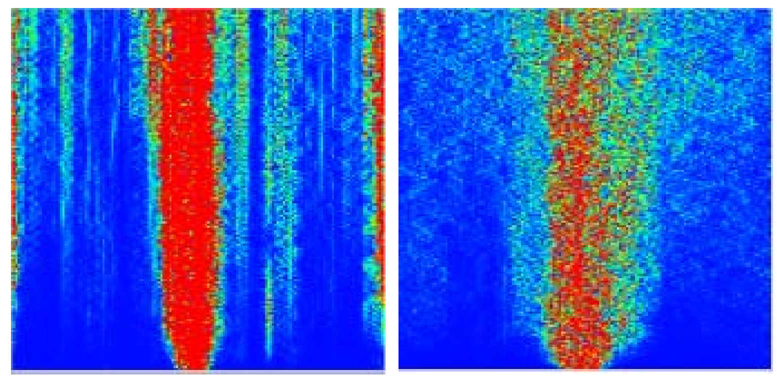
Effects of Decoherence and Imperfections for Quantum Information Processing (EDIQIP)

Coordinator: D.Shepelyansky (www.quantware.ups-tlse.fr)
Nodes: G.Alber (TUD), G.Benenti (INFM), R.Schack(RHUL)

Objective

- Effects of realistic imperfections on quantum computer operability and accuracy
- Decoherence and quantum chaos induced by inter-qubit couplings
- New efficient algorithms for simulation of quantum and classical physical systems
- Numerical codes with up to 30 qubits
- Development and test of error-correcting codes for quantum chaos and noisy gates



Anderson metal-insulator transition with 7 qubits

Objective Approach

- Analytical methods developed for many-body systems (nuclei, atoms, quantum dots)
- Random matrix theory and quantum chaos
- Large-scale numerical simulations of many qubits on modern supercomputers
- Stability of algorithms to quantum errors

Status

- Project started January 1, 2003
- New quantum algorithms and imperfection effects for Anderson transition and quantum wavelet transform; numerics with 7-28 qubits
- Universal law for fidelity decay induced by static imperfections (random matrix theory)
- Quantum trajectories and error-correction

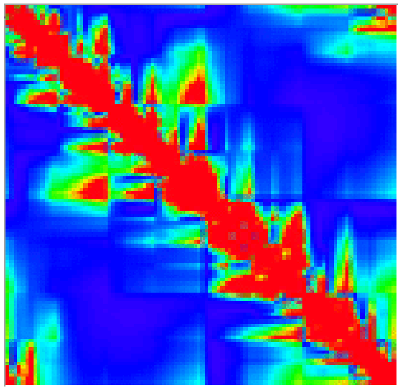
Effects of Decoherence and Imperfections for Quantum Information Processing (EDIQIP)

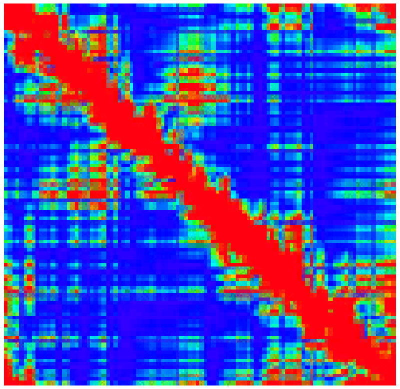
www.quantware.ups-tlse.fr/EDIQIP

(collaboration with ARO/NSA/ARDA QC program)

- First year progress

- New algorithms for classical dynamics: strange attractor (a1), Poincaré recurrences and periodic orbits (a2) => exponential gain
- Quantum chaos algorithms: dynamical localization in saw-tooth map, tent map, Wigner function (a3) => quadratic speed up
- Quantum algorithm (QA) for Anderson metal-insulator transition with quadratic speed-up; static imperfection effects on critical point; numerical tests with 7-12 qubits (future experiments ?) (a4,b2)
- Decoherence law for random errors in quantum gates: strange attractor, quantum chaos maps, wavelets (up to 28 qubits) (b1)
- Universal law for fidelity decay induced by static imperfections: random matrix theory works for 10 orders of magnitude variation, numerics for saw-tooth, tent maps, wavelets (b2,b3)
- Dynamics of entanglement in quantum computer hardware (c1) and quantum chaos algorithm (c2). Additivity implies strong super-additivity of entanglement of formation (Shor – Pomeransky) (c3)
- Quantum teleportation fidelity: quantum trajectories, 24 qubits (c4)
- Quantum sound treatment (d1); effects of measurements (d2)
- Quantum error-correction using knowledge of error location: numerical tests for Grover and saw-tooth algorithms (e1)





Dissemination of results

All information is at **www.quantware.ups-tlse.fr/EDIQIP**

During first year 2003:

- 24 papers including:

$\longrightarrow$ 4 Phys. Rev. Lett.

$\longrightarrow$ 1 book on QIPC

$\longrightarrow$ chapter in a book dedicated to the centennial of A.N.Kolmogorov

-23 talks and posters on international conferences in Europe, Brasil, Japan, Russia, USA

Organization of conferences:

- Quantware workshop, July 2002 (organization meeting)
- Enrico Fermi summer school on QIPC, Varenna, June-July 2005
(directors: G.Casati, D.Shepelyansky, P.Zoller)
- Institut Henri Poincare, QIPC trimestre, Paris, Jan - April 2006
(directors: Ph.Grangier, M.Santha, D.Shepelyansky)

Publications of EDIQIP in 2003

Scientific deliverables are marked by D4,D7,D8,D11,D12.

[1] M.Terraneo, B.Georgeot and D.L.Shepelyansky, *Strange attractor simulated on a quantum computer*, Eur. Phys. J. D **22**, 127 (2003) [quant-ph/0203062] (D4,D8,D12).

[2] G.Benenti, G.Casati, S.Montangero and D.L.Shepelyansky, *Statistical properties of eigenvalues for an operating quantum computer with static imperfections*, Eur. Phys. J. D **22**, 285 (2003) [quant-ph/0206130] (D4,D7).

[3] G.Benenti, G.Casati, S.Montangero and D.L.Shepelyansky, *Dynamical localization simulated on a few qubits quantum computer*, Phys. Rev. A **67**, 052312 (2003) [quant-ph/0210052] (D4,D7,D11,D12).

[4] B.Levi, B.Georgeot and D.L.Shepelyansky, *Quantum computing of quantum chaos in the kicked rotator model*, Phys. Rev. E **67**, 046220 (2003) [quant-ph/0210154] (D4,D8).

[5] S.Bettelli and D.L.Shepelyansky, *Entanglement versus relaxation and decoherence in a quantum algorithm for quantum chaos*, Phys. Rev. A **67**, 054303 (2003) [quant-ph/0301086] (D4,D8).

[6] M.Terraneo and D.L.Shepelyansky, *Imperfection effects for multiple applications of the quantum wavelet transform*, Phys. Rev. Lett. **90**, 257902 (2003) [quant-ph/0303043] (D4,D7,D8).

[7] A.A.Pomeransky, *Strong superadditivity of the entanglement of formation follows from its additivity*, Phys. Rev. A **68**, 032317 (2003) [quant-ph/0305056] (D11).

[8] R.Livi, S.Ruffo and D.L.Shepelyansky, *Le cheminement de Kolmogorov de l'integrabilite au chaos et au-dela*, p.15-45, Eds. R.Livi et A.Vulpiani, in *L'heritage de Kolmogorov en physique* (Belin, Paris, (2003))

(in French); *Kolmogorov pathways from integrability to chaos and beyond*, Eds. R.Livi and A.Vulpiani, in *The Kolmogorov legacy in physics* (Lecture Notes in Physics, Springer, Berlin (2003)) (D4).

[9] S.Montangero, G.Benenti and R.Fazio, *Dynamics of entanglement in quantum computers with imperfections*, Phys. Rev. Lett. **91**, 187901 (2003) [quant-ph/0307036] (D7).

[10] A.A.Pomeransky and D.L.Shepelyansky, *Quantum computation of the Anderson transition in presence of imperfections*, Phys. Rev. A **69**, 014302 (2004) [quant-ph/0306203] (D11,D7).

[11] B.Georgeot et D.L.Shepelyansky, *Les ordinateurs quantiques affrontent le chaos*, to appear in Images de la Physique 2003 [quant-ph/0307103] (D11).

[12] B.Georgeot, *Quantum computing of Poincaré recurrences and periodic orbits*, to appear in Phys. Rev. A [quant-ph/0307233] (D11).

[13] J.W.Lee, A.D.Chepelianskii and D.L.Shepelyansky, *Treatment of sound on quantum computers*, submitted to Eur. Phys. J. D [quant-ph/0309018] (D11,D12).

[14] M.Terraneo and D.L.Shepelyansky, *Dynamical localization and repeated measurements in a quantum computation process*, Phys. Rev. Lett. **92**, 037902 (2004) [quant-ph/0309192] (D4).

[15] S.Bettelli, *A quantitative model for the effective decoherence of a quantum computer with imperfect unitary operations*, submitted to Phys. Rev. A [quant-ph/0310152] (D4,D8).

[16] G.G.Carlo, G.Benenti and G.Casati, *Teleportation in a noisy environment: a quantum trajectories approach*, Phys. Rev. Lett. **91**, 257903 (2003) [quant-ph/0307065] (D8,D12).

[17] G.Casati and S.Montangero, *Measurement and information extraction in complex dynamics quantum computation*, in Proceedings of First International Workshop DICE (Decoherence, Information, Complexity and Entropy), Piombino, Italy, 2002, to appear in Lecture Notes in Physics, Springer-Verlag [quant-ph/0307165]

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[18] D.Rossini, G.Benenti and G.Casati, *Entanglement echoes in quantum computation*, submitted to Phys. Rev. Lett. [quant-ph/0309146] (D4,D8).
[19] W.-G.Wang, G.Casati and B.Li, *Stability of quantum motion: beyond Fermi-golden-rule and Lyapunov decay*, submitted to Phys. Rev. Lett. [quant-ph/0309154] (D7).
[20] G.Benenti, G.Casati and G.Strini, *Principles of quantum computation and information*, Volume I, to be published by World Scientific, Singapore (2003) (D4).
[21] K.M.Frahm, R.Fleckinger and D.L.Shepelyansky, *Quantum chaos and random matrix theory for fidelity decay in quantum computations with static imperfections*, submitted to Eur. Phys. J. D [quant-ph/0312120] (D4,D7,D8,D12).
[22] G.Alber, Th.Beth, Ch.Charnes, A.Delgado, M.Grassl, M.Mussinger, *Detected-jump-error-correcting quantum codes, quantum error designs, and quantum computation*, Phys. Rev. A **68**, 012316 (2003) [quant-ph/0208140] (D8,D12).
[23] Th.Beth, Ch.Charnes, M.Grassl, G.Alber, A.Delgado, M.Mussinger, *A New Class of Designs Which protect against Quantum Jumps*, Designs, Codes and Cryptography **29**, 51 (2003) (D8,D12).
[24] G.Alber, M.Mussinger, A.Delgado, *Quantum information processing and error correction with jump codes*, in *Quantum Information Processing*, edited by Th. Beth and G. Leuchs (Wiley-VCH, Berlin, 2003) (D8,D12).

EDIQIP research group at Toulouse

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15 papers in 2003 (including 2 Phys. Rev. Lett.)

Fidelity decay due to errors (a-b)

Accuracy measure of quantum computation is fidelity: $f(t) = |<\psi(t)|\psi_\varepsilon(t)>|^2$.

Quantum algorithm: $|\psi(t)> = U^t\,|\psi(0)>$, $U = \underbrace{U_{N_g}\cdot \ldots \cdot U_1}_{\text{elementary gates}}$.

Errors: $U_j \;\rightarrow\; U_j\, e^{i\delta H}$, $\delta H \sim \varepsilon$.

(i) Decoherence due to residual couplings of quantum computer to external bath:

δH random and different at each j and t,

e.g.: random phase fluctuations: $\delta\phi \in [-\varepsilon, \varepsilon]$ in phase-shift gates.

(ii) Static imperfections in the quantum computer itself:

δH (random but) constant at each j and t,

e.g.: $\delta H = \sum_{j=0}^{n_q-1} \delta_j\,\sigma_j^{(z)} + 2\sum_{j=0}^{n_q-2} J_j\,\sigma_j^{(x)}\,\sigma_{j+1}^{(x)}$, $J_j, \delta_j \in [-\varepsilon, \varepsilon]$.

(iii) Non-unitary errors in quantum computation:

$e^{i\delta H}$ is non-unitary ($\delta H \neq \delta H^\dagger$, density matrix and quantum trajectories approach)

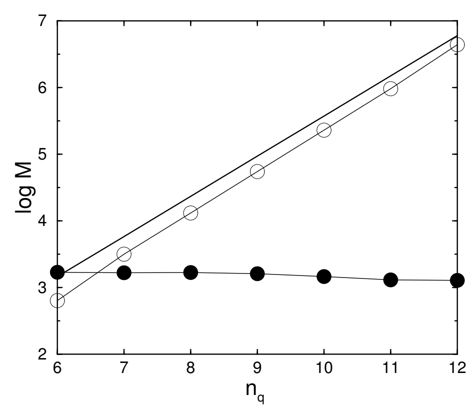
Strange attractor simulated on a quantum computer (a1)

Dissipative dynamical systems often converge to strange attractors, characterized by fractal dimensions and chaotic unstable dynamics of trajectories

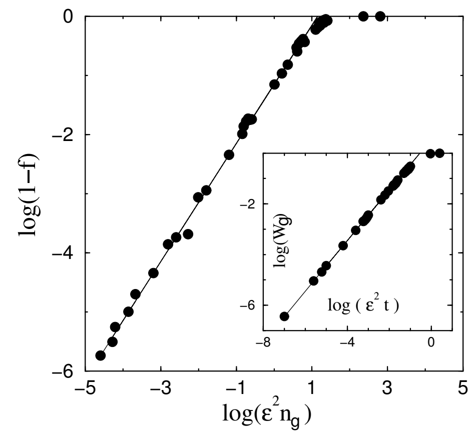
$\longrightarrow$ Applications: turbulence and weather forecast, molecular dynamics, chaotic chemical reactions, multimode solid state lasers, ecology and physiology, etc...

$\longrightarrow$ Examples: Lorenz attractor (1963), Hénon attractor (1976)

- Chaotic dissipative deterministic map: $\bar{y} = \frac{y}{2} + x$ (mod 2) , $\bar{x} = \frac{y}{2} + 2x$ (mod 1) converges to a strange attractor of fractal dimension ≈ 1.543.
- Quantum computation of one iteration on a $2^n \times 2^{n+1}$lattice $\Rightarrow 17n - 10$ gates Efficient measurement of spectrum of phase space correlation functions



Algorithm efficiency

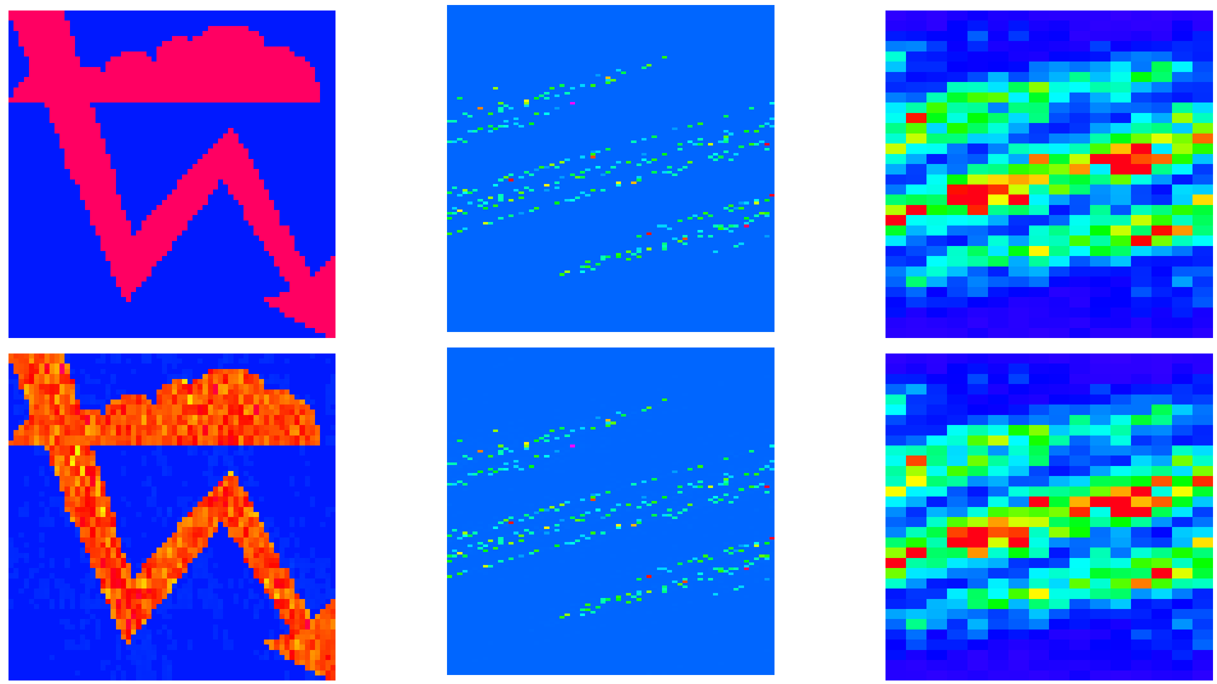


Fidelity decay law

Numerical simulations with 28 qubits

Noise $\epsilon = 0$

$\epsilon = 0.05$ (left, middle)
$\epsilon = 0.025$ (right)



10 forwards and 10 backwards iterations

10 forwards iterations (attractor)

Spectrum of phase space correlations functions

Quantum computing of Poincaré recurrences and periodic orbits (a2)

- Classical bounded conservative systems: Poincaré recurrences

some points from an arbitrary domain A will eventually come back to A after the recurrence time

$\longrightarrow$ Gives information on transport properties, correlation functions

Recurrence time can be very (exponentially) long $\Rightarrow$ hard to find in general

- For more general systems: periodic orbits

$\longrightarrow$ diffusion coefficients, properties of strange attractors

$\longrightarrow$ classical and semiclassical trace formulas

Periodic orbits $=$ very small subset among all classical trajectories $\Rightarrow$ hard to find in general

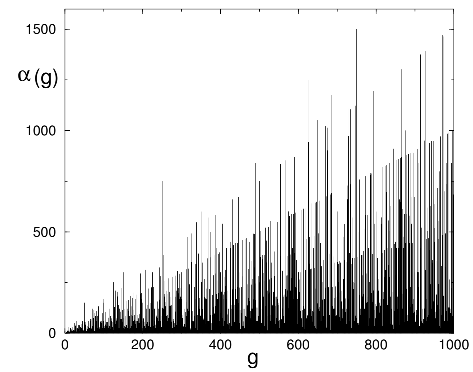
Quantum algorithms

Classical dynamical system described by operator U on discrete lattice of N points

- If U and U^p can be implemented on a quantum computer in polynomial in $\log N$ and $\log p$ number of operations (quantum gates)

  ⇒ Use variant of period-finding (used also in factoring) to find recurrence times or periodic orbits

  ⇒ exponential gain
  (example: Arnold cat map $\bar{y} = y + x$, $\bar{x} = y + 2x$ (mod 1) ;
  $g = N, \alpha =$ period)



cat map recurrence times

- If U requires a polynomial in $\log N$ number of gates but U^p needs a polynomial in p

  ⇒ Use variant of Grover search algorithm to find recurrence times and periodic orbits

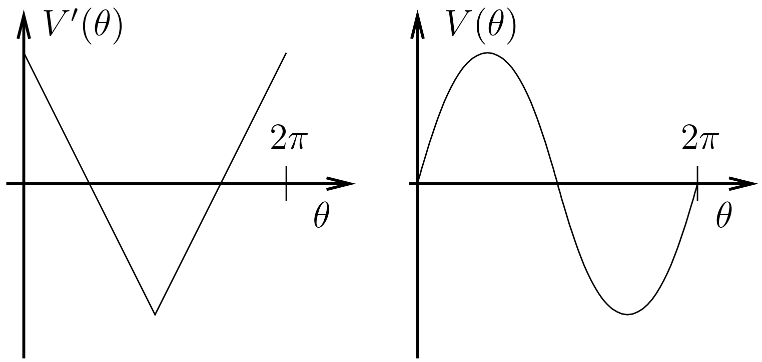
  ⇒ quadratic gain (examples: standard map, classical sawtooth map)

Example: model of quantum tent map (a3,b1-b3)

$$H(t) = \frac{T\,p^2}{2} + V(\theta) \sum_{n=-\infty}^{\infty} \delta(t-n)$$

Classical map :

$$p_{n+1} = p_n - V'(\theta_n)$$
$$\theta_{n+1} = \theta_n + T\,p_{n+1}$$



Quantum map : $p = -i\partial/\partial\theta$

$$|\psi(t+1)>= U\,|\psi(t)>$$
$$U = e^{-iTp^2/2}\,e^{-iV(\theta)}$$

$$V(\theta) = \begin{cases} -\frac{k}{2}\theta(\theta-\pi) \\ \frac{k}{2}(\theta-\pi)(\theta-2\pi) \end{cases} \;,\; V'(\theta) = \begin{cases} k(\frac{\pi}{2}-\theta) & \text{if} \quad 0 \le \theta \le \pi \\ k(-\frac{3\pi}{2}+\theta) & \text{if} \quad \pi \le \theta \le 2\pi \end{cases}$$

Quantum algorithm for tent map (a3,b3)

Quantum register identification: $|p\rangle \equiv |\alpha_0\rangle_0 \, |\alpha_1\rangle_1 \, \ldots \, |\alpha_{n_q-1}\rangle_{n_q-1}$.

$$p = \sum_{j=0}^{n_q-1} \alpha_j \, 2^j \; \in \; \{0, \ldots, N-1\}$$

$N = 2^{n_q}$ = dimension of Hilbert space; n_q = number of qubits; $\alpha_j \in \{0, 1\}$.

Quantum Fourier transform: $p \leftrightarrow \theta$ and $e^{-iTp^2/2} \, |p\rangle = \prod_{j<k} \underbrace{e^{i(\cdots)\alpha_j \alpha_k}}_{B^{(2)}_{jk}(\cdots)} \prod_j \underbrace{e^{i(\cdots)\alpha_j}}_{B^{(1)}_j(\cdots)} \, |p\rangle$.

with simple and controlled phase-shift:

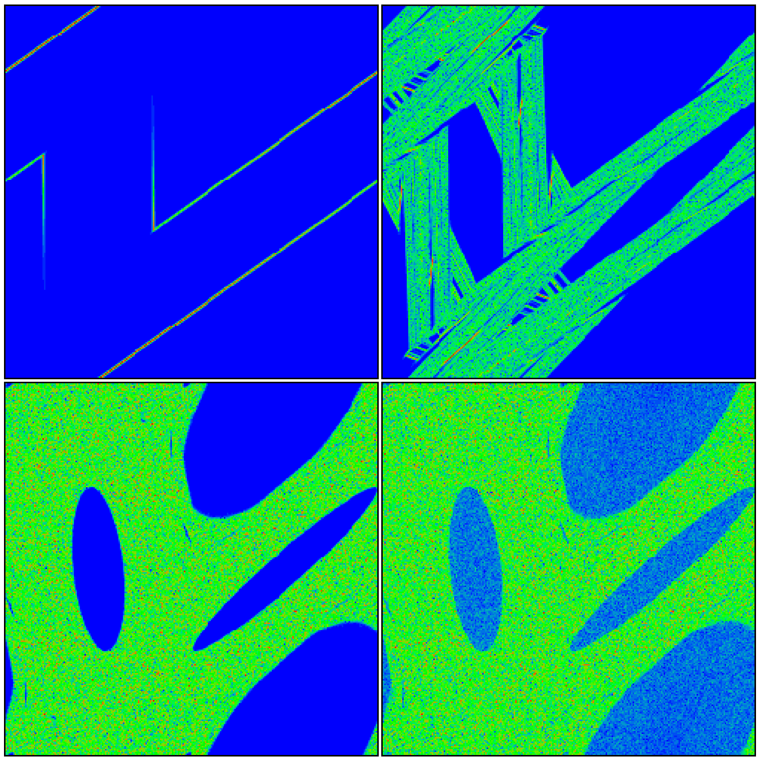
$$B^{(1)}_j(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \quad , \quad B^{(2)}_{jk}(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{pmatrix} .$$

Double controlled phase-shift: $B^{(3)}_{jkl}(\phi) = B^{(2)}_{jl}\left(\frac{\phi}{2}\right) \, B^{(2)}_{jk}\left(\frac{\phi}{2}\right) C^{(N)}_{kl} \, B^{(2)}_{jk}\left(-\frac{\phi}{2}\right) C^{(N)}_{kl}$.

Number of elementary gates: $n_g \approx 9 \, n_q^2/2$

Husimi function

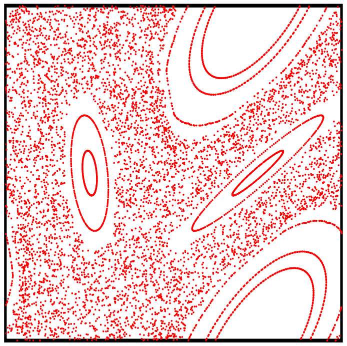
$t = 5$ 16 qubits $t = 15$



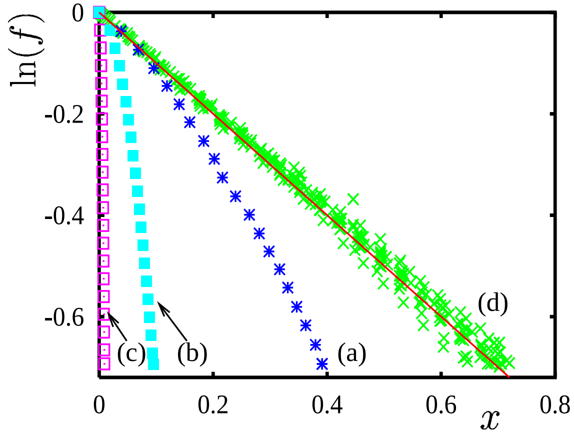
$t = 5625, \varepsilon = 0$ $\varepsilon = 7 \cdot 10^{-7}$

$\hbar_{\rm eff} = T = 2\pi/N, \ N = 2^{nq}$

Poincaré section $(K = kT = 1.7)$



Fidelity decay with errors



(static: $x = t/t_c$, (a), (b), (c),
random: $x = t/t_r$, (d))

Quantum computation of the Anderson transition in presence of imperfections (a4, b2)

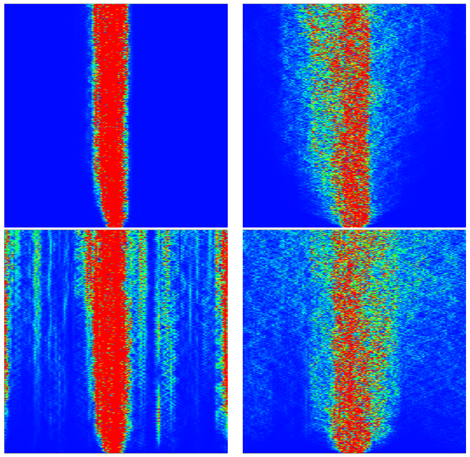
The stationary Schrödinger equation for the Anderson model: a particle on a d-dimensional lattice in a random potential: $\sum_{\vec{m}} V_{\vec{m}} \psi_{\vec{m}+\vec{n}} + E_{\vec{n}} \psi_{\vec{n}} = E\psi_{\vec{n}}$, In $d \geq 3$ dimensions the wave functions are exponentially localized for sufficiently large (compared to $V_{\vec{m}}$) typical value of $E_{\mathbf{n}}$ and delocalized for small typical value of $E_{\mathbf{n}}$ (P.W. Anderson (1958)).
Our model: 1-dim. kicked rotator with frequency modulation.
Anderson localization $\longrightarrow$ dynamical localization of quantum chaos in the kicked rotator model (S. Fishman et al. (1982)).
3 dimensions $\longrightarrow$ 1 dimension plus frequency modulation with 2 incomensurate frequencies. (D.L. Shepelyansky (1983)).

Our Hamiltonian H: $H_0(n) + k(1 + \epsilon \cos(\Omega_1 t) \cos(\Omega_2 t)) \cos\theta \sum_m \delta(t-m)$,
The time evolution: $\bar{\psi} = U_T U_k \psi, \qquad U_T = \exp\{-iH_0(n)\}$
$U_k = \exp\{-ik(1 + 0.75\cos(\Omega_1 t)\cos(\Omega_2 t))\cos\theta\}$.

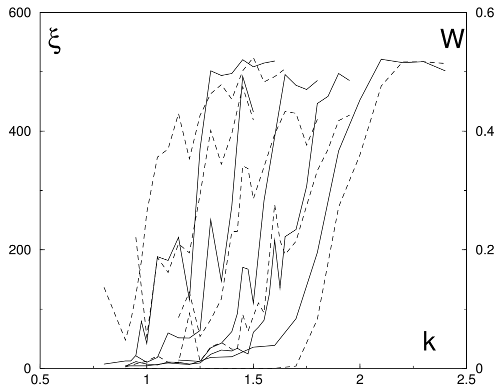
The quantum algorithm (a4, b2)

The quantum states $n = 0, ..., N-1$ are represented by one quantum register with n_q qubits so that $N = 2^{n_q}$. The initial state with all probability at $n_0 = 0$ corresponds to the state $|00...0\rangle$ (momentum n changes on a circle with N levels). The random phase multiplication $U_T = \exp(-iH_0(n))$ in the momentum basis is performed as a random sequence of one-qubit phase shifts and controlled-NOT gates. Then the kick operator $U_k = \exp(-ik(t)\cos\theta)$ is performed as follows. First, one applies the QFT to change the representation. Then θ can be written in the binary representation as $\theta/2\pi = 0.a_1a_2..a_{n_q}$ with $a_i = 0$ or 1. It's convenient to use the notation $\theta = \pi a_1 + \bar{\theta}$ to single out the most significant qubit. Then due to the relation $\cos\theta = (-1)^{a_1}\cos\bar{\theta} = \sigma_1^z\cos\bar{\theta}$ the kick operator takes the form $U_k = e^{-ik(t)\cos\theta} = e^{-i\sigma_1^z k(t)\cos\bar{\theta}}$. This operator can be approximated to an arbitrary precision by a sequence of one-qubit gates applied to the first qubit and the diagonal operators $S^m = e^{ima_1\bar{\theta}}$. We used the following sequence: $R_\gamma(\bar{\theta}) = HS^1H\ e^{-i\frac{\gamma}{4}\sigma_1^z}\ HS^{-2}H\ e^{-i\frac{\gamma}{2}\sigma_1^z}\ HS^2H\ e^{-i\frac{\gamma}{4}\sigma_1^z}\ HS^{-1}H = e^{-i\sigma_1^z\gamma\cos(\bar{\theta})} + O(\gamma^3)$, where $H = (\sigma_1^z + \sigma_1^x)/\sqrt{2}$ is the Hadamard gate. Thus the kick operator is given by $U_k = R_\gamma(\bar{\theta})^l + O(l\gamma^3)$, where the number of steps $l = k/\gamma$ and we used in our numerical simulations the small parameter $\gamma = k/l \approx 0.2$ that gives $l \approx 5 - 10$ for $k \sim 1 - 2$. The number of gates is $\sim k$, so the algorithm is more efficient for moderate k. Then one goes back to the momentum representation by the QFT. One complete iteration of the algorithm requires n_g elementary gates where $n_g = 2[k/\gamma](n_q + 2) + n_q^2 + 12n_q + 9$ with the square brackets denoting the integer part.

Static imperfections in QA for Anderson transition (a4, b2)

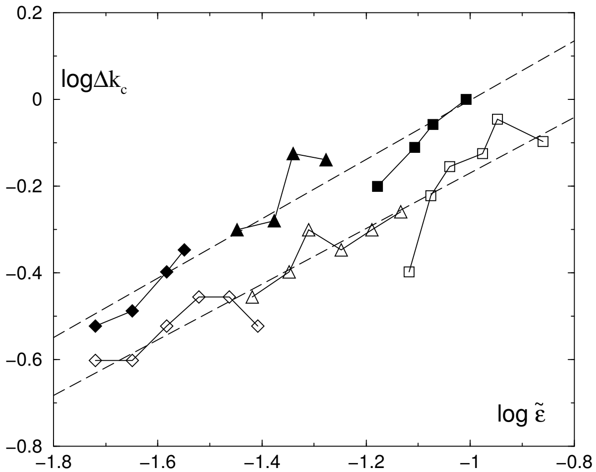


The time evolution of the probability distribution $|\psi_n|^2$ in the localized (left column, $k = 1.2$) and delocalized (right column, $k = 2.4$) phases for $n_q = 7$ qubits ($N = 2^{n_q}$), with $0 \le t \le 400$ (vertical axis) and $-N/2 < n \le N/2$ (horizontal axis); $k_c = 1.8$. The strength of static imperfections is $\epsilon = \mu = 0$ for top row and $\epsilon = \mu = 10^{-4}$ for bottom row.



Dependence of the IPR ξ and the excitation probability: $W = \sum_{n=(N/4,3N/4)} |\psi_n|^2$ (full and dashed curves for left and right scales respectively) on the kick strength k for $n_q = 10$ and $t \ge 10^5$, $\epsilon = 0$; 10^{-5}; $2{\times}10^{-5}$; $4{\times}10^{-5}$; $8{\times}10^{-5}$ (curves from right to left); $\mu = 0$.

Critical point shift (a4, b2)



Dependence of the shift of the critical point $\Delta k_c(\epsilon) = k_c - k_c(\epsilon)$ on rescaled imperfection strength $\tilde{\epsilon} = \epsilon n_g \sqrt{n_q}$ for $\epsilon = 2 \times 10^{-5}$ (diamonds), 4×10^{-5} (triangles) and 8×10^{-5} (squares); open/full symbols are for $\mu = 0$, $8 \le n_q \le 13$ and $\mu = \epsilon$, $8 \le n_q \le 11$ respectively; $k_c = 1.8$. The dashed lines show the scaling relation.

The shift of the critical point $\Delta k_c(\epsilon) = k_c - k_c(\epsilon)$ depends on ϵ, μ and n_q. From the IPR data obtained for various ϵ, μ, n_q we find that the global parameter dependence can be described by the scaling relation $\Delta k_c(\epsilon) = A\tilde{\epsilon}^{\,\alpha}$, $\tilde{\epsilon} = \epsilon n_g \sqrt{n_q}$. The data fit gives $A = 3.0$, $\alpha = 0.64$ for $\mu = 0$ and $A = 4.8$, $\alpha = 0.68$ for $\mu = \epsilon$.

In the vicinity of the critical point the algorithm gives a quadratic speedup in computation of diffusion rate and localization length, comparing to the known classical algorithms.

Imperfection effects for multiple applications of the quantum wavelet transform (b1,b2)

Wavelet Transforms (WT)

- Wavelets obtained by dilations and translations of an original mother function
- Frequency-time analysis
- Continuous and discrete WT.

Applications

- signal treatment - analysis and denoising of time series
- data and image compression
- multifractal analysis

Efficient implementation on Quantum Computers:
Daubeschies and Haar wavelet transforms

Circuit developed in *A. Fijaney and C. Williams, Lecture Notes in Computer Science* **1509**, *10 (Springer, 1998); quant-ph/9809004*

QUANTUM ALGORITHM: the Daubechies wavelets

$$D^{(4)} = (D_4^{(4)} \oplus I_{2^{n_q}-4})(\Pi_8 \oplus I_{2^{n_q}-8}) \dots (D_{2^i}^{(4)} \oplus I_{2^{n_q}-2^i})(\Pi_{2^{i+1}} \oplus I_{2^{n_q}-2^{i+1}}) \dots \Pi_{2^{n_q}} D_{2^{n_q}}^{(4)}$$

- $D_{2^n}^{(4)}$ is the wavelet kernel, acting on vectors of length 2^n

$$D_{2^n}^{(4)} = (I_{2^{n-1}} \otimes C_1) P_{2^n} (N \otimes I_{2^{n-1}})(N \otimes I_{2^{n-2}} \oplus I_{2^{n-1}}) \dots (N \otimes I_2 \oplus I_{2^n-4})(N \oplus I_{2^n-2}) P_{2^n} (I_{2^{n-1}} \otimes C_0)$$

- P_{2^n}: permutation matrix, $P_{2^n}|a_0, a_1, \dots, a_{n-1}\rangle = |a_{n-1}, \dots, a_1, a_0\rangle$
- Π_{2^n}: shuffling matrix, $\Pi_{2^n}|a_0, a_1, \dots, a_{n-2}, a_{n-1}\rangle = |a_{n-1}, a_0, a_1, \dots, a_{n-2}\rangle$

N not gate - C_0, C_1 2×2 matrices related to the Daubechies coefficients

$$\tilde{C}_0 = 2 \begin{pmatrix} c_2 & c_3 \\ c_3 & -c_2 \end{pmatrix} \qquad \tilde{C}_1 = \frac{1}{2} \begin{pmatrix} \frac{c_0}{c_3} & 1 \\ 1 & -\frac{c_0}{c_3} \end{pmatrix}$$

$$C_0 = \frac{1}{\sqrt{\det \tilde{C}_0}} \tilde{C}_0 \qquad C_1 = \frac{1}{\sqrt{\det \tilde{C}_1}} \tilde{C}_1$$

The Model

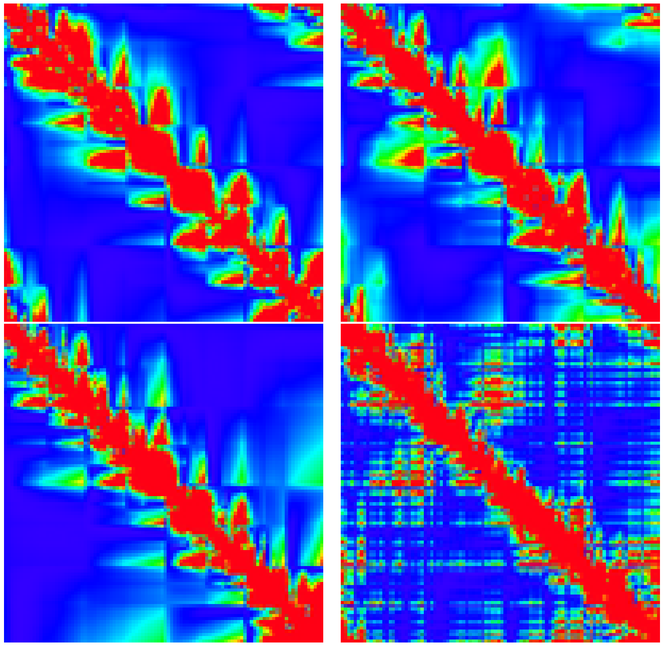
Dynamical model given by repeated applications of WT - efficient simulation on quantum computers. ψ in a Hilbert space of $N = 2^{n_q}$ states - dynamics described by the evolution operator $\hat{U}$: $\overline{\psi} = \hat{U}\psi$ ($O(n_q^3)$ gates and $n_q + 1$ qubits).

$$\hat{U} = D^{(4)\dagger} e^{-ik(x-\pi)^2/2} D^{(4)} e^{-iTn^2/2}$$

n momentum $-N/2 \leq n < N/2$, $x = 2\pi j/N$ with $j = 0, \ldots N-1$ index in the wavelet basis.

Algebraic localization: $|U_{n,n'}|^2 \sim \frac{1}{|n-n'|^\alpha}$

$|n - n'| \gg 5k$, $\alpha = 4$; $|n - n'| \ll 5k$, $\alpha = 2$



Density plot of the $|U_{n,n'}|^2$. Top: $k = 100$ (left), $k = 1000$ (right); bottom is for $k = 1000$: a doubled resolution of left upper quarter (left), perturbed operator with static errors; $n_q = 12$.

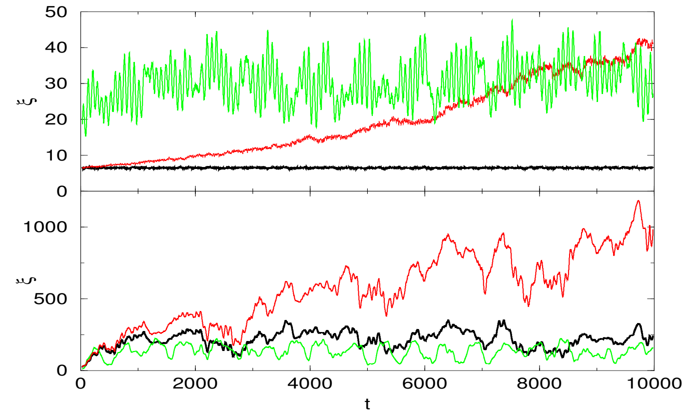
Imperfections (b1, b2)

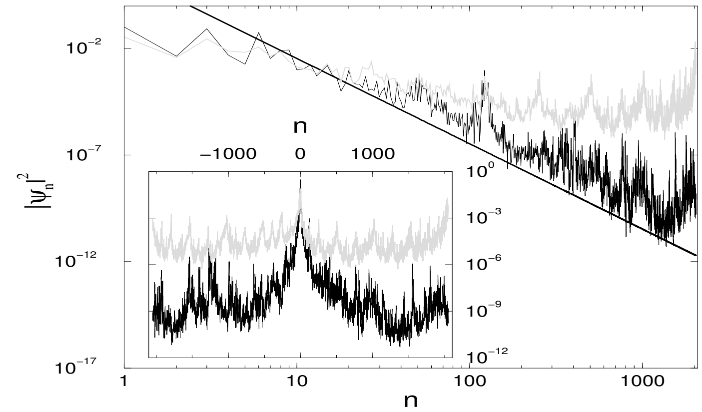
We simulate the gate sequence in the presence of imperfections:

- Noisy gates: each gate is perturbed by a random unitary rotation by an angle η, $-\epsilon/2 \leq \eta \leq \epsilon/2$.

- Static imperfections: given by the Hamiltonian $H = \sum_l \eta_l \sigma_l^z + \mu_l \sigma_l^x \sigma_{l+1}^x$ $(l = 1, \ldots n_q)$. η_l static one-qubit energy shifts, $-\epsilon/2 \leq \eta_l \leq \epsilon/2$, μ_l inter-qubit coupling, $-\mu/2 \leq \mu_l \leq \mu/2$

Effects on IPR ξ and on the wave function ψ_n.





Fidelity time scales (b1, b2)

Fidelity: $f(t) = |\langle\psi_\epsilon(t)|\psi(t)\rangle|^2$

n_g gates per map iteration

$N_g = n_g t_f$ total number of gates

$f(t_f)=0.9$ timescale definition

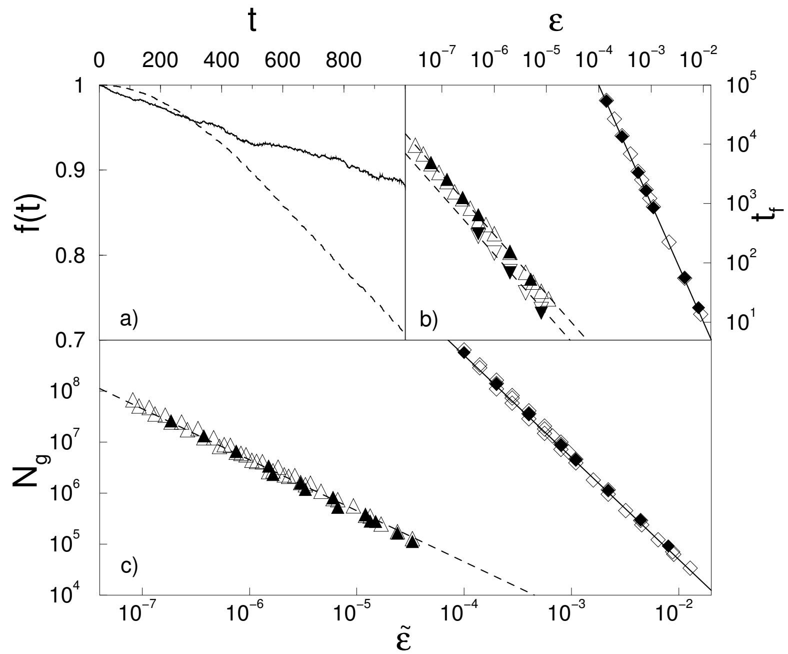
- Noisy gates:

$$f(t) \approx \exp\left(-A\epsilon^2 n_g t\right)$$
$$t_f = C/(\epsilon^2 n_g)$$
$$N_g = C/(\epsilon^2)$$

- Static imperfections:

$$f(t) \approx \exp\left(-n_q(\epsilon n_g t)^2\right)$$
$$t_f = D/(\epsilon n_g \sqrt{n_q})$$
$$N_g = D/(\epsilon \sqrt{n_q})$$

?



Panels: a) fidelity scaling; b) t_f vs. ϵ for noisy gates (diamonds), static imperfections (triangles $\mu = 0$, circles $\mu = \epsilon$). c) N_g scalings: $\tilde{\epsilon} = \epsilon$ for noisy gates, $\tilde{\epsilon} = \epsilon\sqrt{n_q}$

The classical and quantum sawtooth maps (a3, b1, c2)

Time-dependent Hamiltonians with periodicity on q and p. The discretised dynamics, parametrised by K and L, is a "kick" followed by a free evolution:

$$\bar{q} = q + \bar{p} \bmod 2\pi \qquad \bar{p} = p - K\frac{dV}{dq} \bmod 2\pi L \qquad V(q) = \frac{q^2}{2}(\text{sawtooth})$$

In the quantum case one introduces the number of levels N (with $\hbar = 2\pi L/N$). The Floquet operator (evolution operator corresponding to one iteration of the map) is a product of two terms, which are diagonal respectively in $\hat{p}$ and $\hat{q}$.

$$U_F = e^{-i\hat{p}^2/2\hbar} e^{-iK\,V(\hat{q})/\hbar} = \exp\left[-i\frac{\pi L}{N}\hat{n}_p^2\right] \cdot \exp\left[-i\frac{K}{L}\left(\frac{2\pi}{N}\right)^{\alpha-1}\mathcal{P}_\alpha(\hat{n}_q)\right]$$

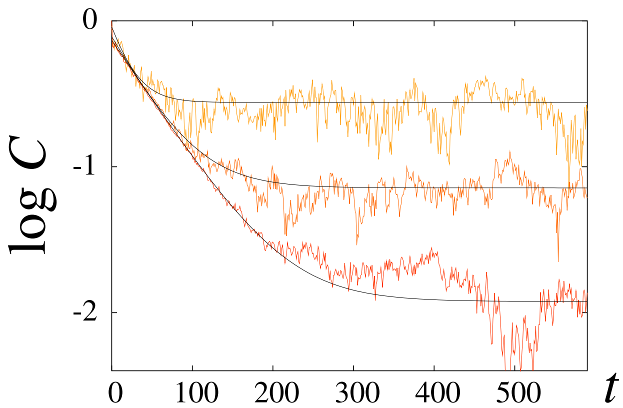
Complexity: $n_g \sim n_q^\alpha \rightarrow n_q^2$. High level primitives are broken down into one or two qubit gates. Algorithm implemented using a quantum language developed by S.Bettelli, numerical experiments performed up to $n_q \sim 20$ qubits.

Characterisation of entanglement

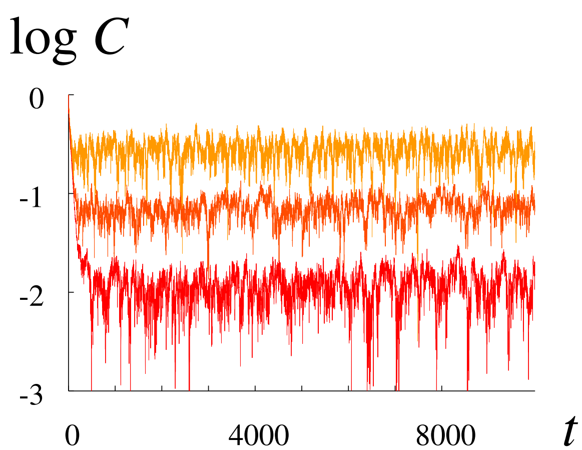
We have chosen to study the evolution of the entanglement of formation of the two most significant qubits in the quantum computer memory while the quantum sawtooth map algorithm is running, using the "concurrence" C. The concurrence depends on the reduced density matrix ρ of the two qubits. If one defines $\widetilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, then C is $\max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ where the λ_i are the square roots of the eigenvalues, in decreasing value order, of $\sqrt{\widetilde{\rho}}\rho\sqrt{\widetilde{\rho}}$ [see Wootters, Phys. Rev. Lett. **80**, 2245 (1998)]. The initial state we use is $|\psi\rangle \propto (|00\rangle + |11\rangle) \otimes |\phi\rangle$, for which $C = 1$, and L is always a multiple of 4.

This quantity does not account for the overall entanglement of the quantum memory, but it has been proven to be linked to interesting physical properties (like quantum phase transitions), and its degradation due to "errors" in the computation should be correlated to the powerfulness of the computation.

Behaviour of the concurrence for an ideal computer

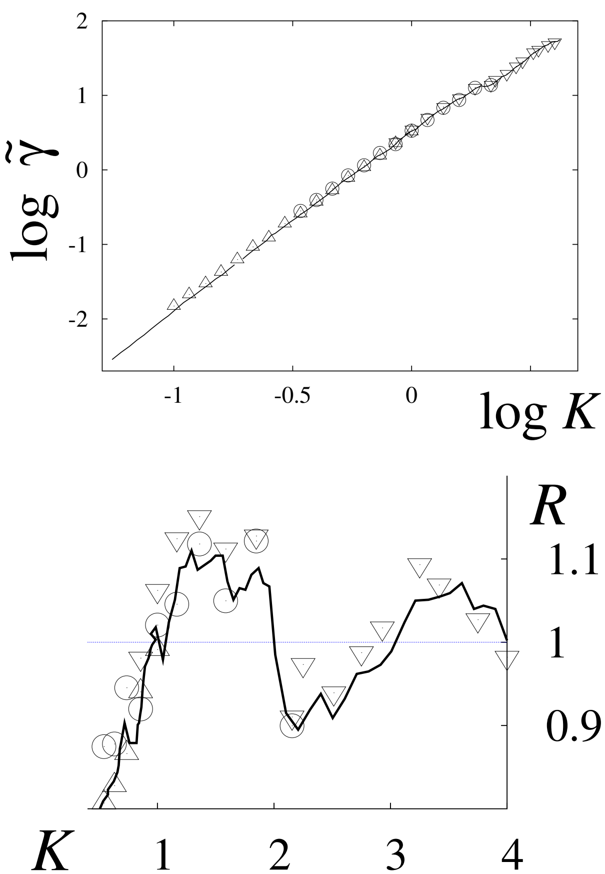


Initial evolution of the concurrence for the sawtooth map at $K = 0.5, L = 4$ and $n_q = 8, 12, 16$ (curves from top to bottom respectively). The smooth curves show the fit $C(t) = A\exp(-\gamma t) + \overline{C}$ of the relaxation to the asymptotic value $\overline{C}$.



Behaviour of $C(t)$ on a larger time scale, showing the asymptotic regime. The initial state is $(|\,00\,\rangle + |\,11\,\rangle)|\,\phi\,\rangle/\sqrt{2}$ where $|\,\phi\,\rangle$ is the uniform superposition of all but the two most significant qubits.

Classical diffusion and concurrence decay

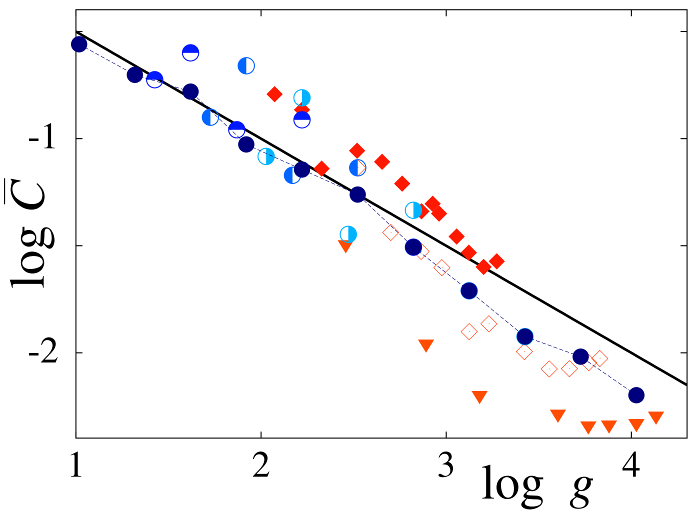


Dependence of the rescaled rate of the concurrence decay, $\tilde{\gamma} = 2\gamma L^2 = (\gamma/\gamma_c)D_0(K)$, on the chaos parameter K for $n_q = 19, L = 16$ (triangles down); $n_q = 18, L = 8$ (circles) and $n_q = 17, L = 4$ (triangles up). The solid curve gives the values of the diffusion rate $D_0(K)$.

This picture shows the data on a larger scale with $R = \tilde{\gamma}/D_{ql}$ (symbols) and $R = D_0(K)/D_{ql}$. It is evident that $\tilde{\gamma}$ follows not only the general trend of D_0 but also its oscillations, showing that γ is almost exactly the classical relaxation rate $\gamma_c = D_0(K)/(2L^2)$.

Numerical results on the residual value of the concurrence

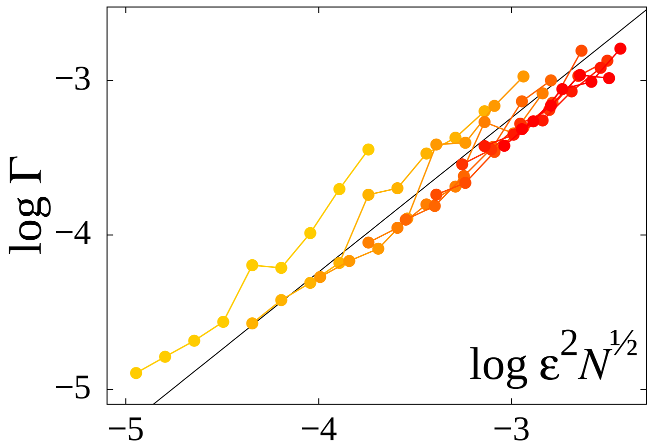
This picture shows the dependence of the residual value of the concurrence $\bar{C}$ versus the conductance $g = ND_0(K)/L^2$: half filled circles show the dependence on $L = 4, 8, 12, 16, 20$ for $K=0.5$ and $n_q = 14, 15, 16$; diamonds and triangles show the variation with K for $n_q = 14, L = 16$; $n_q = 15, L = 8$ and $n_q = 16, L = 4$. The filled circles connected by dashed lines show the dependence on N for $K=0.5, L=4$.



The solid line marks the slope $1/\sqrt{g}$. We attribute the presence of strong fluctuations to the fact that the value $\bar{C}$ is averaged only over the time but there is no averaging over the parameters. Thus, from the point of view of disordered systems $\bar{C}$ represents only one value for one realisation of disorder.

Concurrence decay induced by noisy gates

This picture illustrates the dependence of the decay rate on the error intensity and the number of qubits. The straight line shows the averaged behaviour $\Gamma = 0.58\epsilon^2\sqrt{N}$. Quite naturally we find that $\Gamma \propto \epsilon^2$ [see, for instance, • Phys. Rev. A, **66**, 054301 (2002)]. This scaling becomes better and better for large ϵ values where Γ is larger. However, more surprisingly there is an exponential growth of $\Gamma \propto \sqrt{N}$. This result is very different from those obtained in other papers [see • and Phys. Rev. Lett. **87**, 227901 (2001)], where the time scale for the fidelity and the decoherence rate for tunnelling oscillations varied polynomially with n. A possible explanation is that the eigenstates are exponentially sensitive to imperfections due to the chaotic structure of the wave functions [see e.g. Eur. Phys. J. D **20**, 293 (2002)].



Strong superadditivity of the entanglement of formation follows from its additivity (c3)

Entanglement of formation

Entanglement of two subsystems A and B in a pure state: $E(\psi) = S(\mathrm{Tr}_B(|\,\psi\,\rangle\langle\,\psi\,|))$, where S is the von Neumann entropy: $S(\rho) = -\mathrm{Tr}\rho\log_2\rho$.
For mixed states: the *entanglement of formation (EoF)*:
$E_F(\rho) = \min_{\{p_i,\psi_i\}} \sum_i p_i E(\psi_i)$, with $\rho = \sum_i p_i|\,\psi_i\,\rangle\langle\,\psi_i\,|$ (C.H. Bennett et al. (1996)).

Additivity

Let us consider two separate systems 1 and 2 (each is a bipartite system with the parts $1A$, $1B$ and $2A$, $2B$ respectively, we always consider entanglement between A and B). What is the EoF of the state $\rho_1 \otimes \rho_2$ of the composite system? It has been conjectured that it is the sum of the EoFs of the parts 1 and 2 (the EoF is additive): $E_F(\rho_1 \otimes \rho_2) \stackrel{?}{=} E_F(\rho_1) + E_F(\rho_2)$. This is trivially true for pure states.There are proofs for particular classes of states (G. Vidal et al. (2002)).

Strong superadditivity

Within the same setting, it is natural to compare the EoF of a system with the sum of the EoF's of its subsystems. It has been conjectured (K.G.H. Vollbrecht and R.F. Werner (2001)) that the former is not less than the latter: $E_F(\rho) \overset{?}{\geq} E_F(\mathrm{Tr}_2\rho) + E_F(\mathrm{Tr}_1\rho)$. This property is called strong superadditivity. It is sufficient to proof this conjecture for pure states. The strong superadditivity implies both the additivity of the EoF and that of the Holevo classical channel capacity (K. Matsumoto et al. (2002)).

Equivalence of the conjectures

We show that, conversely, the additivity of the EoF implies the strong superadditivity, that is the two conjectures are equivalent (see PRA **68**, 032317 for the details). We used the methods of convex analysis, introduced in this context by K.M.R. Audenaert and S.L. Braunstein (quant-ph/0303045), most notably the notion of the conjugated function. We use also some properties of optimal decompositions of density matrices which are known from the work of F. Benatti and H. Narnhofer (2001). P.W. Shor (quant-ph/0305035) has proved that the addititivity of the EoF, the strong superadditivity of the EoF, the additivity of the Holevo classical channel capacity and the additivity of the minimal output entropy conjectures are all equivalent.

Random matrix theory for fidelity decay (b2)

Fidelity with average initial state: $f(t) = \left| \frac{1}{N} \mathtt{tr} \left(U^{-t} \left(U\, e^{i\delta H_{\text{eff}}} \right)^t \right) \right|^2$

Regime $(1-f) \ll 1$: $f(t) \approx 1 - \frac{t}{t_c} - \frac{2}{t_c} \sum_{\tau=1}^{t-1} (t-\tau)\, C(\tau)$

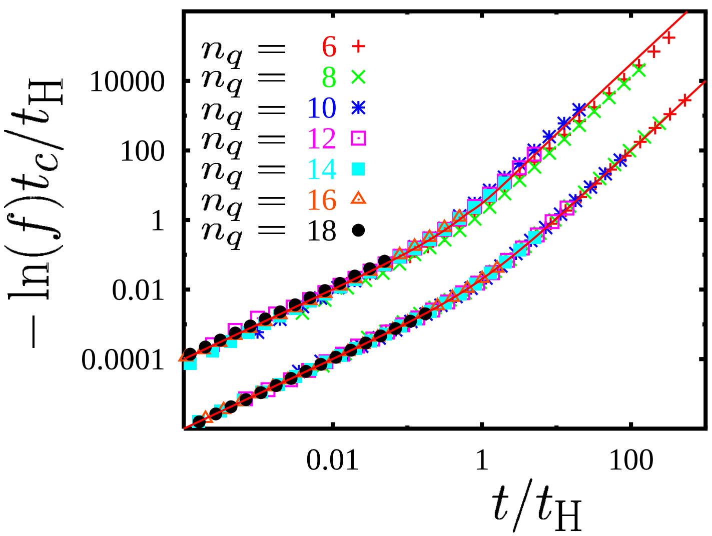
with: $\frac{1}{t_c} = \frac{1}{N} \mathtt{tr} \left(\delta H_{\text{eff}}{}^2 \right)$, $C(\tau) = \frac{t_c}{N} \mathtt{tr} \Big(\underbrace{U^{-\tau}\, \delta H_{\text{eff}}\, U^{\tau}}_{\delta H_{\text{eff}}(\tau)}\, \delta H_{\text{eff}} \Big)$

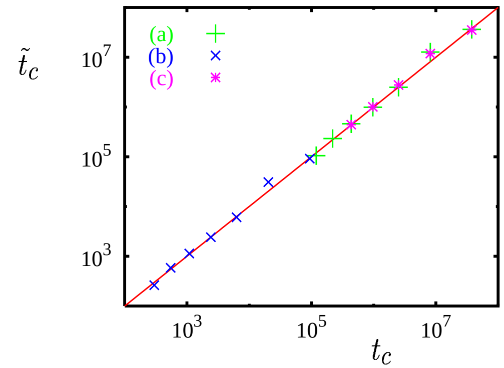
$U \in$ COE (CUE) $\Rightarrow$ Scaling law:

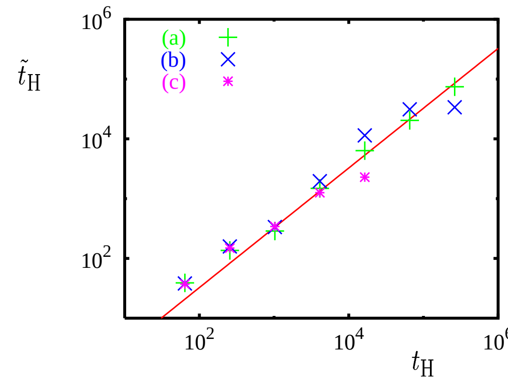
$$-\langle \ln f(t) \rangle_U \approx \frac{N}{t_c} \chi\left(\frac{t}{N}\right) \quad , \quad \chi(s) = s + \frac{2}{\beta} s^2 - 2 \int_0^s d\tilde{\tau}\, (s - \tilde{\tau})\, b_2(\tilde{\tau}) \; .$$

with the “two-level form factor”: $b_2(\tilde{\tau})$.

Scaling analysis for chaotic dynamics (b2-b3)







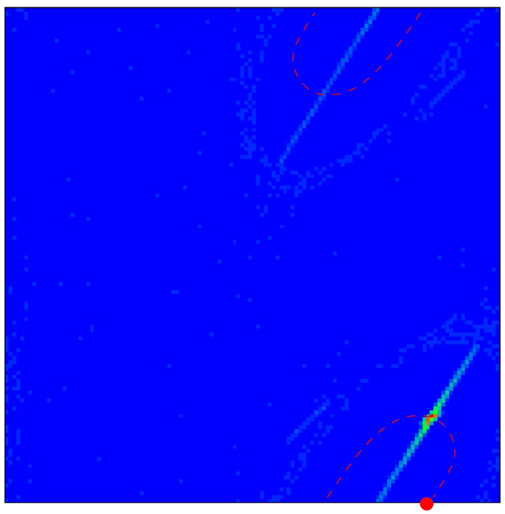
Upper curve: with theoretical values:
$t_{\rm H} = 2^{n_q}$ and $t_c = 1/(\varepsilon^2\, n_q\, n_g^2)$

Lower curve: with fit values $\tilde{t}_c$ and $\tilde{t}_{\rm H}$ from: $-\ln(f(t)) = \dfrac{t}{\tilde{t}_c} + \dfrac{t^2}{\tilde{t}_c \tilde{t}_{\rm H}} \quad (\tilde{t}_{\rm H} \approx t_{\rm H}/3)$

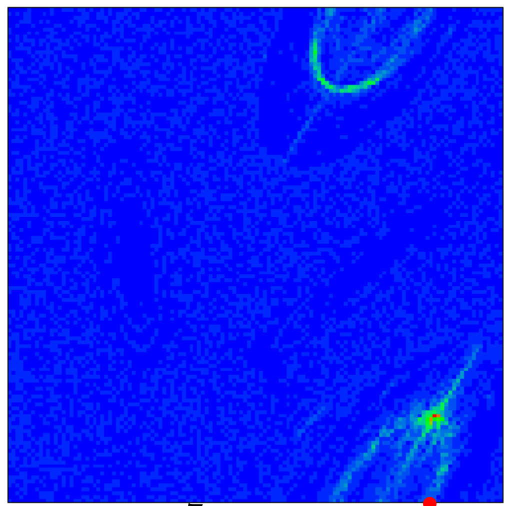
Integrable dynamics (b3)

$t = 22783$, Fit: $-\ln(f(t)) = \frac{t}{\tilde{t}_c} + \frac{t^2}{\tilde{t}_c \tilde{t}_{\mathrm{H}}}$.

$n_q = 14$

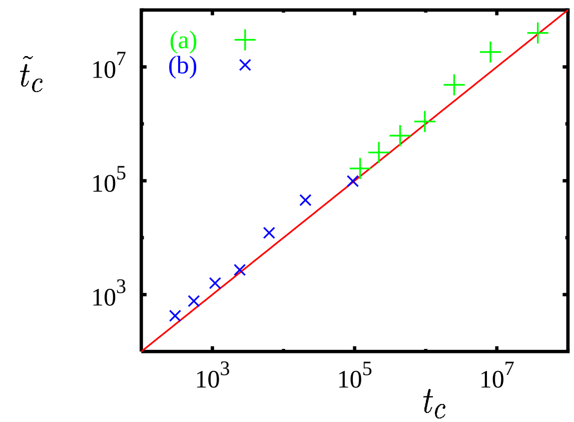


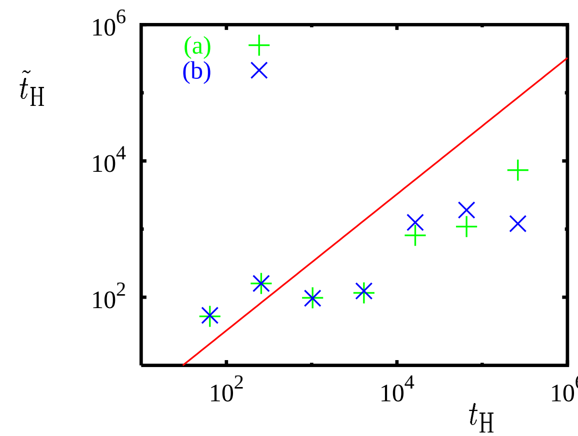
$\varepsilon = 0$



$\varepsilon = 5 \cdot 10^{-7}$, $f = 0.5$

Position of initial gaussian wave packet





Time scale of reliable quantum computations (b1-b3)

Time scale t_f with $f(t_f) = 0.9$:

Theory from RMT-approach:

If $\varepsilon \gg (2^{n_q}\, n_g^2 n_q)^{-1/2}$:

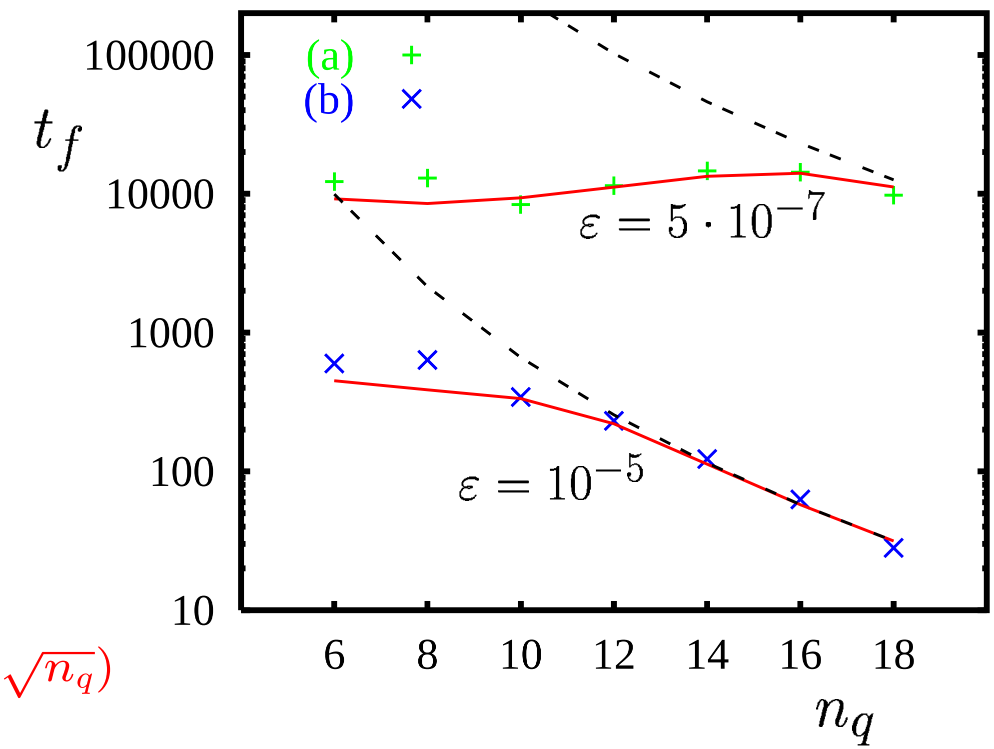
$$t_f \approx 0.1\, t_c \approx 1/(10\varepsilon^2\, n_q\, n_g^2)$$

$$N_g = t_f\, n_g \approx 1/(10\varepsilon^2\, n_q\, n_g)$$

If $\varepsilon \ll (2^{n_q}\, n_g^2 n_q)^{-1/2}$:

$$t_f \approx 0.2\,\sqrt{t_c t_{\rm H}} \approx 2^{n_q/2}/(5\varepsilon\, n_g\,\sqrt{n_q})$$

Random errors: $N_g \approx 5/\varepsilon^2$



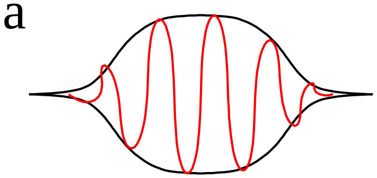
Box-Husimi functions (a3)

Circle state: $|\psi > \sim \sum_{(p,\theta)\in\bigcirc} |\varphi(p,\theta)>$

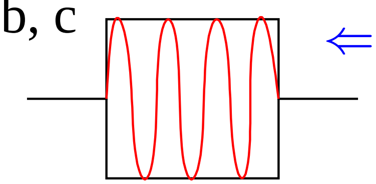
Husimi function: $\rho_H(p,\theta) = |<\varphi(p,\theta)\,|\,\psi>|^2$

Coherent state: $|\varphi(p,\theta)>$

a

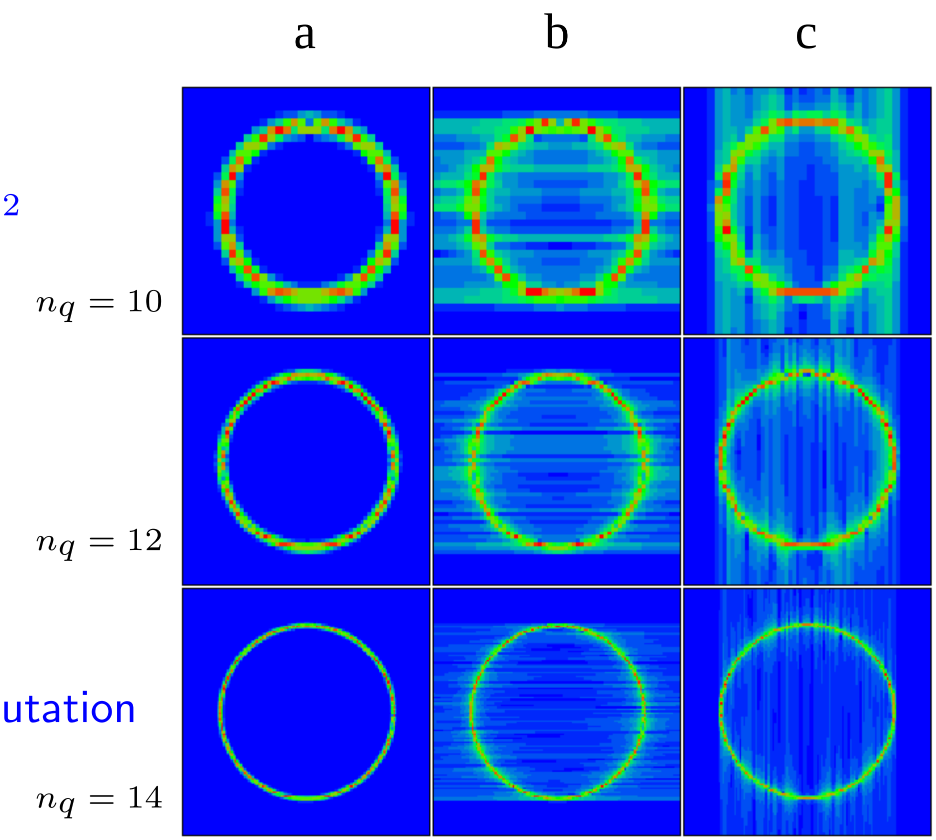


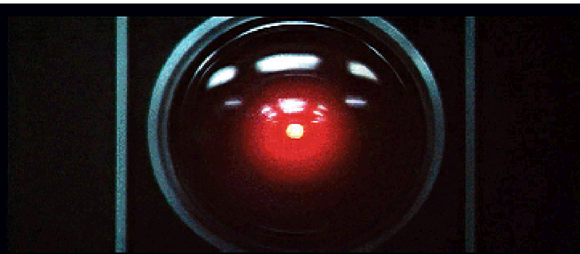
b, c



$\Leftarrow$ Available by quantum computation

with partial QFT





The HAL Quantum Speech

- original HAL speech

measurements in frequency domain
- 10 measurements per frame

measurements in frequency domain with
quantum errors (amplitude of gate errors 0.05)
- 10 measurements per frame

measurements in time domain
- 10 measurements per frame

quantum sound of coarse grained Wigner function
- 10 measurements per frame

- download all sound files (2 Mb)

**"Good afternoon, gentlemen. I am a HAL 9000 computer.
I became operational at the H. A. L. lab in Urbana, Illinois
on the 12th of January"**

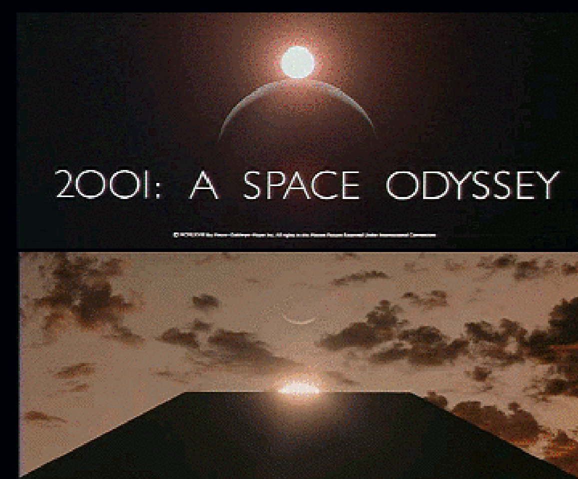
This sound is enconded on an 18 qubit quantum computer and it is recovered with a finite number of quantum measurments performed in time or frequency domain (MP3 like) in presence of gate imperfections.

We can also listen the sound of a complex quantum wavefunction generated by an efficient quantum algorithm (14 qubits).

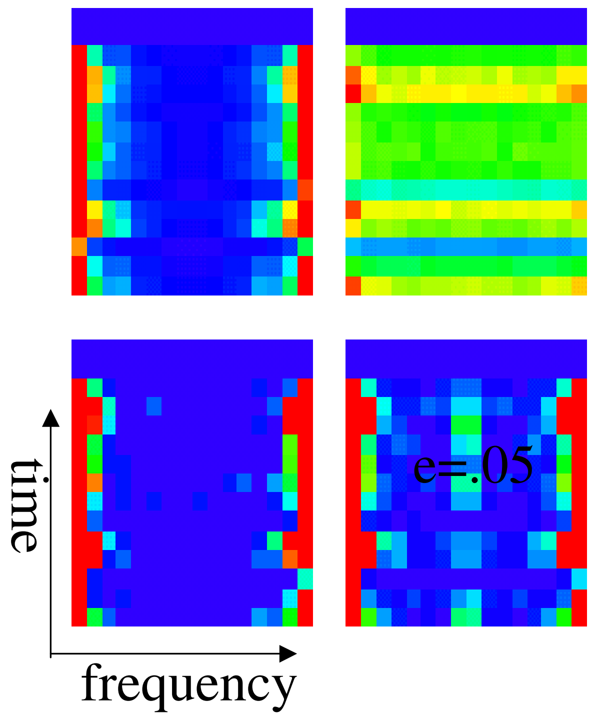
Quantum computations are done
on Quantium I simulated numerically on Pentium IV

The paper is at http://xxx...

The original HAL speech and *2001 A Space odyssey* pictures are taken from http://www.palantir.net/2001/



Quantum MP3-like treatment of sound on quantum computers (d1)



Original Spectrum (upper left)
Time domain measurement(upper right)
Frequency domain measurement(lower left)
Same with noise=0.05(lower right)

- **We can store 1000 yrs sound data within 50 qubits**
- **HAL speech 26 sec. coded in 18 qubits**
- **Using quantum Fourier transform (QFT), it is possible to reduce the number of measurements.**

Direct(time domain) measurement

$$\psi = \sum_n s_n |n\rangle \longrightarrow \tilde{s}_n = | s_n |$$

After QFT and frequency domain measurement

$$\psi = \sum_{k,j} S_{k,j} |k, j\rangle \longrightarrow$$

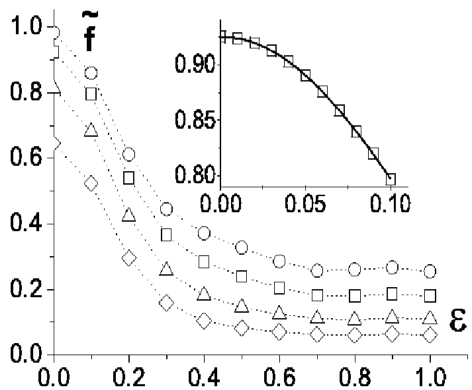
$$\tilde{s}'_n (M, \varepsilon) = \sum_j | S_{k,j} (M, \varepsilon) | \exp(2\pi ijm / \Delta n)$$

- **M=5 measurements per frame with noise level ε**
- **Frequency domain measurement is more effective**

$n_q = 18$, 512 frames(each containing 2^{n_f} =512 data), sampling rate 8kHz

J. Lee, A. Chepelianskii, D. Shepelyansky

Dependence of fidelity $\tilde{f}$ on **M** and noise ε

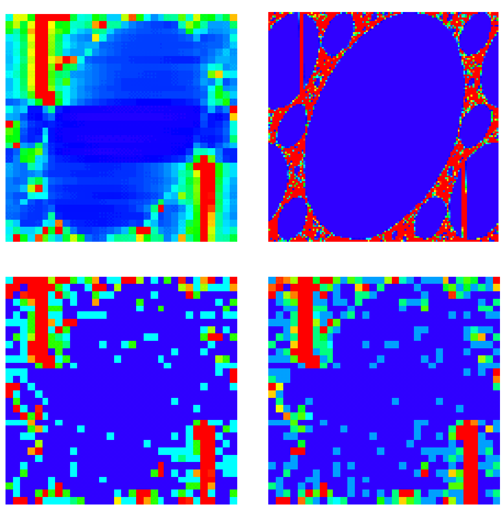


- **Each gate transfers about of ε^2 amount of probability to all other states**

$$1-\tilde{f} \sim \varepsilon^2 n_f^2, \quad 5 \le M \le 1000$$ (from bottom to top)

- **MP3-like strategy allows to recover voice with a reduction factor of up to 50**

Hearing sound of quantum chaos



- **Quantum sawtooth map gives an example of quantum data for sound**

$$U\psi = \exp(-iT\hat{n}^2/2)\exp(ik\hat{\theta}^2/2)\psi$$

with $K = 0.5, n_q = 14, n_f = 9, t = 100$

- **Coarse grained spectrum resembles Husimi distribution**

$$h(l,\theta) = \sum_{l'=l-N/2}^{l+N/2} G(l'-l)\psi(l')\exp(il'\theta)$$

J. Lee, A. Chepelianskii, D. Shepelyansky

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Università Insubria and INFM

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Giuliano Benenti (researcher)

Gabriel Carlo (postdoc, supported by EC IST-FET project EDIQIP)

Carlos Mejía-Monasterio (postdoc, supported by US ARO/NSA/ARDA grant)

Davide Rossini (former diploma student, now PhD student in Pisa)

9 papers in 2003 (including 2 Phys. Rev. Lett. and 1 book)

(a3) Quantum computing of dynamical localization

We have shown that quantum computers can simulate efficiently the quantum localization of classical chaos

Quantum dynamical localization: suppression of diffusion due to quantum interference effects

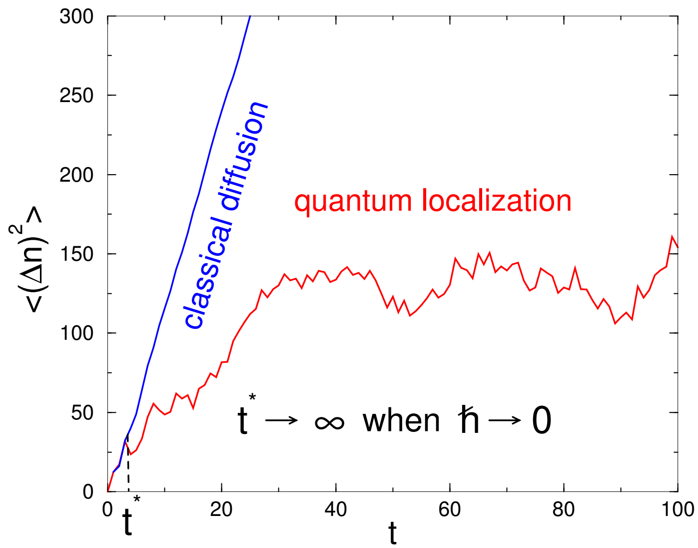
Dynamical localization is one of the most interesting phenomena that characterize the quantum behavior of classically chaotic systems: quantum interference effects suppress classical diffusion leading to exponentially localized wave functions

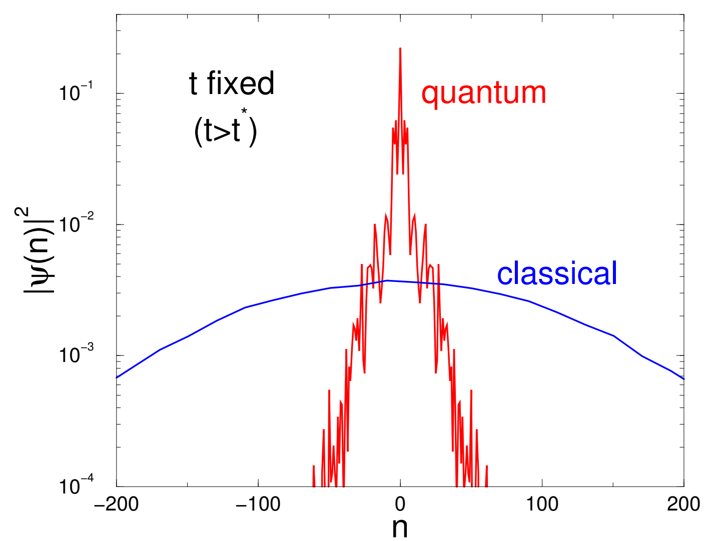
This phenomenon was first observed in the quantum kicked rotor model and has profound analogies with Anderson localization of electronic transport in disordered materials

Experiments: Dynamical localization has been observed experimentally in the microwave ionization of Rydberg atoms and with cold atoms

Dynamical localization observed in periodically driven Hamiltonian systems (a perturbed oscillator, an hydrogen atom under a microwave field, cold ions in an optical lattice, ...)

$$H = H_0(n) + \epsilon V(\theta, t), \quad V(\theta, t + T) = V(\theta, t)$$





- Exponential quantum localization
- No disorder in the Hamiltonian
- Also in conservative systems

Computing dynamical localization in the sawtooth map model

Using the quantum algorithm for the sawtooth map, it is possible to study localization effects on a quantum computer with exponentially large system sizes $N = 2^{n_q}$ (n_q number of qubits)

The sawtooth map is a driven Hamiltonian system. We write, in the action-angle variables (n, θ)

$$H = \frac{n^2}{2} + V(\theta, t) \sum_m \delta(t - mT) = \frac{n^2}{2} - \frac{k}{2}(\theta - \pi)^2 \sum_m \delta(t - mT),$$

Classical sawtooth map:

$$\overline{n} = n + k(\theta - \pi), \quad \overline{\theta} = \theta + T\overline{n}.$$

Quantum sawtooth map:

$$\overline{\psi} = \hat{U}\psi = e^{ik(\hat{\theta}-\pi)^2/2} e^{-iT\hat{n}^2/2}\psi, \quad (\hat{n} = -i\partial/\partial\theta, \hbar = 1)$$

The Floquet operator U (evolution operator corresponding to one iteration of the map) is a product of two terms, which are diagonal in $\hat{\theta}$ and $\hat{n}$, respectively:

$$\hat{U} = \hat{U}_k \hat{U}_T = \exp[ik(\hat{\theta}-\pi)^2/2]\exp(-iT\hat{n}^2/2)$$

On a classical computer, time evolution is simulated via forward/backward Fast Fourier Transforms, in $O(N \log N) = O(2^{n_q} n_q$ operations per kick. We found a quantum algorithm, based on the Quantum Fourier Transform, that requires $O(n_q^2)$ elementary quantum gates ($3n_q^2 - n_q$ controlled-phase shift and $2n_q$ Hadamard gates) [Phys. Rev. Lett. **87**, 227901 (2001)]

Advantages of this quantum algorithm

- It is exponentially faster than classical computation: it requires $n_g = O(n_q^2 = (\log_2 N)^2)$ quantum gates per map iteration instead of $O(N \log_2 N)$ elementary operations

- Optimum use of qubits: no extra work space qubits

- Complex dynamics can be simulated with less than 10 qubits (less than 40 qubits would be sufficient to perform simulations inaccessible to present-day supercomputers)

Extracting information: the localization length

Step 1. Classical diffusion in momentum stops after a time $t^\star \sim \ell$

Step 2. After this time, perform a standard projective measurement in the computational (momentum) basis and store results in histograms bins of width $\delta n \propto \ell$

Step 3. Extract ℓ from the exponential decay of this coarse-grained distribution

Thus the localization length can be obtained with accuracy ν in order of $1/\nu^2$ computer runs

QUADRATIC SPEED UP: If $\ell \sim N$, a classical computer requires $O(N^2(\log N))$ operations to extract the localization length, a quantum computer $O(N(\log N)^2)$ elementary gates
However, there is an exponential advantage in memory requirements

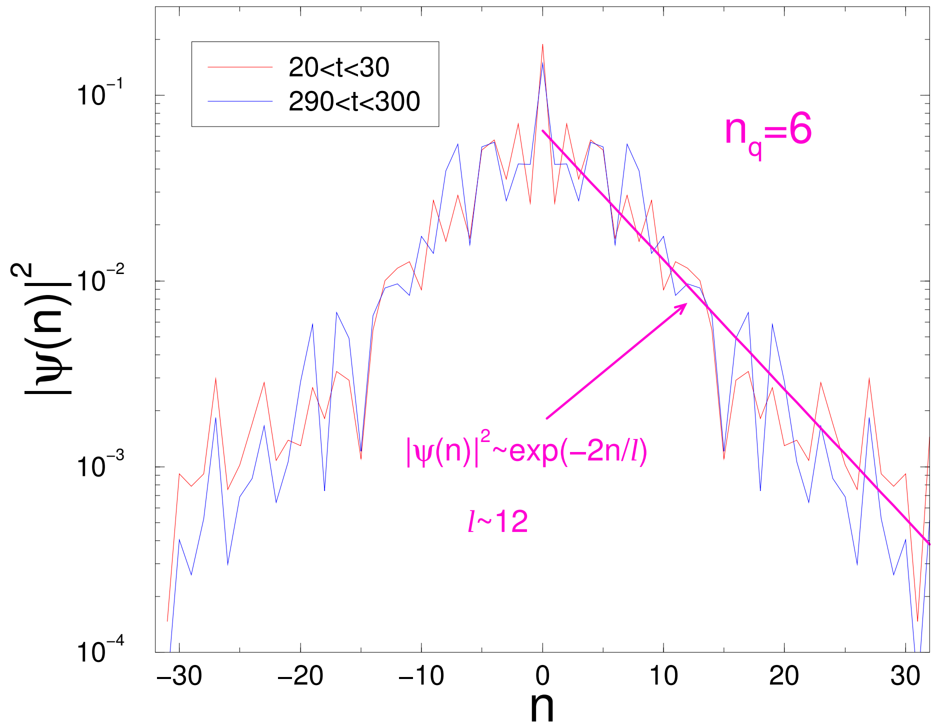
EXPONENTIAL SPEED UP in the measurement of CORRELATION FUNCTIONS of the form

$$C(t) = \langle \psi | T^\dagger A^\dagger T B | \psi \rangle$$

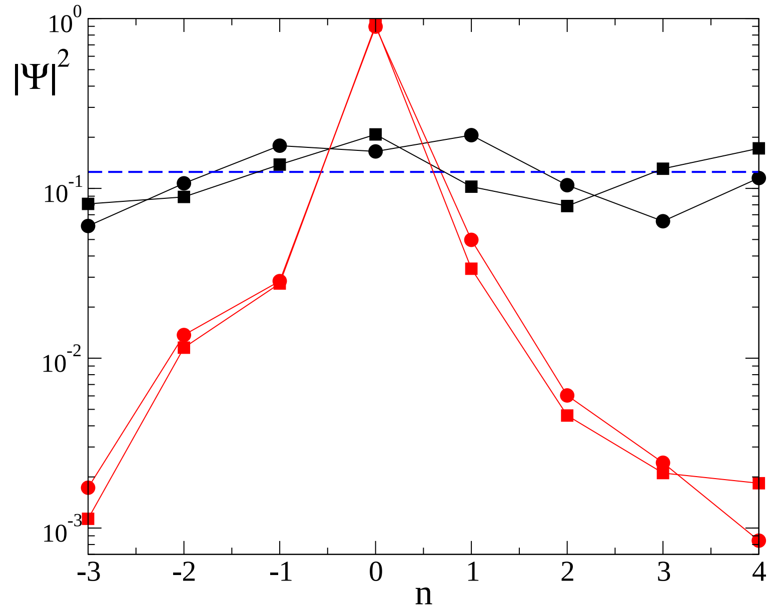
(A and B unitary operators, $T = \exp(-iHt)$ time evolution operator, and $t \ll N$)

Simulating localization on a few-qubit quantum computer

$n_q = 6, k = \sqrt{3}, K = \sqrt{2}$



A case close to experimental possibilities: $n_q = 3$ qubits, $t = 3$ kicks in less than 100 quantum gates



Localized (red) vs. ergodic (black) case, $t = 3$ (circles) and $t = 50$ (squares) (Average over different values of K between 1.4 and 1.5)

Localization is a purely quantum effect, quite fragile in the presence of noise: an ideal testing ground for the coming generation of quantum processors

Effects of static imperfections

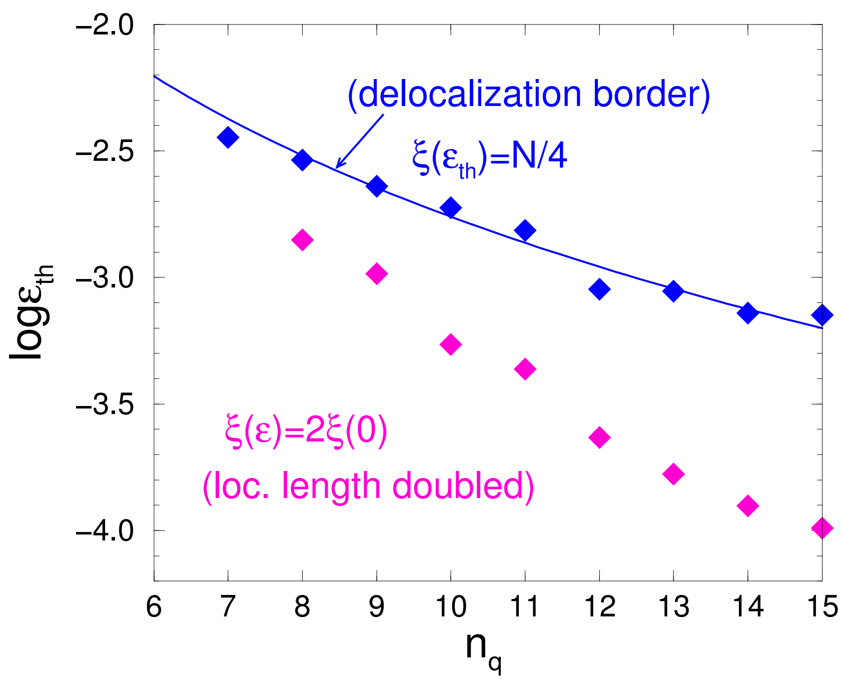
Consider the effect of static imperfections in the quantum computer hardware (unitary errors)

Stability of the inverse participation ratio

$$\xi = (\sum_n |\psi(n)|^4)^{-1}$$

(This quantity gives a good estimate of the localization length: $\xi = 1$ for localization on a single momentum value, $\xi = N$ for complete delocalization)

Analytical estimate: $\epsilon_{\rm th} \propto n_q^{-5/2}$
(algebraic drop)



(c1) Dynamics of entanglement in quantum computers with imperfections

Objective: understand the time scales for the stability of entanglement under decoherence and imperfection effects

Let us first consider this question for a quantum computer hardware model [introduced by Georgeot and Shepelyansky, Phys. Rev. E **62**, 3504 (2000)], in which ystem imperfections generate unwanted interqubit couplings and energy fluctuations. We have n_q qubits on a two dimensional lattice:

$$H = \sum_{i=1}^{n_q} \Delta_i \sigma_i^z + \sum_{i<j=1}^{n_q} J_{ij} \left(\frac{1+\gamma}{2} \sigma_i^x \sigma_j^x + \frac{1-\gamma}{2} \sigma_i^y \sigma_j^y \right),$$

$$\Delta_i = \Delta_0 + \delta_i \; \delta_i \in [-\delta/2, \delta/2], \quad J_{ij} \in [-J, J], \quad 0 \le \gamma \le 1$$

We study the evolution of the entanglement of formation between two nearest neighbor qubits in the lattice, which initially are:

- maximally entangled

$$|\Psi_B(0)\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \otimes |0101\ldots01\rangle$$

- separable

$$|\Psi_S(0)\rangle = |01\rangle \otimes |0101\ldots01\rangle$$

Procedure to compute Wootter's concurrence:

- Compute the reduced density matrix

$$\rho_{12}(t) = \mathrm{Tr}_{3,\ldots,n_q} \left(|\psi(t)\rangle\langle\psi(t)|\right)$$

- Compute concurrence

$$C(t) = \max[\lambda_1(t) - \lambda_2(t) - \lambda_3(t) - \lambda_4(t), 0]$$

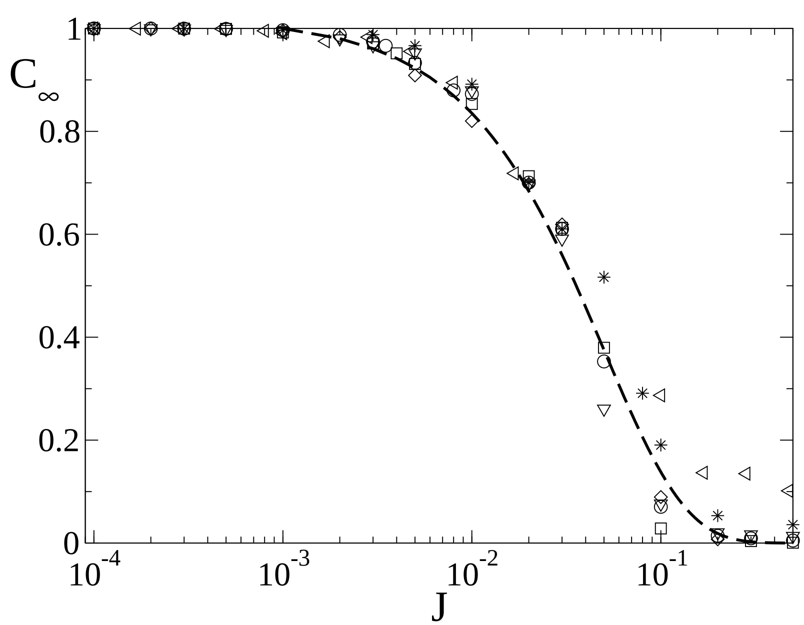
(the λ_i's are the square roots of the eigenvalues of the matrix $R = \rho_{12}\tilde{\rho}_{12}$, in decreasing order)
$\tilde{\rho}_{12} = (\sigma_y \otimes \sigma_y)\, \rho^{\star}_{12}\, (\sigma_y \otimes \sigma_y)$ is the spin flipped matrix of ρ_{12},

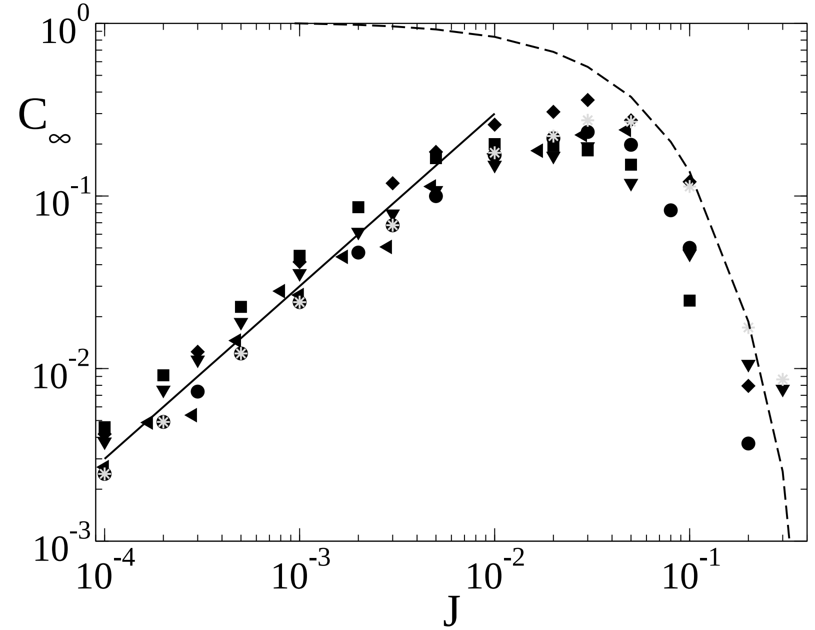
- The pairwise entanglement of formation is given by $E = h((1 + \sqrt{1 - C^2})/2)$, where $h(x) = -x \log_2 x - (1-x)\log_2(1-x)$

Different regimes for the dynamics of pairwise entanglement

We have characterized three different regimes:

- Perturbative regime $(J < \delta/n_q)$ - the entanglement is stable against imperfections
- Crossever regime $(\delta/n_q < J < \delta)$ - imperfections degrade the concurrence of an initially entangled pair but can also drive a significant entanglement generation
- Ergodic regime $(J \sim \delta)$ - a pair of qubits becomes entangled with the rest of the lattice and the concurrence of the pair drops to zero





Concurrence saturation values for different number of qubits, starting from a Bell state (left) or a separable state (right)

(c1-c2) Entanglement echoes in quantum computation

We study the stability of pairwise entanglement for an operating quantum computer

Entanglement echo: Start from an initial state with two maximally entangled qubits:

$$|\psi_0\rangle = |\Phi_B\rangle \otimes |\chi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |00\ldots0\rangle$$

and perform a forward evolution (for instance, of the quantum sawtooth map) up to time $t = t_r$; then compute the time reversal evolution up to the echo time $t_e = 2t_r$

Due to noise and imperfections, the initial state is not exactly recovered

We consider noisy gates introducing unitary errors; we also assume that there are no correlations between consecutive noisy gates

The first two qubits are no longer in a Bell state: their pairwise entanglement echo is reduced; conversely, this pair of qubits becomes entangled with the other qubits, thus generating multipartite entanglement

We compute numerically:

1) The pairwise entanglement of formation $E(t)$ of the two qubits which are initially maximally entangled

2) The reduced Von Neumann entropy

$$S(t) = -\mathrm{Tr}\left[\rho_{12}(t)\log_2 \rho_{12}(t)\right]$$

of the two qubits: this quantity measures the entanglement of the two qubits with the other qubits of the quantum computer

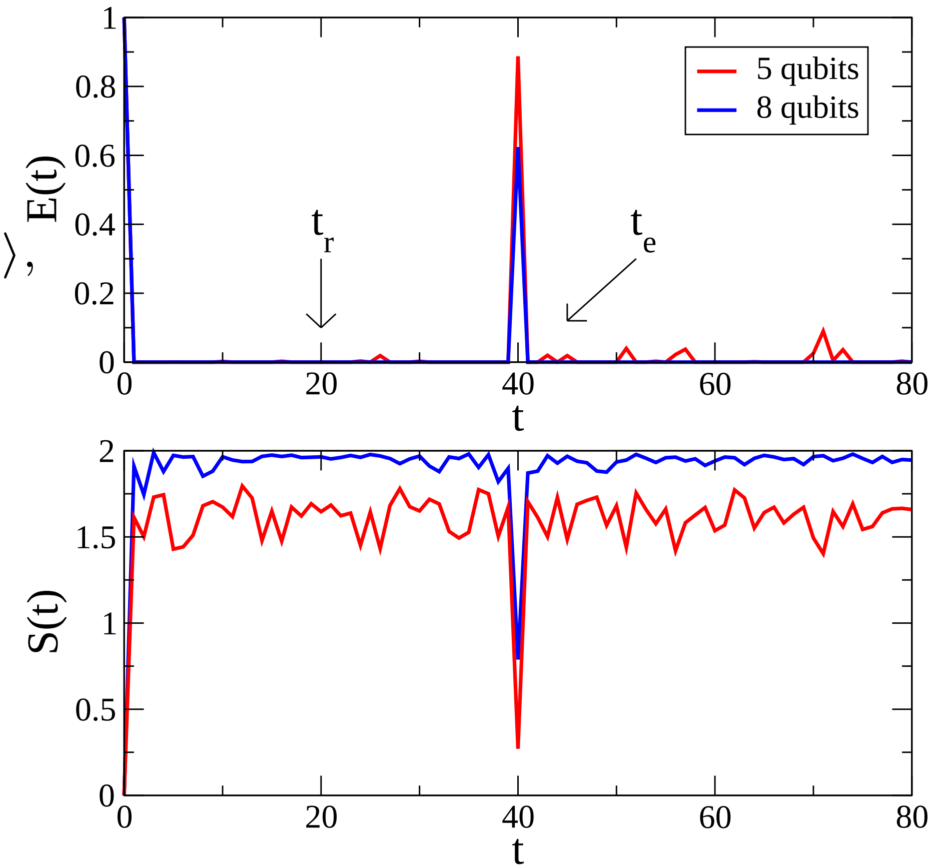
The chaotic dynamics destroys the pairwise entanglement and maximizes the Von Neumann entropy

Indeed, for an ergodic state

$$|\psi(t)\rangle = \sum_{\alpha_1 \ldots \alpha_{n_q}} c_{\alpha_1,\ldots,\alpha_{n_q}}(t) |\alpha_1 \ldots \alpha_{n_q}\rangle,$$

$$(\alpha_i = 0, 1, \; i = 1, \ldots, n_q),$$

the coefficients $c_{\alpha_1,\ldots,\alpha_{n_q}}(t)$ have random phases and amplitudes $|c_{\alpha_1,\ldots,\alpha_{n_q}}| \sim 1/\sqrt{N}$, and it follows that the reduced two-qubit density matrix is essentially diagonal, and leads to $E(t) = 0$ and $S(t) \approx 2 - 8/(N \ln 2)$, where $N = 2^{n_q}$.



Time scales for the stability of pairwise entanglement

Assume that each noisy gate transfers a probability of order ϵ^2 from the ideal state to all other states. Since we assume that there are no correlations between consecutive noisy gates, the population of the initial state decays exponentially, and we can write the echo state as follows:

$$|\psi(t_e)\rangle \approx e^{-C\epsilon^2 n_g t_e/2}|\psi_0\rangle + \sum_{\alpha \neq \alpha_A, \alpha_B} a_\alpha(t_e)|\alpha\rangle,$$

where C is a constant to be determined numerically, $n_g t_e$ is the total number of gates required to perform the echo experiment ($n_g = 3n_q^2 + n_q$ being the number of gates per map iteration), and the sum runs over all the states of the computational basis (except for the two states involved in the initial wave vector), with random signs and equal amplitudes

The above *ansatz* implies that

1) $E(t_e) \approx 1 - (3/2\ln 2)C\epsilon^2 n_g t_e$

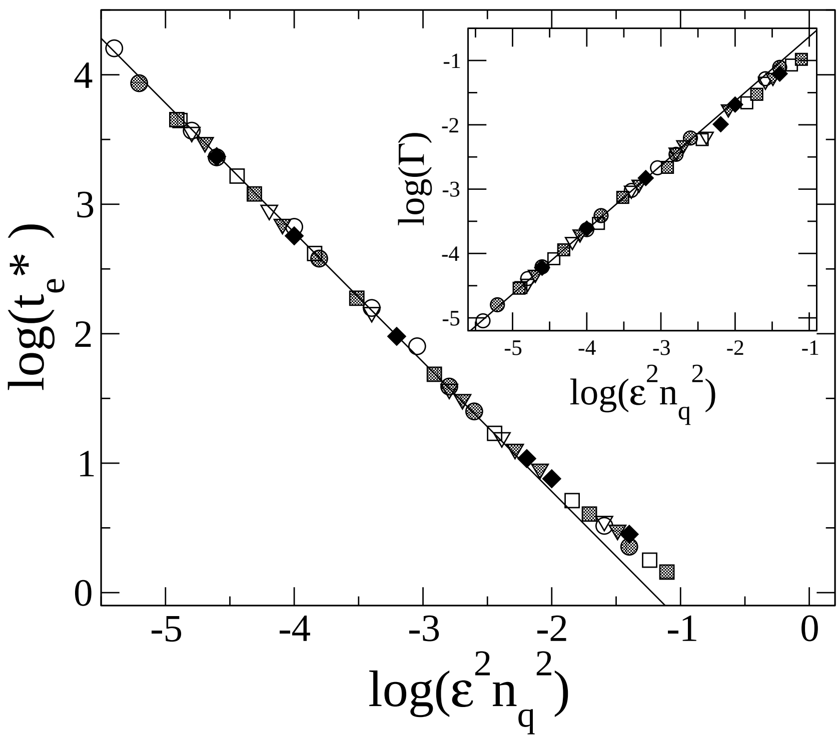
2) $S_\infty - S(t_e) \propto \exp(-2C\epsilon^2 n_g t)$

3) $f(t_e) \approx \exp(-C\epsilon^2 n_g t_e)$

Therefore the fidelity decay, the entanglement echo decay and the approach to equilibrium for the reduced Von Neumann entropy take place in the same time scale $\propto 1/(\epsilon^2 n_g)$

Thus noisy gates degrade the entanglement echo after a number $n_e^\star$ of elementary gates which is independent of the number of qubits

Numerical confirmation



(d1) Simulating noisy quantum protocols with quantum trajectories

The implementation of quantum information protocols has to face the problem of the coupling of quantum processors with the environment: Knowing the sources and strength of quantum noise and controlling it is a fundamental goal to be achieved in quantum computation

Quantum trajectories allow storing only a stochastically evolving state vector, instead of a density matrix

This has an enormous advantage in memory requirements: if the Hilbert space has size N, we store only a state vector of size N instead of a density matrix of size $N \times N$

By averaging over many runs we get the same probabilities (within statistical errors) as the ones obtained by solving the density matrix directly

The theory of quantum trajectories is of widespread use in quantum optics and quantum foundations, but with the exception of a few examples [see Barenco, Brun, Schack, Spiller, Phys. Rev. A **56**, 1177 (1997)] it has not been explored as a tool in quantum computation and information

Quantum trajectories in the Markov approximation

If a system interacts with the environment, its state is described by a density operator ρ. Under the Markov assumption, the dynamics of the system is described by a (Lindblad) master equation:

$$\dot{\rho} = -\frac{i}{\hbar}[H_s, \rho] - \frac{1}{2}\sum_k \{L_k^\dagger L_k, \rho\} + \sum_k L_k \rho L_k^\dagger,$$

H_s is the system's Hamiltonian, $\{\ ,\ \}$ denotes the anticommutator and L_k are the Lindblad operators, with $k \in [1, \ldots, M]$ (the number M depending on the particular model of interaction with the environment)

- The first two terms of the above equation can be regarded as the evolution performed by an effective non-hermitian Hamiltonian, $H_{\text{eff}} = H_s + iK$, with $K = -\hbar/2 \sum_k L_k^\dagger L_k$:

$$-\frac{i}{\hbar}[H_s, \rho] - \frac{1}{2}\sum_k \{L_k^\dagger L_k, \rho\} = -\frac{i}{\hbar}[H_{\text{eff}}\rho - \rho H_{\text{eff}}^\dagger].$$

- The last term is the one responsible for the so called quantum jumps

If the initial density matrix describes a pure state ($\rho(t_0) = |\phi(t_0)\rangle\langle\phi(t_0)|$), then, after an infinitesimal time dt, it evolves into the statistical mixture

$$\rho(t_0 + dt) = (1 - \sum_k dp_k)|\phi_0\rangle\langle\phi_0| + \sum_k dp_k|\phi_k\rangle\langle\phi_k|,$$

where $dp_k = dt\langle\phi(t_0)|L_k^\dagger L_k|\phi(t_0)\rangle$, and the new states are defined by

$$|\phi_0\rangle = \frac{(I - iH_{\text{eff}}dt/\hbar)|\phi(t_0)\rangle}{\sqrt{1 - \sum_k dp_k}}$$

and

$$|\phi_k\rangle = \frac{L_k|\phi(t_0)\rangle}{||L_k|\phi(t_0)\rangle||}.$$

Therefore with probability dp_k a jump occurs and the system is prepared in the state $|\phi_k\rangle$. With probability $1 - \sum_k dp_k$ there are no jumps and the system evolves according to the effective Hamiltonian H_{eff}. (normalization is included because the evolution is non-hermitian)

Numerical method

• Start the time evolution from a pure state $|\phi(t_0)\rangle$

• At intervals dt much smaller than the time scales relevant for the evolution of the system, choose a random number ϵ from a uniform distribution in the unit interval $[0, 1]$

• 1) If $\epsilon \leq dp$, where $dp = \sum_k dp_k$, the state of the system jumps to one of the states $|\phi_k\rangle$ (to $|\phi_1\rangle$ if $0 \leq \epsilon \leq dp_1$, to $|\phi_2\rangle$ if $dp_1 < \epsilon \leq dp_1 + dp_2$, and so on)
2) if $\epsilon > dp$ the evolution with the non-hermitian Hamiltonian H_{eff} takes place and we end up in the state $|\phi_0\rangle$

• Repeat this process as many times as $n_{\text{steps}} = \Delta t/dt$, where Δt is the total evolution time

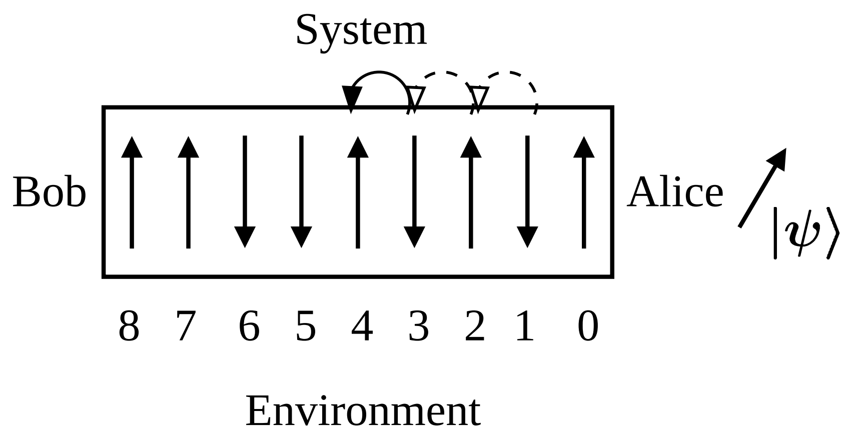
This procedure describes a stochastically evolving wave vector, and we say that a single evolution is a quantum trajectory

• Average over different runs to recover, up to statistical errors, the probabilities obtained using the density operator. Given an operator A, we can write the mean value $\langle A \rangle_t = \mathrm{Tr}[A\rho(t)]$ as the average over $\mathcal{N}$ trajectories:

$$\langle A \rangle_t = \lim_{\mathcal{N}\to\infty} \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \langle \phi_i(t) | A | \phi_i(t) \rangle$$

Teleportation in a noisy environment

We assume that the delivery of one of the qubits of the EPR pair required in the teleportation protocol is done by means of SWAP gates along a noisy chain of qubits



- Assume that the initial state of the chain is given by

$$\sum_{i_{n-1},\ldots,i_2} c_{i_{n-1},\ldots,i_2} |i_{n-1} \ldots i_2\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

where $i_k = 0, 1$ denotes the down or up state of qubit k

- In order to deliver one of the qubits of the EPR pair to Bob, we implement a protocol consisting of $n-2$ SWAP gates, each one exchanging the states of a pair of qubits:

$$\sum_{i_{n-1},\ldots,i_2} \frac{c_{i_{n-1},\ldots,i_2}}{\sqrt{2}} (|i_{n-1}\ldots i_2 00\rangle + |i_{n-1}\ldots i_2 11\rangle)$$

$$\rightarrow \sum_{i_{n-1},\ldots,i_2} \frac{c_{i_{n-1},\ldots,i_2}}{\sqrt{2}} (|i_{n-1}\ldots 0 i_2 0\rangle + |i_{n-1}\ldots 1 i_2 1\rangle) \rightarrow$$

$$\ldots \rightarrow \sum_{i_{n-1},\ldots,i_2} \frac{c_{i_{n-1},\ldots,i_2}}{\sqrt{2}} (|0 i_{n-1}\ldots i_2 0\rangle + |1 i_{n-1}\ldots i_2 1\rangle).$$

- To model the transmission of the qubit through a through a chaotic quantum chain we take random coefficients $c_{i_{n-1},\ldots,i_2}$, that is they have amplitudes of the order of $1/\sqrt{2^{n-2}}$ (to assure wave function normalization) and random phases

Numerical results with up to 24 qubits

We have computed the fidelity $\bar{F} = F - F_\infty$ of teleportation in the presence of a dissipative environment, as a function of the dimensionless damping rate γ and for up to $n_q = 24$ qubits