Degree-degree correlations in directed networks with heavy-tailed degrees

Pim van der Hoorn
Stochastic Operations Research Group, University of Twente

EU FP7 grant 288956, NADINE

June 13, 2013
Introduction
Degree-degree correlations
Pearson correlation coefficients
Rank correlations
Results
Example
Future research
Introduction
Introduction

▶ Newman 2002
Introduction

- Newman 2002
- Nelly Litvak, Remco van de Hofstad 2013
Introduction
Degree-degree correlations
Pearson correlation coefficients
Rank correlations
Results
Example
Future research
Four types of correlations
Four types of correlations
Four types of correlations

Out → In
In → Out
Out → Out
In → In
Four types of correlations

- Out - In
- In - Out
- Out - Out
- In - In
Four types of correlations

- Out - In
- In - Out
- Out - Out
- In - In
Four types of correlations

- Out - In
- In - Out
- Out - Out
- In - In
Four types of correlations

- Out - In
- In - Out
- Out - Out
- In - In
Some notations

\[ G = (V, E) \]
Some notations

\[ G = (V, E) \quad G_n = (V_n, E_n) \]
Some notations

\[ G = (V, E) \quad G_n = (V_n, E_n) \]

\[ D^+ e_* e^* \quad D^- \]

\[ P(\alpha > x) = L_{\alpha}(x - \gamma) \]
Some notations

\[ G = (V, E) \quad \text{and} \quad G_n = (V_n, E_n) \]

\[ \alpha, \beta \in \{+, -\} \]
Some notations

\[ G = (V, E) \quad G_n = (V_n, E_n) \]

\[ D^\alpha(e_*) \quad D^\beta(e^*) \]

\( \alpha, \beta \in \{+,-\} \)

\( P^\alpha(e) > x \Rightarrow L^\alpha(x) - \gamma^\alpha \)
Some notations

\[ G = (V, E) \quad G_n = (V_n, E_n) \]

\[ \alpha, \beta \in \{+, -\} \quad D^\alpha(e_\ast), D^\beta(e_\ast) \quad \mathbb{P}(D^\alpha > x) = L_\alpha(x)x^{-\gamma_\alpha} \]
Sequences of graphs

Definition

Let $\mathcal{G}_{\gamma-\gamma_+}$ denote the space of all sequences of graphs $(G_n)_{n \in \mathbb{N}}$ with the following properties:

G1 $|V_n| = n$

G2 For all $p \geq \gamma_+$ or $q \geq \gamma_-$,

$$\sum_{v \in V_n} D_n^+(v)^p D_n^-(v)^q = \Theta\left(n^{\max\left(p/\gamma_+, q/\gamma_-, 1\right)}\right).$$

G3 There exist two independent regular varying random variables $D^+, D^-$ such that for all $p < \gamma_+$ and $q < \gamma_-$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{v \in V_n} D_n^+(v)^p D_n^-(v)^q = \mathbb{E}\left[(D^+)^p\right] \mathbb{E}\left[(D^-)^q\right].$$
Introduction
Degree-degree correlations
Pearson correlation coefficients
Rank correlations
Results
Example
Future research
General formula edges

\[ \rho_\alpha^\beta(G) = \frac{1}{\sigma_\alpha(G) \sigma_\beta(G)} \frac{1}{|E|} \sum_{e \in E} D^\alpha(e^*) D^\beta(e^*) - \hat{\rho}_\alpha^\beta(G) \]
General formula edges

\[
\rho_{\alpha}^{\beta}(G) = \frac{1}{\sigma_{\alpha}(G)\sigma_{\beta}(G)} \frac{1}{|E|} \sum_{e \in E} D_{\alpha}(e^*) D_{\beta}(e^*) - \hat{\rho}_{\alpha}^{\beta}(G)
\]

\[
\hat{\rho}_{\alpha}^{\beta}(G) = \frac{1}{\sigma_{\alpha}(G)\sigma_{\beta}(G)} \frac{1}{|E|^2} \sum_{e \in E} D_{\alpha}(e^*) \sum_{e \in E} D_{\beta}(e^*)
\]

\[
\sigma_{\alpha}(G) = \sqrt{\frac{1}{|E|} \sum_{e \in E} D_{\alpha}(e^*)^2 - \frac{1}{|E|^2} \left( \sum_{e \in E} D_{\alpha}(e^*) \right)^2}
\]

\[
\sigma_{\beta}(G) = \sqrt{\frac{1}{|E|} \sum_{e \in E} D_{\beta}(e^*)^2 - \frac{1}{|E|^2} \left( \sum_{e \in E} D_{\beta}(e^*) \right)^2}
\]
From edges to vertices

\[ \sum_{e \in E} D^\alpha(e^*) = \sum_{v \in V} D^+(v)D^\alpha(v) \]

\[ \sum_{e \in E} D^\alpha(e^*) = \sum_{v \in V} D^-(v)D^\alpha(v) \]
General formula vertices

\[ \rho_\alpha^\beta(G) = \frac{1}{\sigma_\alpha \sigma_\beta} \frac{1}{|E|} \sum_{e \in E} D_\alpha(e^*) D_\beta(e^*) - \hat{\rho}_\alpha^\beta(G) \]
General formula vertices

\[
\rho^\beta_\alpha(G) = \frac{1}{\sigma_\alpha \sigma_\beta} \frac{1}{|E|} \sum_{e \in E} D^\alpha(e_*) D^\beta(e^*) - \hat{\rho}^\beta_\alpha(G)
\]

\[
\hat{\rho}^\beta_\alpha(G) = \frac{1}{\sigma_\alpha \sigma_\beta} \frac{1}{|E|^2} \sum_{v \in V} D^+(v) D^\alpha(v) \sum_{v \in V} D^-(v) D^\beta(v)
\]
General formula vertices

\[ \rho_\alpha^\beta(G) = \frac{1}{\sigma_\alpha \sigma^\beta} \frac{1}{|E|} \sum_{e \in E} D^\alpha(e^*) D^\beta(e^*) - \hat{\rho}_\alpha^\beta(G) \]

\[ \hat{\rho}_\alpha^\beta(G) = \frac{1}{\sigma_\alpha \sigma^\beta} \frac{1}{|E|^2} \sum_{v \in V} D^+(v) D^\alpha(v) \sum_{v \in V} D^-(v) D^\beta(v) \]

\[ \sigma_\alpha(G) = \sqrt{\frac{1}{|E|} \sum_{v \in V} D^+(v) D^\alpha(v)^2 - \frac{1}{|E|^2} \left( \sum_{v \in V} D^+ D^\alpha(v) \right)^2} \]

\[ \sigma^\beta(G) = \sqrt{\frac{1}{|E|} \sum_{v \in V} D^-(v) D^\beta(v)^2 - \frac{1}{|E|^2} \left( \sum_{v \in V} D^- D^\beta(v) \right)^2} \]
General formula vertices

\[
\rho_\alpha^\beta(G) = \frac{1}{\sigma_\alpha \sigma_\beta} \frac{1}{|E|} \sum_{e \in E} D_\alpha(e^*) D_\beta(e^*) - \hat{\rho}_\alpha^\beta(G)
\]

\[
\hat{\rho}_\alpha^\beta(G) = \frac{1}{\sigma_\alpha \sigma_\beta} \frac{1}{|E|^2} \sum_{v \in V} D^+(v) D_\alpha(v) \sum_{v \in V} D^-(v) D_\beta(v)
\]

\[
\sigma_\alpha(G) = \sqrt{\frac{1}{|E|} \sum_{v \in V} D^+(v) D_\alpha(v)^2 - \frac{1}{|E|^2} \left( \sum_{v \in V} D^+ D_\alpha(v) \right)^2}
\]

\[
\sigma_\beta(G) = \sqrt{\frac{1}{|E|} \sum_{v \in V} D^-(v) D_\beta(v)^2 - \frac{1}{|E|^2} \left( \sum_{v \in V} D^- D_\beta(v) \right)^2}
\]
General formula vertices

\[ \rho^\beta_\alpha(G) = \frac{1}{\sigma^\alpha \sigma^\beta |E|} \sum_{e \in E} D^\alpha(e_*) D^\beta(e^*) - \hat{\rho}^\beta_\alpha(G) \]

\[ \hat{\rho}^\beta_\alpha(G) = \frac{1}{\sigma^\alpha \sigma^\beta |E|^2} \sum_{v \in V} D^+(v) D^\alpha(v) \sum_{v \in V} D^-(v) D^\beta(v) \]

\[ \sigma^\alpha(G) = \sqrt{\frac{1}{|E|} \sum_{v \in V} D^+(v) D^\alpha(v)^2 - \frac{1}{|E|^2} \left( \sum_{v \in V} D^+ D^\alpha(v) \right)^2} \]

\[ \sigma^\beta(G) = \sqrt{\frac{1}{|E|} \sum_{v \in V} D^-(v) D^\beta(v)^2 - \frac{1}{|E|^2} \left( \sum_{v \in V} D^- D^\beta(v) \right)^2} \]
Convergence to a non-negative value

**Theorem**

Let $\alpha, \beta \in \{+, -\}$, then there exists an area $A^\beta_\alpha \subset \mathbb{R}^2$ such that for $(\gamma_+, \gamma_-) \in A^\beta_\alpha$ and $\{G_n\}_{n \in \mathbb{N}} \in \mathcal{G}_{\gamma_-, \gamma_+}$

$$\lim_{n \to \infty} \hat{\rho}^\beta_\alpha(G_n) = 0$$

and hence

$$\lim_{n \to \infty} \rho^\beta_\alpha(G_n) \geq 0.$$
Convergence areas $A^\beta_\alpha$
Convergence areas $A_\alpha^\beta$
Convergence areas $A_{\alpha}^\beta$
Convergence areas $A^\beta_\alpha$
Outline of the proof
Outline of the proof

\[ \hat{\rho}^\beta_{\alpha}(G_n)^2 = \left( \frac{1}{|E_n|} \sum_{v \in V} D_n^+(v) D_n^{\alpha}(v) \right)^2 \left( \frac{1}{|E_n|} \sum_{v \in V} D_n^-(v) D_n^{\beta}(v) \right)^2 \]
\[ \sigma^{\alpha}(G_n)^2 \sigma^{\beta}(G_n)^2 \]
Outline of the proof

\[
\hat{\rho}_\beta(G_n)^2 = \left( \frac{1}{|E_n|} \sum_{v \in V} D^+_n(v) D^\alpha_n(v) \right)^2 \left( \frac{1}{|E_n|} \sum_{v \in V} D^-_n(v) D^\beta_n(v) \right)^2 \frac{\sigma_\alpha(G_n)^2 \sigma_\beta(G_n)^2}{\sigma_\alpha(G_n)^2 \sigma_\beta(G_n)^2}
\]

\[
= \frac{a_n}{a_n + b_n - c_n - d_n}
\]
Outline of the proof

\[ \hat{\rho}_\beta^\alpha(G_n)^2 = \left( \frac{1}{|E_n|} \sum_{v \in V} D_n^+(v) D_n^\alpha(v) \right)^2 \left( \frac{1}{|E_n|} \sum_{v \in V} D_n^-(v) D_n^\beta(v) \right)^2 \frac{\sigma_\alpha(G_n)^2 \sigma_\beta(G_n)^2}{\sigma_\alpha(G_n)^2 \sigma_\beta(G_n)^2} \]

\[ = \frac{a_n}{a_n + b_n - c_n - d_n} \]

\[ \frac{a_n}{b_n} = \Theta \left( \frac{n^a}{n^b} \right) \quad \frac{c_n + d_n}{b_n} = \Theta \left( \frac{n^c + n^d}{n^d} \right) \]

\[ \frac{a_n}{c_n + d_n} = \Theta \left( \frac{n^a}{n^c + n^d} \right) \quad \frac{b_n}{c_n + d_n} = \Theta \left( \frac{n^b}{n^c + n^d} \right) \]
Outline of the proof continued...
Outline of the proof continued...

\[(a < b \land \max(c, d) < b) \lor (a < \max(c, d) \land b < \max(c, d))\]
Outline of the proof continued...

\[(a < b \land \max(c, d) < b) \lor (a < \max(c, d) \land b < \max(c, d))\]

\[
\lim_{{n \to \infty}} \frac{a_n}{b_n} = 0 \text{ and } \lim_{{n \to \infty}} \frac{c_n + d_n}{b_n} = 0
\]

or

\[
\lim_{{n \to \infty}} \frac{a_n}{c_n + d_n} = 0 \text{ and } \lim_{{n \to \infty}} \frac{b_n}{c_n + d_n} = 0
\]
Outline of the proof continued...

\[(a < b \land \max(c, d) < b) \lor (a < \max(c, d) \land b < \max(c, d))\]

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{c_n + d_n}{b_n} = 0
\]

or

\[
\lim_{n \to \infty} \frac{a_n}{c_n + d_n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{b_n}{c_n + d_n} = 0
\]

\[\Rightarrow \lim_{n \to \infty} \frac{a_n}{a_n + b_n - c_n - d_n} = 0\]
Outline of the proof continued...

\((a < b \land \max(c, d) \leq b) \lor (a < \max(c, d) \land b \leq \max(c, d))\)

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{c_n + d_n}{b_n} = 0
\]

or

\[
\lim_{n \to \infty} \frac{a_n}{c_n + d_n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{b_n}{c_n + d_n} = 0
\]

\[
\Rightarrow \lim_{n \to \infty} \frac{a_n}{a_n + b_n - c_n - d_n} = 0
\]
Outline of the proof continued...

\((a < b \land \max(c, d) \leq b) \lor (a < \max(c, d) \land b \leq \max(c, d))\)

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = 0 \text{ and } \lim_{n \to \infty} \frac{c_n + d_n}{b_n} = 0
\]

or

\[
\lim_{n \to \infty} \frac{a_n}{c_n + d_n} = 0 \text{ and } \lim_{n \to \infty} \frac{b_n}{c_n + d_n} = 0
\]

\[\Rightarrow \lim_{n \to \infty} \frac{a_n}{a_n + b_n - c_n - d_n} = 0\]
Issues
Issues

- Graph model with heavy tails have non-negative degree-degree correlation limit
Issues

- Graph model with heavy tails have non-negative degree-degree correlation limit
- Degree-degree correlations cannot be compared for different sizes
Introduction
Degree-degree correlations
Pearson correlation coefficients
Rank correlations
Results
Example
Future research
Spearman’s Rho

\[
\rho = \frac{\sum_{i=1}^{n} (r_{X_i} - \frac{n+1}{2})(r_{Y_i} - \frac{n+1}{2})}{\sum_{i=1}^{n} (r_{X_i} - \frac{n+1}{2})^2 - \frac{1}{n^2}(\sum_{i=1}^{n} r_{X_i})^2 - \frac{1}{n^2}(\sum_{i=1}^{n} r_{Y_i})^2}
\]

Here, \( r_{X_i} \) and \( r_{Y_i} \) are the ranks of \( X_i \) and \( Y_i \) respectively, and \( n \) is the sample size.
Spearman’s Rho

\{X_i\}_{1 \leq i \leq n}, \{Y_i\}_{1 \leq i \leq n}, \text{i.i.d. samples of } X, Y

r_i^X, r_i^Y \text{ ranks of } X_i, Y_i.
Spearman’s Rho

\[ \{X_i\}_{1 \leq i \leq n}, \{Y_i\}_{1 \leq i \leq n}, \text{i.i.d. samples of } X, Y \]

\( r^X_i, r^Y_i \) ranks of \( X_i, Y_i \).

\[ \rho_s^n \left( r^X_i, r^Y_i \right) := \frac{\sum_{i=1}^{n} \left( r_i^X - \frac{n+1}{2} \right) \left( r_i^Y - \frac{n+1}{2} \right)}{\sum_{i=1}^{n} \left( r_i^X - \frac{n+1}{2} \right)^2 \sum_{i=1}^{n} \left( r_i^Y - \frac{n+1}{2} \right)} \]
Spearman’s Rho

\(\{X_i\}_{1 \leq i \leq n}, \{Y_i\}_{1 \leq i \leq n}, \) i.i.d. samples of \(X, Y\)

\(r_i^X, r_i^Y\) ranks of \(X_i, Y_i\).

\[\rho[n]^S (r_i^X, r_i^Y) := \frac{\sum_{i=1}^{n} (r_i^X - \frac{n+1}{2})(r_i^Y - \frac{n+1}{2})}{\sum_{i=1}^{n} (r_i^X - \frac{n+1}{2})^2 \sum_{i=1}^{n} (r_i^Y - \frac{n+1}{2})}\]

\[\rho_S(X, Y) = \frac{\mathbb{E}[F_X(X)F_Y(Y)] - \mathbb{E}[F_X(X)]\mathbb{E}[F_Y(Y)]}{\sqrt{\mathbb{E}[F_X(X)^2] - \mathbb{E}[F_X(X)]^2} \sqrt{\mathbb{E}[F_Y(Y)^2] - \mathbb{E}[F_Y(Y)]^2}}\]
Spearman’s Rho

\[\{X_i\}_{1\leq i \leq n}, \{Y_i\}_{1\leq i \leq n}, \text{i.i.d. samples of } X, Y\]

\(r_i^X, r_i^Y\) ranks of \(X_i, Y_i\).

\[
\rho_S^{[n]} \left( r_i^X, r_i^Y \right) := \frac{\sum_{i=1}^{n} (r_i^X - \frac{n+1}{2})(r_i^Y - \frac{n+1}{2})}{\sum_{i=1}^{n} (r_i^X - \frac{n+1}{2})^2 \sum_{i=1}^{n} (r_i^Y - \frac{n+1}{2})}
\]

\[
\rho_S(X, Y) = \frac{\mathbb{E}[F_X(X)F_Y(Y)] - \mathbb{E}[F_X(X)]\mathbb{E}[F_Y(Y)]}{\sqrt{\mathbb{E}[F_X(X)^2] - \mathbb{E}[F_X(X)]^2} \sqrt{\mathbb{E}[F_Y(Y)^2] - \mathbb{E}[F_Y(Y)]^2}}
\]

\[
= \frac{\mathbb{E}[F_X(X)F_Y(Y)] - \frac{1}{4}}{1/12}
\]
Spearman’s Rho

\[
\{X_i\}_{1 \leq i \leq n}, \{Y_i\}_{1 \leq i \leq n}, \text{ i.i.d. samples of } X, Y
\]

\[ r_i^X, r_i^Y \text{ ranks of } X_i, Y_i. \]

\[
\rho_S^{[n]} \left( r_i^X, r_i^Y \right) := \frac{\sum_{i=1}^{n} (r_i^X - \frac{n+1}{2})(r_i^Y - \frac{n+1}{2})}{\sum_{i=1}^{n} (r_i^X - \frac{n+1}{2})^2 \sum_{i=1}^{n} (r_i^Y - \frac{n+1}{2})}
\]

\[
\rho_S(X, Y) = \frac{\mathbb{E} \left[ F_X(X) F_Y(Y) \right] - \mathbb{E} \left[ F_X(X) \right] \mathbb{E} \left[ F_Y(Y) \right]}{\sqrt{\mathbb{E} \left[ F_X(X)^2 \right] - \mathbb{E} \left[ F_X(X) \right]^2} \sqrt{\mathbb{E} \left[ F_Y(Y)^2 \right] - \mathbb{E} \left[ F_Y(Y) \right]^2}}
\]

\[
= \frac{\mathbb{E} \left[ F_X(X) F_Y(Y) \right] - \frac{1}{4}}{1/12}
\]

\[ F_X(X) := F_X \circ X \text{ is a uniform random variable on } (0,1). \]
Resloving ties, uniform at random

\[ \tilde{X} = X + U \]
\[ \tilde{Y} = Y + V \]

\[ \tilde{X}_i = X_i + U_i \]
\[ \tilde{Y}_i = Y_i + V_i \]

\[ \rho[n] S(\tilde{r}X_i, \tilde{r}Y_i) \]
Resloving ties, uniform at random

Turn discrete random variables $X, Y$ into continuous random variables $\tilde{X}, \tilde{Y}$
Resolving ties, uniform at random

Turn discrete random variables \( X, Y \) into continuous random variables \( \tilde{X}, \tilde{Y} \)

Two uniform random variables \( U, V \) on \((0,1)\)

\[
\begin{align*}
\tilde{X} &:= X + U & \tilde{Y} &:= Y + V \\
\tilde{X}_i &:= X_i + U_i & \tilde{Y}_i &:= Y_i + V_i
\end{align*}
\]
Resloving ties, uniform at random

Turn discrete random variables $X$, $Y$ into continuous random variables $\tilde{X}$, $\tilde{Y}$

Two uniform random variables $U$, $V$ on $(0,1)$

$$\tilde{X} := X + U \quad \tilde{Y} := Y + V$$

$$\tilde{X}_i := X_i + U_i \quad \tilde{Y}_i := Y_i + V_i$$

$\tilde{r}_i^X$, $\tilde{r}_i^Y$ ranking.
Resolving ties, uniform at random

Turn discrete random variables $X$, $Y$ into continuous random variables $\tilde{X}$, $\tilde{Y}$

Two uniform random variables $U$, $V$ on $(0,1)$

$$\tilde{X} := X + U \quad \tilde{Y} := Y + V$$

$$\tilde{X}_i := X_i + U_i \quad \tilde{Y}_i := Y_i + V_i$$

$\tilde{r}_i^X$, $\tilde{r}_i^Y$ ranking.

$$\rho_S^{[n]} (\tilde{r}_i^X, \tilde{r}_i^Y)$$
Resolving ties, take average
Resolving ties, take average

\[ r_i^X = \frac{1}{|\{ k | X_k = X_i \}|} \sum_{j: X_j = X_i} |\{ k | X_k > X_i \}| + j \]

\((1, 2, 1, 3, 3) \rightarrow (1.5, 3, 1.5, 4.5, 4.5)\)
Resolving ties, take average

\[ r_i^X = \frac{1}{|\{k | X_k = X_i\}|} \sum_{j:X_j=X_i} |\{k | X_k > X_i\}| + j \]

\[(1, 2, 1, 3, 3) \rightarrow (1.5, 3, 1.5, 4.5, 4.5)\]

- Average unchanged
Resolving ties, take average

\[
 r_i^X = \frac{1}{\left|\{k \mid X_k = X_i\}\right|} \sum_{j : X_j = X_i} \left|\{k \mid X_k > X_i\}\right| + j
\]

\[(1, 2, 1, 3, 3) \rightarrow (1.5, 3, 1.5, 4.5, 4.5)\]

- Average unchanged
- No randomness
Resolving ties, take average

\[ r^X_i = \frac{1}{|\{k|X_k = X_i\}|} \sum_{j:X_j=X_i} |\{k|X_k > X_i\}| + j \]

\((1, 2, 1, 3, 3) \rightarrow (1.5, 3, 1.5, 4.5, 4.5)\)

- Average unchanged
- No randomness
- Variance?
Kendall Tau
Kendall Tau

\[ \{(X_i, Y_i)\}_{1 \leq i \leq n} \text{ observations} \]
Kendall Tau

\{(X_i, Y_i)\}_{1 \leq i \leq n} \text{ observations}

\frac{\text{number of concordant pairs} - \text{number of disconcordant pairs}}{\frac{1}{2} n(n - 1)}
Kendall Tau

\[ \{(X_i, Y_i)\}_{1 \leq i \leq n} \text{ observations} \]

\[
\frac{\text{number of concordant pairs} - \text{number of disconcordant pairs}}{\frac{1}{2} n(n - 1)}
\]

\( \{X_i\}_{1 \leq i \leq n}, \{Y_i\}_{1 \leq i \leq n} \) i.i.d. samples of X and Y
Kendall Tau

\[ \{(X_i, Y_i)\}_{1 \leq i \leq n} \text{ observations} \]

\[
\frac{\text{number of concordant pairs} - \text{number of disconcordant pairs}}{\frac{1}{2}n(n-1)}
\]

\{X_i\}_{1 \leq i \leq n}, \{Y_i\}_{1 \leq i \leq n} \text{ i.i.d. samples of } X \text{ and } Y

\[
\tau := \sum_{i=1, j \neq i}^{n} P((X_i - X_j)(Y_i - Y_j) > 0) - P((X_i - X_j)(Y_i - Y_j) < 0)
\]
Introduction
Degree-degree correlations
Pearson correlation coefficients
Rank correlations
Results
Example
Future research
## Results for Wikipedia

<table>
<thead>
<tr>
<th>Graph</th>
<th>Exponent(^1)</th>
<th>Assortativity</th>
<th>Pearson Average</th>
<th>Spearman's Rho Uniform</th>
<th>Kendall Tau</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\gamma_-)</td>
<td>(\gamma_+)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DE wiki</td>
<td>1.7</td>
<td>1.9</td>
<td>+/-</td>
<td>-0.0552</td>
<td>-0.1435</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>+/-</td>
<td>0.0154</td>
<td>0.0484</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>+/-</td>
<td>-0.0323</td>
<td>-0.0640</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>+/-</td>
<td>-0.0123</td>
<td>0.0120</td>
</tr>
<tr>
<td>EN wiki</td>
<td>1.9</td>
<td>2.5</td>
<td>+/-</td>
<td>-0.0557</td>
<td>-0.1999</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>+/-</td>
<td>-0.0007</td>
<td>0.0240</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>+/-</td>
<td>-0.0713</td>
<td>-0.0855</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>+/-</td>
<td>-0.0074</td>
<td>-0.0666</td>
</tr>
<tr>
<td>ES wiki</td>
<td>1.4</td>
<td>2.5</td>
<td>+/-</td>
<td>-0.1031</td>
<td>-0.1429</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>+/-</td>
<td>-0.0033</td>
<td>-0.0417</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>+/-</td>
<td>-0.0272</td>
<td>0.0178</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>+/-</td>
<td>-0.0262</td>
<td>-0.1669</td>
</tr>
<tr>
<td>FR wiki</td>
<td>1.5</td>
<td>2.6</td>
<td>+/-</td>
<td>-0.0536</td>
<td>-0.1065</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>+/-</td>
<td>0.0048</td>
<td>0.0121</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>+/-</td>
<td>-0.0512</td>
<td>-0.0126</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>+/-</td>
<td>-0.0094</td>
<td>-0.0267</td>
</tr>
</tbody>
</table>

Table: Results on the wikipedia graphs obtained from the http://law.di.unimi.it/ database
\(^1\) determined using Hill’s estimator
<table>
<thead>
<tr>
<th>Graph</th>
<th>Exponent(^1)</th>
<th>Assortativity</th>
<th>Pearson</th>
<th>Spearman’s Rho</th>
<th>Kendall Tau</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\gamma_-)</td>
<td>(\gamma_+)</td>
<td>Average</td>
<td>Uniform</td>
<td></td>
</tr>
<tr>
<td>HU wiki</td>
<td>1.3</td>
<td>2.2</td>
<td>++/−</td>
<td>−0.1048</td>
<td>−0.1280</td>
</tr>
<tr>
<td></td>
<td>−/−</td>
<td></td>
<td>0.0120</td>
<td>0.0595</td>
<td>0.0525</td>
</tr>
<tr>
<td></td>
<td>−/+</td>
<td></td>
<td>−0.0579</td>
<td>−0.0207</td>
<td>−0.0207</td>
</tr>
<tr>
<td></td>
<td>++/+</td>
<td></td>
<td>−0.0279</td>
<td>0.0060</td>
<td>0.0051</td>
</tr>
<tr>
<td>IT wiki</td>
<td>1.4</td>
<td>2.5</td>
<td>++/−</td>
<td>−0.0711</td>
<td>−0.0964</td>
</tr>
<tr>
<td></td>
<td>−/−</td>
<td></td>
<td>0.0048</td>
<td>0.0469</td>
<td>0.0468</td>
</tr>
<tr>
<td></td>
<td>−/+</td>
<td></td>
<td>−0.0704</td>
<td>−0.0277</td>
<td>−0.0277</td>
</tr>
<tr>
<td></td>
<td>++/+</td>
<td></td>
<td>−0.0115</td>
<td>−0.0429</td>
<td>−0.0428</td>
</tr>
<tr>
<td>NL wiki</td>
<td>1.3</td>
<td>1.8</td>
<td>++/−</td>
<td>−0.0585</td>
<td>−0.3018</td>
</tr>
<tr>
<td></td>
<td>−/−</td>
<td></td>
<td>0.0100</td>
<td>0.0730</td>
<td>0.0727</td>
</tr>
<tr>
<td></td>
<td>−/+</td>
<td></td>
<td>−0.0628</td>
<td>0.0016</td>
<td>0.0016</td>
</tr>
<tr>
<td></td>
<td>++/+</td>
<td></td>
<td>−0.0233</td>
<td>−0.1505</td>
<td>−0.1498</td>
</tr>
<tr>
<td>KO wiki</td>
<td>−</td>
<td>−</td>
<td>++/−</td>
<td>−0.0805</td>
<td>−0.2733</td>
</tr>
<tr>
<td></td>
<td>−/−</td>
<td></td>
<td>0.0157</td>
<td>0.2323</td>
<td>0.1760</td>
</tr>
<tr>
<td></td>
<td>−/+</td>
<td></td>
<td>−0.1697</td>
<td>0.0191</td>
<td>0.0175</td>
</tr>
<tr>
<td></td>
<td>++/+</td>
<td></td>
<td>−0.0138</td>
<td>−0.0618</td>
<td>−0.0493</td>
</tr>
<tr>
<td>RU wiki</td>
<td>−</td>
<td>−</td>
<td>++/−</td>
<td>−0.0911</td>
<td>−0.1084</td>
</tr>
<tr>
<td></td>
<td>−/−</td>
<td></td>
<td>0.0398</td>
<td>0.2200</td>
<td>0.1977</td>
</tr>
<tr>
<td></td>
<td>−/+</td>
<td></td>
<td>0.0082</td>
<td>0.2480</td>
<td>0.2472</td>
</tr>
<tr>
<td></td>
<td>++/+</td>
<td></td>
<td>−0.0242</td>
<td>0.0255</td>
<td>0.0236</td>
</tr>
</tbody>
</table>

Table: Results on the wikipedia graphs obtained from the [http://law.di.unimi.it/](http://law.di.unimi.it/) database

\(^1\) determined using Hill’s estimator

UNIVERSITY OF TWENTE.
Introduction
Degree-degree correlations
Pearson correlation coefficients
Rank correlations
Results
Example
Future research
In-Out correlation
In-Out correlation

\[ G_n = \rho + \rho - 2n + 3n^2 - n^3 + 1 \rightarrow 1 \]
In-Out correlation

\[ \rho^+(G_n) = \frac{2n^3 - 3n^2}{2n^3 - n^2 + 1} \]
In-Out correlation

\[ \rho_\pm(G_n) = \frac{2n^3 - 3n^2}{2n^3 - n^2 + 1} \rightarrow 1 \]
Convergence to random variable
Convergence to random variable

\[ X_i := W_i + W'_i \]
\[ Y_i := W_i + aW'_i \]

\( W_i, W'_i \) i.i.d samples

\( a > 0 \)

\( a \gg 1 \) \( \Rightarrow \rho_{\text{im}} \rightarrow -1 \)
Convergence to random variable

\[
\begin{align*}
X_1 &:= W_i + W'_i \\
Y_1 &:= W_i + aW'_i \\
X_n, W_i, W'_i &\text{ i.i.d samples } W, W' \\
\text{heavy tailed, same exponent.} \\
a &> 0
\end{align*}
\]
Convergence to random variable

$X_i := W_i + W_i'$

$Y_i := W_i + aW_i'$

$W_i, W_i'$ i.i.d samples $W, W'$

heavy tailed, same exponent.

$a > 0$

$a >> 1 \Rightarrow \rho_+^+ \rightarrow -1$
Convergence to random variable, continued…
Convergence to random variable, continued...

\[ \rho^+ \rightarrow \frac{Z_1 + aZ_2}{\sqrt{Z_1 + \bar{Z}_2} \sqrt{Z_1 + a^2 Z_2}} \]

\[ Z_1, Z_2 \text{ stable random variables} \]
Convergence to random variable, continued...

\[ \rho^+ \rightarrow \frac{Z_1 + aZ_2}{\sqrt{Z_1 + Z_2} \sqrt{Z_1 + a^2 Z_2}} \]

\( Z_1, Z_2 \) stable random variables

\[ T := \frac{Z_2}{Z_1} \]
Convergence to random variable, continued...

\[ \rho^+ \rightarrow \frac{Z_1 + aZ_2}{\sqrt{Z_1 + Z_2} \sqrt{Z_1 + a^2Z_2}} \]

\[ Z_1, Z_2 \text{ stable random variables} \]

\[ T := \frac{Z_2}{Z_1} \quad \rho^+ \rightarrow \frac{1 + aT}{\sqrt{1 + T} \sqrt{1 + a^2T}} \]
Convergence to random variable, continued...

\[ \rho^+ \rightarrow \frac{Z_1 + aZ_2}{\sqrt{Z_1 + Z_2} \sqrt{Z_1 + a^2 Z_2}} \]

\( Z_1, Z_2 \) stable random variables

\( T := \frac{Z_2}{Z_1} \quad \rho^+ \rightarrow \frac{1 + aT}{\sqrt{1 + T} \sqrt{1 + a^2 T}} \)

\[ 0 < \varepsilon < 1 \quad a = \frac{2(1 \pm \sqrt{1 - \varepsilon^2})}{\varepsilon^2} - 1 \]

\( \rho^+ \) has support on \((\varepsilon, 1)\).
Introduction
Degree-degree correlations
Pearson correlation coefficients
Rank correlations
Results
Example
Future research
Possible topics
Possible topics

- Investigate null model
Possible topics

▶ Investigate null model
▶ Include null model in code framework
Possible topics

- Investigate null model
- Include null model in code framework
- Lower bounds on average ties
Possible topics

- Investigate null model
- Include null model in code framework
- Lower bounds on average ties
- Directed models [Preferential Attachment]
Possible topics

- Investigate null model
- Include null model in code framework
- Lower bounds on average ties
- Directed models [Preferential Attachment]
- Angular measure (dependence between large nodes)