k, t_k)$.

The algorithm (2)

Prepare register in the state

$$|\Psi^0\rangle = (2^a N)^{-1/2} \sum_{x=0}^{2^a N-1} |x\rangle \,,$$

then apply the Grover operators successively to create the state

$$|\Psi^T\rangle = \hat{G}(O_T, t_T) \cdots \hat{G}(O_1, t_1)|\Psi^0\rangle \;.$$

Now measure the a auxiliary qubits in the computational basis. If one of the outcomes is 1, this stage of the algorithm has failed, and one has to start over.

Performance

Given: • classical algorithm for computing p: $p(x) \leq 1/(\eta N)$.
• target state $|\Psi_p\rangle = \sum_{x=0}^{N-1} \sqrt{p(x)}\,|x\rangle$.

We prepare: $|\Psi_{\tilde{p}}\rangle = \sum_{x=0}^{N-1} \sqrt{\tilde{p}(x)}\,|x\rangle$, such that with probability $1-\nu$, $|\langle\Psi_{\tilde{p}}|\Psi_p\rangle| > 1-\lambda$ and choose $\epsilon < \lambda\eta/3$.

	oracle calls	auxiliary qubits
counting	$\frac{27(1+4\nu)}{\nu\epsilon^6}$	$\log_2 \frac{27(1+4\nu)}{\nu\epsilon^5}$
preparing $\|\Psi_{\tilde{p}}\rangle$	$\frac{3\pi}{\epsilon^3\sqrt{\epsilon}}$	$3 + 3\log_2 \frac{1}{\epsilon}$

Binary version

For $k = 1, \ldots, T$, we define $O_k(x) = c_k(x)$, where

$$\sqrt{\eta N p(x)} = \sum_{k=1}^{\infty} c_k(x) 2^{-k} ,$$

and then proceed as before.

If $N_k = \sum_x O_k(x)$ are known, this is much more efficient than the stepfunction version of the algorithm.

For bounds on fidelity and number of oracle calls, see quant-ph/0411010.

Work in progress

- Eliminate the need for auxiliary qubits by introducing phase oracles (i.e., controlled rotations rather than controlled phase flips).

- Make detailed proposal for 3-qubit NMR demonstration experiment based on binary version of the algorithm and phase oracles.

Quantum Bayesian updating

Bayes rule:

$$p(h|d) = \frac{p(d|h)p(h)}{\sum_h p(d|h)p(h)} \,.$$

Quantum Bayesian updating:

$$\sum_{h=0}^{N-1} \sqrt{p(h)} \, |h\rangle \longmapsto \sum_{h=0}^{N-1} \sqrt{p(h|d)} \, |h\rangle \,.$$

In general, this evolution is nonunitary.

Hypothesis elimination

In quant-ph/0412025, we describe and analyse an algorithm for hypothesis elimination, which is a special case of Bayesian updating where the model is of the form

$$p(d|h) = \begin{cases} 0 & \text{if } h \text{ is ruled out by } d\,, \\ c_h & \text{otherwise.} \end{cases}$$

It is based on Grover's algorithm.

Generalization: Work in progress!

Decoherence

Definition 1: Projective Partition

A set of projectors $\{P_\mu\}$ on a Hilbert space $\mathcal{H}$ is called a projective *partition* of $\mathcal{H}$, if $\forall\, \mu, \mu' :\; P_\mu P_{\mu'} = \delta_{\mu\mu'} P_\mu$ and $\sum_\mu P_\mu = 1_{\mathcal{H}}$. We call a projective partition *fine-grained* if all projectors are one-dimensional, i.e., $\forall\, \mu\;\; \dim\big(\text{supp}(P_\mu)\big) = 1$, and *coarse-grained* otherwise.

Definition 2: Quantum Histories

Histories are defined to be ordered sequences of projection operators, corresponding to quantum-mechanical propositions:

$$h_{\boldsymbol{\alpha}} = \left(P_{\alpha_1}, P_{\alpha_2}, \ldots, P_{\alpha_k}\right) \ .$$

Definition 3: Decoherence functional

The *decoherence functional* is defined by

$$\mathcal{D}_{U,\,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}] := \mathsf{Tr}\left[C_{\boldsymbol{\alpha}}\,\rho\,C_{\boldsymbol{\beta}}^{\dagger}\right]\,, \quad C_{\boldsymbol{\alpha}} := U^{\dagger\,k} P_{\alpha_k} U P_{\alpha_{k-1}} U \ldots P_{\alpha_2} U P_{\alpha_1} U\ .$$

A set of histories is said to be *decoherent* with respect to a given unitary map $U : \mathcal{H} \to \mathcal{H}$ and a given initial state ρ, if

$$\forall h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}} \;:\; \mathcal{D}_{U,\,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}] \propto \delta_{\boldsymbol{\alpha}\boldsymbol{\beta}} \equiv \prod_{j=1}^{k} \delta_{\alpha_j \beta_j}$$

Definition 4: Classical States

A state ρ is called *classical with respect to a partition* $\{P_\mu\}$ of the Hilbert space $\mathcal{H}$, if it is block-diagonal w.r.t. $\{P_\mu\}$, i.e., if

$$\rho = \sum_\mu P_\mu \, \rho \, P_\mu \ .$$

We will denote the set of all states that are classical w.r.t. a given partition $\{P_\mu\}$ by $\mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}$.

Decoherence of arbitrarily long histories

Decoherence for all initial states ρ_0 that are naturally induced by the projectors $\{P_\mu\}$ of a given projective partition via normalization, i.e. decoherence for all

$$\rho_0 \in \mathcal{S}_{\{P_\mu\}} := \left\{ \frac{P_\nu}{\mathsf{Tr}[P_\nu]} \, : \, P_\nu \in \{P_\mu\} \right\} \ ,$$

implies decoherence *for arbitrary initial states*.

Necessary and sufficient single-iteration decoherence condition

Fine-grained histories decohere for all classical initial states and arbitrarily many iterations *if and only if* the unitary map U preserves classicality of states, i.e., *if and only if*

$$\forall\, \rho \in \mathcal{S}^{\text{cl}}_{\{P_\mu\}} \; : \; U\rho U^\dagger \in \mathcal{S}^{\text{cl}}_{\{P_\mu\}}$$

By contrast, decoherence of coarse-grained histories does not, in general, imply that the unitary evolution preserves classicality of states.

Coarse grained partitions

The following is a necessary condition for the decoherence of arbitrary (i.e., not necessarily fine-grained) partitions for all classical initial states and arbitrarily long histories.

$$\forall\, P_{\mu'}, P_{\mu''} \in \{P_\mu\}\ :$$

$$\left[U P_{\mu'} U^\dagger\,,\, P_{\mu''}\right] = 0$$

Application

Analysis of the symbolic dynamics for the quantum baker's map on a qubit register (work in progress).