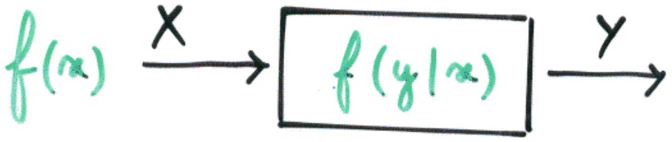
SHANNON INFORMATION THEORY
WITH CONTINUOUS VARIABLES



continuous random variables (e.g. voltages)

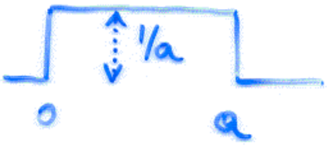
DIFFERENTIAL ENTROPY :

$$h(f) \equiv -\int_S f(x)\,\log_{\substack{2\\e}} f(x)\,dx$$

(bits)
(nats)

with $S = \{x : f(x) > 0\}$

Uniform distribution



$$h(f) = -\int_0^a \frac{1}{a}\log\frac{1}{a}\,dx$$
$$= \log a$$

Gaussian distribution

$\mathcal{N}_{0,\sigma^2}$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right)$$

$$h(f) = -\int_{-\infty}^{+\infty} f(x)\left\{\log_e\frac{1}{\sqrt{2\pi\sigma^2}} - \frac{x^2}{2\sigma^2}\right\}dx$$

$$= \frac{1}{2}\log_e(2\pi\sigma^2) + \frac{1}{2}\log_e(e)$$

$$= \frac{1}{2}\log_{\substack{e\\2}}(2\pi e\sigma^2)$$

(nats)
(bits)

$\sim \log\sigma$

$h(f)$ MEASURES "SPREADING" OF $f(x)$

... AKIN TO DISCRETE-VARIABLE ENTROPY $H(x)$ <u>BUT</u>

1°) DEFINED UP TO ARBITRARY CONSTANT !

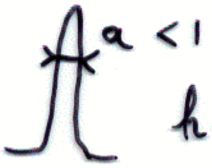
$$Y = aX$$

$$\rightarrow f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y}{a}\right)$$

$$\Rightarrow \underset{\substack{\text{variable}\\ Y}}{h(f_Y)} = \underset{\substack{\text{variable}\\ X}}{h(f_X)} + \underset{\text{SCALING TERM}}{\underline{\underline{\log|a|}}}$$

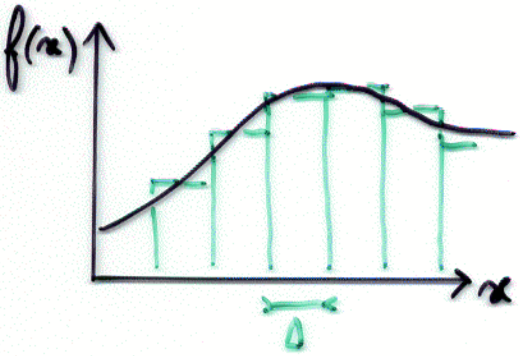
2°) CAN BE NEGATIVE !

NARROW DISTR.



$h \propto \log a < 0$

3°) LINKED TO SHANNON ENTROPY OF DISCRETIZED SIGNAL <u>UP TO CONSTANT</u> !



X^{Δ} = discretized version of X

$$\lim_{\Delta\to 0} \underset{\substack{\downarrow\\ +\infty}}{H(X^{\Delta})} + \underset{\substack{\downarrow\\ -\infty}}{\log\Delta} = h(x)$$

.... DIFFERENTIAL ENTROPIES $h(f)$ HAVE NO <u>ABSOLUTE</u> MEANING

.... ONLY <u>DIFFERENCES</u> OF ENTROPIES ARE MEANINGFUL

$$f(x)\ \xrightarrow{X}\ \boxed{f(y|x)}\ \xrightarrow{Y}$$

- $h(X) \simeq H(X^\Delta) + \log \Delta$
- $h(X|Y) = \int f(y)\, h(X|Y=y)\, dy$

$$\simeq \int f(y) \left\{ H(X^\Delta|Y=y) + \log \Delta \right\} dy$$

$$\simeq \sum_{y^\Delta} p(y^\Delta)\, H(X^\Delta|Y^\Delta = y^\Delta) + \log \Delta$$

$$\simeq H(X^\Delta|Y^\Delta) + \log \Delta$$

- $I(X:Y) = h(X) - h(X|Y)$

$$\simeq H(X^\Delta) - H(X^\Delta|Y^\Delta)$$ <u>ABSOLUTE</u> MEANING

MUTUAL INFORMATION BETWEEN
THE DISCRETIZED VERSIONS OF
INPUT AND OUTPUT SIGNALS

<u>RELATIVE ENTROPY</u> BETWEEN $f(x)$ and $g(x)$
(same support)

$$D(f\|g) = \int_S f(x) \log \frac{f(x)}{g(x)}\, dx$$

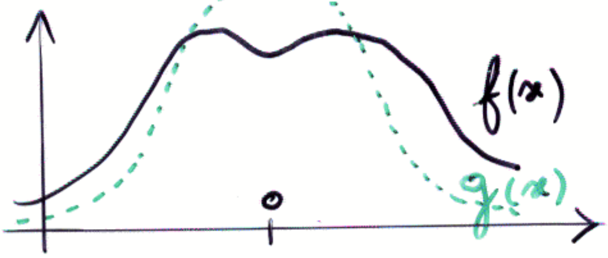
i) $D(f\|g) \geq 0$ Same demonstration

ii) $D(f\|g) = 0$ iif $f(x) = g(x)$ almost everywhere

- AGAIN $I(X:Y) = D\big(f(x,y) \,\|\, f(x)\cdot f(y)\big) \geq 0$

measures how far from independence.

USEFUL PROPERTY : GAUSSIAN DISTRIBUTION HAS HIGHEST ENTROPY FOR GIVEN VARIANCE



$g(x) \equiv$ GAUSSIAN DISTR. WITH VARIANCE $E[x^2]_{X\sim f}$

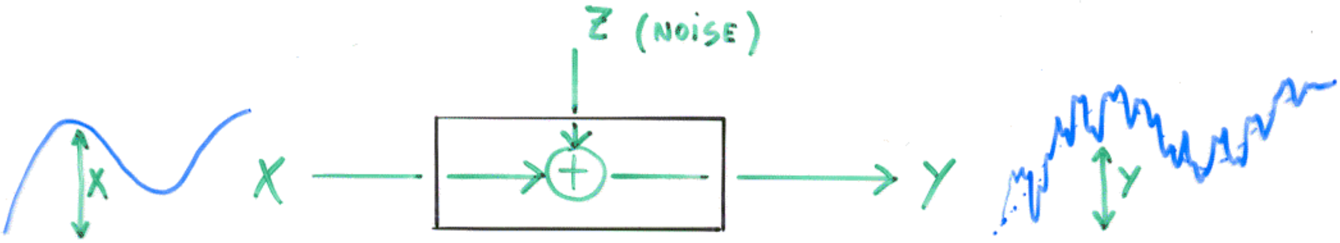
$$\begin{cases} f(x) \\ g(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \end{cases} \quad \text{with } \sigma^2 = \int x^2 f(x)\, dx$$

$$\begin{aligned} D(f\|g) &= \int f(x)\left[\log f(x) - \log g(x)\right] dx \\ &= -h(f) - \int f(x)\left[\log \frac{1}{\sqrt{2\pi\sigma^2}} - \frac{x^2}{2\sigma^2}\right] dx \\ &= -h(f) + \frac{1}{2}\log_e(2\pi\sigma^2) + \frac{1}{2}\log_e e \\ &= -h(f) + \underbrace{\frac{1}{2}\log_e(2\pi e\sigma^2)}_{h(g)} \quad \geq 0 \end{aligned}$$

$$\Rightarrow \quad h(f) \leq h(g)$$

GAUSSIAN DISTRIBUTION IS MOST DISORDERED AMONG ALL DISTRIBUTIONS OF SAME VARIANCE

GAUSSIAN CHANNELS



$Z \sim \mathcal{N}_{0,N}$ Gaussian noise independent of X

$$\mathcal{N}_{\underset{\text{mean}}{0},\ \underset{\text{variance}}{N}}$$

$$C = \max_{f(x)} I(X:Y)$$ DIVERGES IF NO ADDITIONAL CONSTRAINT !

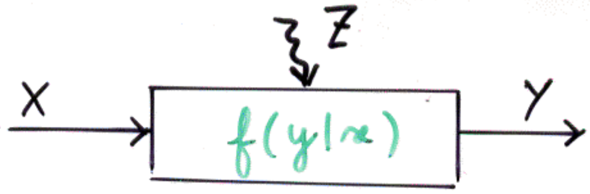
$\Rightarrow$ CONSTRAINT ON <u>VARIANCE (POWER)</u> OF INPUT SIGNAL

$$\begin{cases} E[X] = 0 \\ E[X^2] \leqslant P \end{cases}$$

$$\Rightarrow \boxed{C = \max_{f(x)\,:\,E[X^2] \leqslant P} I(X:Y)}$$

$$\begin{aligned}
I(X:Y) &= h(Y) - h(Y|X) \\
&= h(Y) - h(X+Z|X) \\
&= h(Y) - \int f(x)\, h(Z+\underline{x}\,|X=x)\, dx \\
&= h(Y) - \int f(x)\, h(Z|X=x)\, dx \\
&= h(Y) - h(Z|X) \\
&= h(Y) - h(Z)
\end{aligned}$$ since Z indep of X

$f(x) = ?$



$$I(X:Y) = h(Y) - h(Z)$$

- $Z \sim \mathcal{N}_{0,N} \rightarrow h(Z) = \frac{1}{2} \log(2\pi e N)$
- $Y \sim ?$ but entropy can be upper bounded !

$$h(Y) \leq \frac{1}{2} \log(2\pi e \underbrace{E[Y^2]}_{= \underbrace{E[X^2]}_{\leq P} + \underbrace{E[Z^2]}_{= N} \leq P + N})$$

$$h(Y) \leq \frac{1}{2} \log(2\pi e (P+N))$$

$$\Rightarrow I(X:Y) \leqslant \frac{1}{2} \log\left(\frac{P+N}{N} \right)$$

.... BOUND IS SATURATED IFF $Y \sim \mathcal{N}_{0,P+N}$

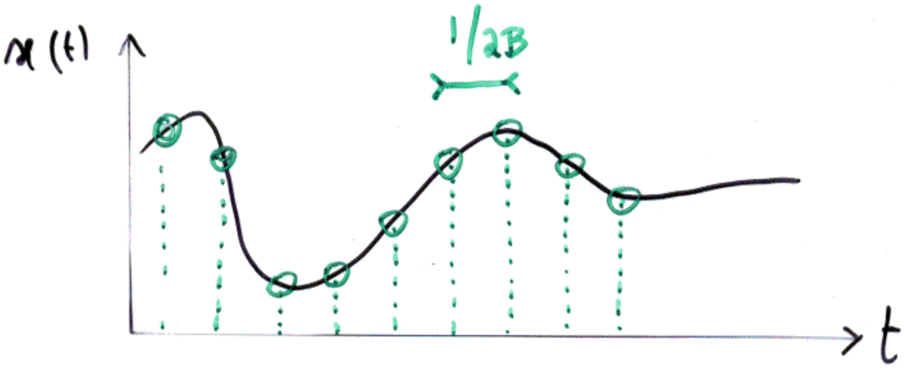
WHICH IS POSSIBLE IFF $X \sim \mathcal{N}_{0,P}$

$$\Rightarrow \boxed{C = \frac{1}{2} \log\left(1 + \frac{P}{N}\right)} \quad \text{(bits per symbol)}$$

IF INPUT IS GAUSSIAN-DISTRIBUTED WITH HIGHEST ALLOWED VARIANCE P, THEN OUTPUT IS GAUSSIAN-DISTRIBUTED WITH VARIANCE $P+N$ AND <u>CAPACITY IS ATTAINED</u>.

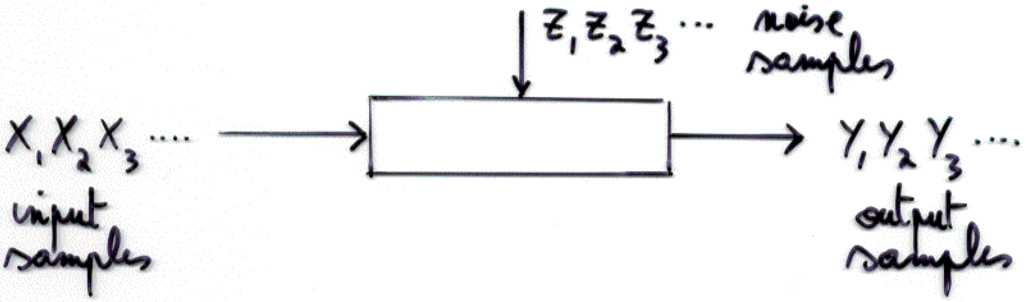
$$C = \frac{1}{2}\log\left(1+\frac{P}{N}\right)$$

CONTINUOUS SIGNAL
<u>BUT</u> DISCRETE TIME



NYQUIST THEOREM :

IF SIGNAL $x(t)$ HAS FINITE BANDWIDTH B
IT IS COMPLETELY DEFINED BY ITS SAMPLES
WITH A SAMPLING FREQUENCY = 2B



$$C = 2B \text{ (samples/s)} \cdot \frac{1}{2}\log\left(1+\frac{P}{N}\right) \text{ (bits/sample)}$$

$$\boxed{C = B\log\left(1+\frac{P}{N}\right)} \quad \text{(bits per second)}$$

QUANTUM CONTINUOUS VARIABLES

PAIR OF CANONICALLY CONJUGATE OBSERVABLES
IN AN INFINITE-DIMENSIONAL HILBERT SPACE

$$\{x, p\} \quad \text{such that} \quad [x,p] = i$$

E.G. QUADRATURE COMPONENTS OF LIGHT FIELD (BOSONIC FIELD)

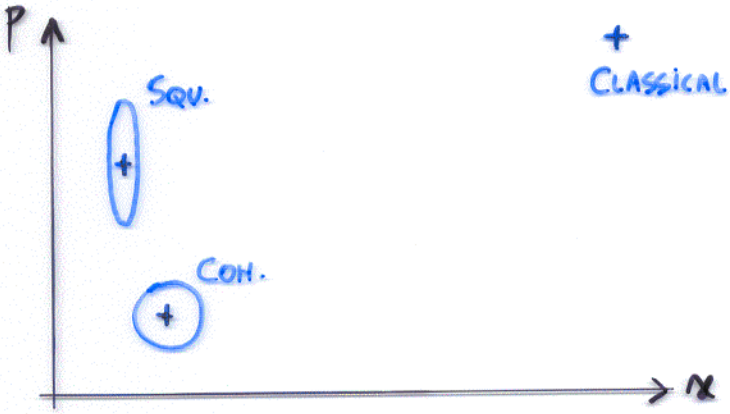
$$E = x \cos \omega t + p \sin \omega t$$

non-commuting observables
for quantized light field

UNCERTAINTY PRINCIPLE

$$\Delta x \cdot \Delta p \geqslant \frac{1}{2} \left| \langle [x,p] \rangle \right| = \frac{1}{2}$$

Saturated for
{ coherent st.
{ squeezed st.



COH. $\Delta x^2 = \Delta p^2 = \frac{1}{2}$ (shot-noise unit)

SQU. $\begin{cases} \Delta x^2 = e^{-2r} \cdot \frac{1}{2} \\ \Delta p^2 = e^{+2r} \cdot \frac{1}{2} \end{cases}$ (r = squeezing parameter)

FIRST- & SECOND-ORDER MOMENTS

n MODES → ASSOCIATED WITH $R = (x_1 \; p_1 \; \cdots \; x_n \; p_n)^T$

with $[x_j , p_k] = i \, \delta_{j,k}$

- 1^{st}- ORDER : DISPLACEMENT VECTOR d

$$d_j = \langle R_j \rangle$$

- 2^{nd}- ORDER : COVARIANCE MATRIX γ

$$\gamma_{j,k} = \langle \{ R_j - d_j , R_k - d_k \}_+ \rangle$$
$$= \langle R_j R_k + R_k R_j \rangle - 2 \, d_j d_k$$

E.G. 1 MODE
$$d = \begin{pmatrix} \langle x \rangle \\ \langle p \rangle \end{pmatrix} \qquad \gamma = \begin{pmatrix} 2\langle x^2 \rangle & \langle xp + px \rangle \\ \langle xp + px \rangle & 2 \langle p^2 \rangle \end{pmatrix}$$

VACUUM
$$d = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

SQUEEZED VACUUM
$$d = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \gamma = \begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix}$$

UNCERTAINTY RELATION : $\underline{\det \gamma \geq 1}$ (for 1 mode)

or $\gamma \geq \underbrace{\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \oplus \cdots \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}}_{n}$ (for n modes)

GAUSSIAN STATES

FULLY CHARACTERIZED BY d AND γ

WIGNER FUNCT. $$W(r) = \frac{1}{\pi^n \sqrt{\det \gamma}} \exp\left\{ -(r-d)^T \gamma^{-1} (r-d) \right\}$$

↳ FOURIER TRANSFORM OF CHARACTERISTIC FUNCTION

$$\chi(\xi) = \langle e^{i \xi R} \rangle$$ depends on all moments of R.

- SINGLE-MODE SQUEEZED STATE $\gamma = \begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix}$ r = squeezing parameter
- SINGLE-MODE THERMAL STATE $\gamma = \begin{pmatrix} 1+2\bar{n} & 0 \\ 0 & 1+2\bar{n} \end{pmatrix}$ $\bar{n}$ = n. of thermal photons

$$n \equiv a^\dagger a = \frac{1}{\sqrt{2}}(x - ip) \cdot \frac{1}{\sqrt{2}}(x + ip)$$
$$= \frac{1}{2}\left(x^2 + p^2 + i \underbrace{[x, p]}_{i}\right)$$
$$= \frac{1}{2}\left(x^2 + p^2 - 1\right)$$

$$\langle x^2 \rangle = \langle p^2 \rangle = \frac{1}{2}(1 + 2\bar{n}) \Rightarrow \langle n \rangle = \bar{n}$$

- TWO-MODE SQUEEZED STATE $\equiv$ EPR STATE

$$\begin{tikzcd} & x_1 + x_2 \arrow[r] & \text{SQUEEZED} & (\, p_1 + p_2 \ \text{ANTI-SQUEEZED} \,) \\ \text{COMMUTE} \arrow[ur] \arrow[dr] & & & \\ & p_1 - p_2 \arrow[r] & \text{SQUEEZED} & (\, x_1 - x_2 \ \text{ANTI-SQUEEZED} \,) \end{tikzcd}$$

$$\begin{cases} x_1 = \frac{1}{\sqrt{2}}\left(\underline{x_1'} + x_2'\right) \\ p_1 = \frac{1}{\sqrt{2}}\left(p_1' + \underline{p_2'}\right) \end{cases}$$

$$\begin{cases} x_2 = \frac{1}{\sqrt{2}}\left(\underline{x_1'} - x_2'\right) \\ p_2 = \frac{1}{\sqrt{2}}\left(p_1' - \underline{p_2'}\right) \end{cases}$$

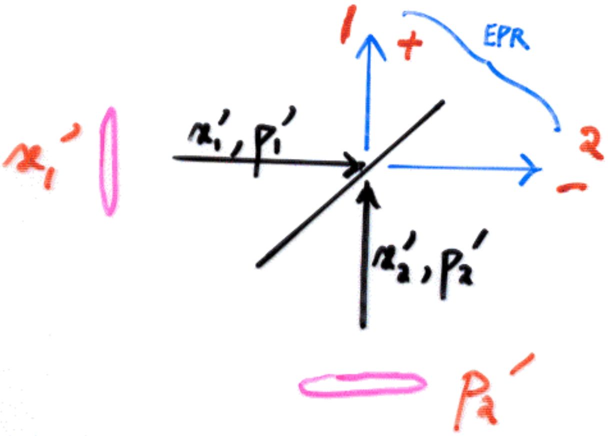
SQUEEZED

$2\langle x_1'^2\rangle = e^{-2s}$ SQUEEZED

$\hookrightarrow \langle p_1'^2\rangle = e^{2s}$

$2\langle p_2'^2\rangle = e^{-2s}$ SQUEEZED

$\hookrightarrow \langle x_2'^2\rangle = e^{2s}$



$$2\langle x_1^2\rangle = \langle x_1'^2\rangle + \langle x_2'^2\rangle = \frac{e^{-2s}+e^{2s}}{2} = \cosh(2s)$$

NOTE : $= \cosh^2 s + \sinh^2 s = 1 + 2\underbrace{\sinh^2 s}_{\bar{n}}$

$$\langle x_1 p_1 + p_1 x_1\rangle = \frac{1}{2}\left(\langle x_1'p_1' + p_1'x_1'\rangle + \langle x_2'p_2' + p_2'x_2'\rangle\right) = 0$$

⇒ REDUCED STATE is THERMAL

$$2\langle x_1 x_2\rangle = \langle x_1'^2\rangle - \langle x_2'^2\rangle = \frac{e^{-2s}-e^{2s}}{2} = -\sinh(2s)$$

ANTI-CORRELATED

$$2\langle p_1 p_2\rangle = \langle p_1'^2\rangle - \langle p_2'^2\rangle = \frac{e^{2s}-e^{-2s}}{2} = \sinh(2s)$$

CORRELATED

$$\gamma = \begin{pmatrix} \cosh 2s & 0 & -\sinh 2s & 0 \\ 0 & \cosh 2s & 0 & \sinh 2s \\ -\sinh 2s & 0 & \cosh 2s & 0 \\ 0 & \sinh 2s & 0 & \cosh 2s \end{pmatrix}$$

SYMPLECTIC TRANSFORMATIONS

PHYSICALLY : UNITARY GAUSSIAN TRANSFORMATION ON STATES

$\equiv$ SYMPLECTIC TRANSFORMATION ON γ

i.e. CONSERVES $[x_j, p_k] = i\,\delta_{j,k}$

Symplectic matrix : $\sigma_{j,k} \equiv [R_j, R_k] = \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

Symplectic transformation : $R' = S R$

such that S leaves σ invariant

$\leftrightarrow \sigma' = S \sigma S^T = \sigma$

Symplectic eigenvalues of γ : $\{\lambda_i\}$

one can find S such that $S \gamma S^T = \bigoplus_{i=1}^{n} \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{pmatrix}$

$\Rightarrow$ one transforms state into <u>product of thermal states</u> using Gaussian unitary transformation S.

The symplectic eigenvalues λ_i are obtained by solving

$$\det \left[\gamma - \lambda \cdot \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right] = 0$$

... invariant under symplectic transformation S

ENTROPY OF THERMAL STATE

2-MODE SQUEEZED STATE IN FOCK BASIS

$$\sqrt{1-\lambda^2}\sum_{n=0}^{\infty}\lambda^n |n\rangle|n\rangle \qquad \text{with } \lambda = \tanh s$$

THERMAL STATE

$$(1-\lambda^2)\sum_{n=0}^{\infty}\lambda^{2n}|n\rangle\langle n|$$

$$\text{use } \lambda^2 = \frac{\sinh^2 s}{\cosh^2 s} = \frac{\sinh^2 s}{1+\sinh^2 s} = \frac{\bar{n}}{1+\bar{n}}$$

$$\frac{1}{1+\bar{n}}\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n |n\rangle\langle n|$$

ENTROPY

$$S(\rho) = H[p_n] \qquad \text{with } p_n = \frac{1}{1+\bar{n}}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n$$

$$= -\sum_n p_n\left\{-\log(1+\bar{n}) + n\log\left(\frac{\bar{n}}{1+\bar{n}}\right)\right\}$$

$$= \log(\bar{n}+1) - \bar{n}\log\left(\frac{\bar{n}}{1+\bar{n}}\right)$$

$$= (\bar{n}+1)\log(\bar{n}+1) - \bar{n}\log\bar{n}$$

$\Rightarrow$ The entropy of any Gaussian state can be computed from the symplectic eigenvalues of γ

$$S = \sum_{i=1}^{n} g\left(\frac{\lambda_i - 1}{2}\right) \qquad \text{with } \lambda_i = \text{symplectic eigenvalues}$$

$$g(x) = (x+1)\log(x+1) - x\log x$$

ENTROPY OF ARBITRARY (NON-GAUSSIAN) STATE

USE QUANTUM RELATIVE ENTROPY (BETWEEN 2 STATES)

$$S(\rho\,\|\,\sigma) \equiv \mathrm{Tr}\left(\rho\log\rho - \rho\log\sigma\right) \geq 0$$

Let $\begin{cases} \rho = \text{arbitrary density oper.} \\ \rho_G = \text{density oper. of Gaussian state with same } d \text{ and } \gamma \end{cases}$

$$\begin{aligned} S(\rho_G) - S(\rho) &= \mathrm{Tr}(\rho\log\rho) - \mathrm{Tr}(\rho_G\log\rho_G) \\ &= \mathrm{Tr}(\rho\log\rho) - \mathrm{Tr}(\rho\log\rho_G) \\ &\qquad + \mathrm{Tr}(\rho\log\rho_G) - \mathrm{Tr}(\rho_G\log\rho_G) \\ &= \underbrace{S(\rho\,\|\,\rho_G)}_{\geq 0} + \underbrace{\mathrm{Tr}(\rho\log\rho_G) - \mathrm{Tr}(\rho_G\log\rho_G)}_{\substack{= 0 \text{ since } \rho \text{ and } \rho_G \\ \text{have same 1-st order} \\ \text{and 2-nd order moments}}} \end{aligned}$$

ρ_G is Gaussian

↔ $\log\rho_G$ is polynomial of order 2 in field operators $R_j = (x, p, \ldots)$

⇒ Entropy of arbitrary state is upper bounded by entropy of Gaussian state with the same covariance matrix.