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Degree-degree correlations in directed networks with heavy-tailed degrees

Pim van der Hoorn

Stochastic Operations Research Group, University of Twente

EU FP7 grant 288956, NADINE

June 13, 2013

Introduction

Introduction

- Newman 2002

Introduction

- Newman 2002
- Nelly Litvak, Remco van de Hofstad 2013

- Introduction
- **Degree-degree correlations**
- Pearson correlation coefficients
- Rank correlations
- Results
- Example
- Future research

Four types of correlations

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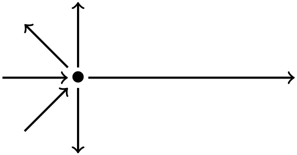
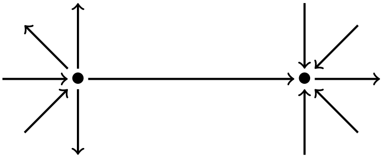


Four types of correlations



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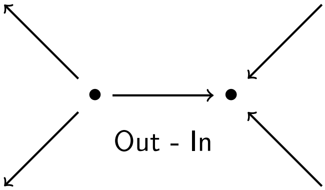

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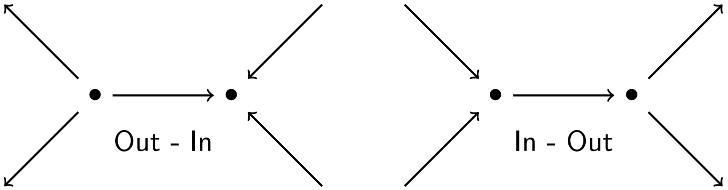

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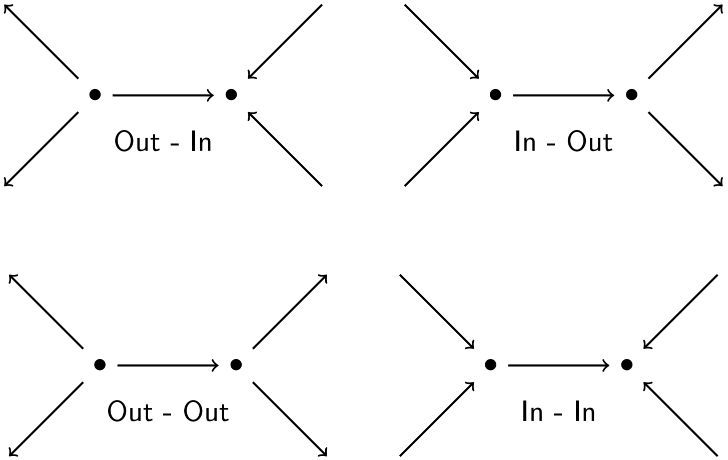

Some notations

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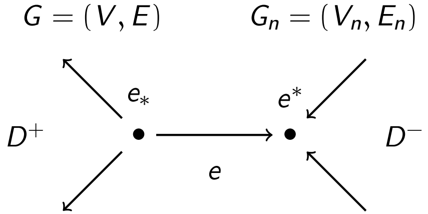

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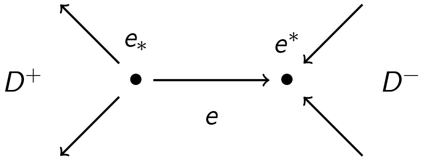

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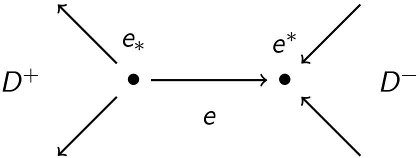

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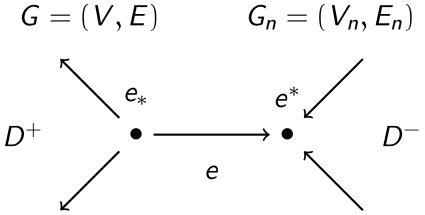

$\alpha, \beta \in \{+, -\}$ $\quad D^{\alpha}(e_*), D^{\beta}(e^*)$ $\quad \mathbb{P}(D^{\alpha} > x) = L_{\alpha}(x) x^{-\gamma_{\alpha}}$

Sequences of graphs

Definition

Let $\mathcal{G}_{\gamma_-\gamma_+}$ denote the space of all sequences of graphs $(G_n)_{n\in\mathbb{N}}$ with the following properties:

G1 $|V_n| = n$

G2 For all $p \geq \gamma_+$ or $q \geq \gamma_-$,

$$\sum_{v\in V_n} D_n^+(v)^p D_n^-(v)^q = \Theta(n^{\max(p/\gamma_+, q/\gamma_-, 1)}).$$

G3 There exist two independent regular varying random variables D^+, D^- such that for all $p < \gamma_+$ and $q < \gamma_-$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{v\in V_n} D_n^+(v)^p D_n^-(v)^q = \mathbb{E}\left[(D^+)^p\right] \mathbb{E}\left[(D^-)^q\right].$$

General formula edges

$$\rho_\alpha^\beta(G) = \frac{1}{\sigma_\alpha(G)\sigma^\beta(G)} \frac{1}{|E|} \sum_{e \in E} D^\alpha(e_*) D^\beta(e^*) - \hat{\rho}_\alpha^\beta(G)$$

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$$\sigma^\beta(G) = \sqrt{\frac{1}{|E|} \sum_{e \in E} D^\beta(e^*)^2 - \frac{1}{|E|^2} \left(\sum_{e \in E} D^\beta(e^*) \right)^2}$$

From edges to vertices

$$\sum_{e \in E} D^{\alpha}(e_*) = \sum_{v \in V} D^{+}(v) D^{\alpha}(v)$$

$$\sum_{e \in E} D^{\alpha}(e^*) = \sum_{v \in V} D^{-}(v) D^{\alpha}(v)$$

General formula vertices

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Convergence to a non-negative value

Theorem

Let $\alpha, \beta \in \{+, -\}$, then there exists an area $A_\alpha^\beta \subset \mathbb{R}^2$ such that for $(\gamma_+, \gamma_-) \in A_\alpha^\beta$ and $\{G_n\}_{n \in \mathbb{N}} \in \mathcal{G}_{\gamma_-, \gamma_+}$

$$\lim_{n \to \infty} \hat{\rho}_\alpha^\beta(G_n) = 0$$

and hence

$$\lim_{n \to \infty} \rho_\alpha^\beta(G_n) \geq 0.$$

Convergence areas A_α^β

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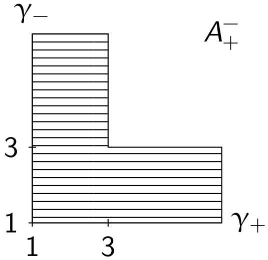

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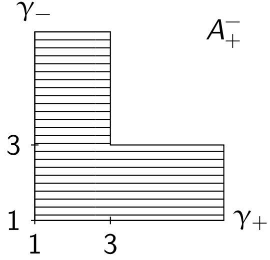
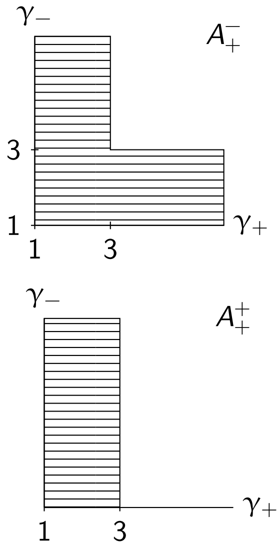


Convergence areas A_{α}^{β}



Outline of the proof

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$$\hat{\rho}_{\alpha}^{\beta}(G_n)^2 = \frac{\left(\frac{1}{|E_n|}\sum_{v \in V} D_n^{+}(v) D_n^{\alpha}(v)\right)^2 \left(\frac{1}{|E_n|}\sum_{v \in V} D_n^{-}(v) D_n^{\beta}(v)\right)^2}{\sigma_{\alpha}(G_n)^2 \sigma^{\beta}(G_n)^2}$$

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$$= \frac{a_n}{a_n + b_n - c_n - d_n}$$

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$$\frac{a_n}{b_n} = \Theta\left(\frac{n^a}{n^b}\right) \qquad \frac{c_n + d_n}{b_n} = \Theta\left(\frac{n^c + n^d}{n^b}\right)$$

$$\frac{a_n}{c_n + d_n} = \Theta\left(\frac{n^a}{n^c + n^d}\right) \qquad \frac{b_n}{c_n + d_n} = \Theta\left(\frac{n^b}{n^c + n^d}\right)$$

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$$(a < b \wedge \max(c, d) < b) \vee (a < \max(c, d) \wedge b < \max(c, d))$$

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$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \text{ and } \lim_{n\to\infty} \frac{c_n + d_n}{b_n} = 0$$

or

$$\lim_{n\to\infty} \frac{a_n}{c_n + d_n} = 0 \text{ and } \lim_{n\to\infty} \frac{b_n}{c_n + d_n} = 0$$

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Issues

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- Degree-degree correlations cannot be compared for different sizes

Spearman's Rho

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$\{X_i\}_{1 \leq i \leq n}, \{Y_i\}_{1 \leq i \leq n}$, i.i.d. samples of X, Y

r_i^X, r_i^Y ranks of X_i, Y_i.

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$$\rho_S^{[n]}\left(r_i^X, r_i^Y\right) := \frac{\sum_{i=1}^n (r_i^X - \frac{n+1}{2})(r_i^Y - \frac{n+1}{2})}{\sum_{i=1}^n (r_i^X - \frac{n+1}{2})^2 \sum_{i=1}^n (r_i^Y - \frac{n+1}{2})}$$

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$$\rho_S(X,Y) = \frac{\mathbb{E}\left[F_X(X)F_Y(Y)\right] - \mathbb{E}\left[F_X(X)\right]\mathbb{E}\left[F_Y(Y)\right]}{\sqrt{\mathbb{E}\left[F_X(X)^2\right] - \mathbb{E}\left[F_X(X)\right]^2}\sqrt{\mathbb{E}\left[F_Y(Y)^2\right] - \mathbb{E}\left[F_Y(Y)\right]^2}}$$

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$$\begin{aligned}\rho_S(X, Y) &= \frac{\mathbb{E}\left[F_X(X)F_Y(Y)\right] - \mathbb{E}\left[F_X(X)\right]\mathbb{E}\left[F_Y(Y)\right]}{\sqrt{\mathbb{E}\left[F_X(X)^2\right] - \mathbb{E}\left[F_X(X)\right]^2}\sqrt{\mathbb{E}\left[F_Y(Y)^2\right] - \mathbb{E}\left[F_Y(Y)\right]^2}} \\ &= \frac{\mathbb{E}\left[F_X(X)F_Y(Y)\right] - \frac{1}{4}}{1/12}\end{aligned}$$

$F_X(X) := F_X \circ X$ is a uniform random variable on (0,1).

Resloving ties, uniform at random

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Two uniform random variables U, V on (0,1)

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$$\rho_S^{[n]}\left(\tilde{r}_i^X, \tilde{r}_i^Y\right)$$

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$$r_i^X = \frac{1}{|\{k|X_k = X_i\}|} \sum_{j:X_j=X_i} |\{k|X_k > X_i\}| + j$$

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- Variance?

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$$\tau := \sum_{i=1, j \neq i}^{n} \mathbb{P}((X_i - X_j)(Y_i - Y_j) > 0) - \mathbb{P}((X_i - X_j)(Y_i - Y_j) < 0)$$

Introduction

Degree-degree correlations

Pearson correlation coefficients

Rank correlations

Results

Example

Future research

Results for Wikipedia

Graph	Exponent[1] γ_-	γ_+	Assortativity	Pearson	Spearman's Rho Average	Uniform	Kendall Tau
DE wiki	1.7	1.9	+/-	-0.0552	-0.1435	-0.1434	-0.0986
			-/+	0.0154	0.0484	0.0481	0.0326
			+/+	-0.0323	-0.0640	-0.0640	-0.0446
			-/-	-0.0123	0.0120	0.0119	0.0074
EN wiki	1.9	2.5	+/-	-0.0557	-0.1999	-0.1999	-0.1364
			-/+	-0.0007	0.0240	0.0239	0.0163
			+/+	-0.0713	-0.0855	-0.0855	-0.0581
			-/-	-0.0074	-0.0666	-0.0664	-0.0457
ES wiki	1.4	2.5	+/-	-0.1031	-0.1429	-0.1429	-0.0972
			-/+	-0.0033	-0.0417	-0.0407	-0.0294
			+/+	-0.0272	0.0178	0.0178	0.0119
			-/-	-0.0262	-0.1669	-0.1627	-0.1174
FR wiki	1.5	2.6	+/-	-0.0536	-0.1065	-0.1065	-0.0720
			-/+	0.0048	0.0121	0.0119	0.0085
			+/+	-0.0512	-0.0126	-0.0126	-0.0087
			-/-	-0.0094	-0.0267	-0.0262	-0.0186

Table: Results on the wikipedia graphs obtained from the `http://law.di.unimi.it/` database
[1] determined using Hill's estimator

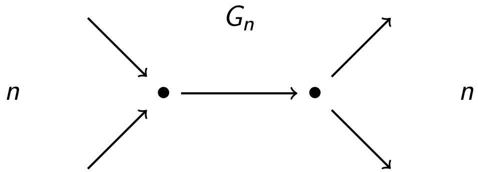
Graph	Exponent[1]		Assortativity	Pearson	Spearman's Rho		Kendall Tau
	γ_-	γ_+			Average	Uniform	
HU wiki	1.3	2.2	+/-	-0.1048	-0.1280	-0.1280	-0.0877
			-/+	0.0120	0.0595	0.0525	0.0442
			+/+	-0.0579	-0.0207	-0.0207	-0.0140
			-/-	-0.0279	0.0060	0.0051	0.0050
IT wiki	1.4	2.5	+/-	-0.0711	-0.0964	-0.0964	-0.0653
			-/+	0.0048	0.0469	0.0468	0.0319
			+/+	-0.0704	-0.0277	-0.0277	-0.0189
			-/-	-0.0115	-0.0429	-0.0428	-0.0296
NL wiki	1.3	1.8	+/-	-0.0585	-0.3018	-0.3017	-0.2089
			-/+	0.0100	0.0730	0.0727	0.0504
			+/+	-0.0628	0.0016	0.0016	0.0015
			-/-	-0.0233	-0.1505	-0.1498	-0.1048
KO wiki	-	-	+/-	-0.0805	-0.2733	-0.2696	-0.1985
			-/+	0.0157	0.2323	0.1760	0.1902
			+/+	-0.1697	0.0191	0.0175	0.0170
			-/-	-0.0138	-0.0618	-0.0493	-0.0463
RU wiki	-	-	+/-	-0.0911	-0.1084	-0.1080	-0.0755
			-/+	0.0398	0.2200	0.1977	0.1655
			+/+	0.0082	0.2480	0.2472	0.1736
			-/-	-0.0242	0.0255	0.0236	0.0187

Table: Results on the wikipedia graphs obtained from the `http://law.di.unimi.it/` database

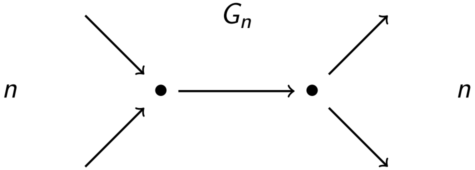
[1] determined using Hill's estimator

In-Out correlation

In-Out correlation

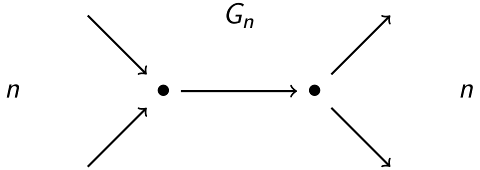


In-Out correlation



$$\rho^{+}_{-}(G_n) = \frac{2n^3 - 3n^2}{2n^3 - n^2 + 1}$$

In-Out correlation



$$\rho_{-}^{+}(G_n) = \frac{2n^3 - 3n^2}{2n^3 - n^2 + 1} \to 1$$

Convergence to random variable

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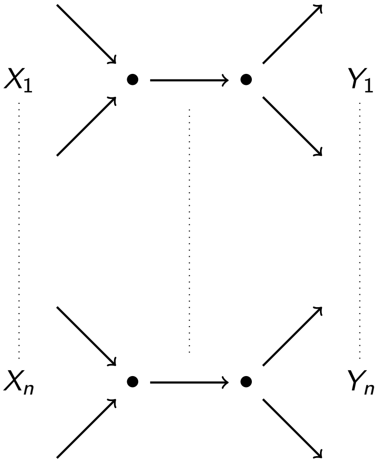
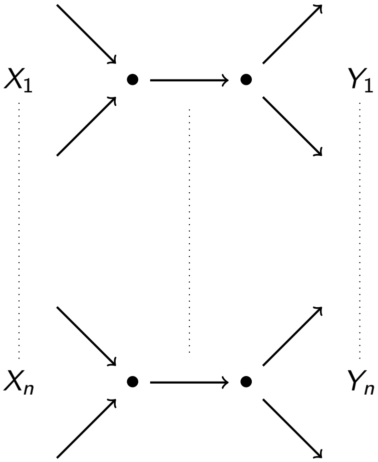


Convergence to random variable

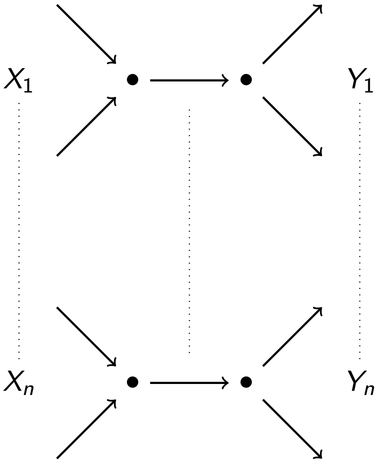


$$X_i := W_i + W_i'$$

$$Y_i := W_i + aW_i'$$

W_i, W_i' i.i.d samples W, W' heavy tailed, same exponent.
$a > 0$

Convergence to random variable



$$X_i := W_i + W_i'$$

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$$a >> 1 \Rightarrow \rho_-^+ \to -1$$

Convergence to random variable, continued...

Convergence to random variable, continued...

$$\rho_-^+ \to \frac{Z_1 + aZ_2}{\sqrt{Z_1 + Z_2}\sqrt{Z_1 + a^2 Z_2}}$$

Z_1, Z_2 stable random variables

Convergence to random variable, continued...

$$\rho_{-}^{+} \to \frac{Z_1 + aZ_2}{\sqrt{Z_1 + Z_2}\sqrt{Z_1 + a^2 Z_2}}$$

Z_1, Z_2 stable random variables

$$T := \frac{Z_2}{Z_1}$$

Convergence to random variable, continued...

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Z_1, Z_2 stable random variables

$$T := \frac{Z_2}{Z_1} \qquad \rho_{-}^{+} \to \frac{1 + aT}{\sqrt{1 + T}\sqrt{1 + a^2 T}}$$

Convergence to random variable, continued...

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Z_1, Z_2 stable random variables

$$T := \frac{Z_2}{Z_1} \qquad \rho_-^+ \to \frac{1 + aT}{\sqrt{1 + T}\sqrt{1 + a^2 T}}$$

$$0 < \varepsilon < 1 \qquad a = \frac{2(1 \pm \sqrt{1 - \varepsilon^2})}{\varepsilon^2} - 1$$

ρ_-^+ has support on $(\varepsilon, 1)$.

Possible topics

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- Investigate null model

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- Angular measure (dependence between large nodes)