Quantum cloning and key distribution with continuous variables of light

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“Franco-Belgian” collaboration

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Quantum continuous variables

Thrust: Use, as an information carrier, a physical quantity that has a continuous spectrum instead of a binary spectrum

Focus on optical quantum continuous variables ...

* Quantization of the electromagnetic field:
  - Discrete degree of freedom (polarization of a single photon)
  - Continuous degree of freedom (amplitude of quadrature X or P)

* Advantages :
  - Use of standard telecommunication techniques
  - No need for single-photon sources and/or detectors
  - Prospect for higher data rates and simpler processing tools

* Disadvantages :
  - Vacuum noise (shot noise)
  - Sensitivity to optical losses (limits the achievable range)

Optical continuous variables

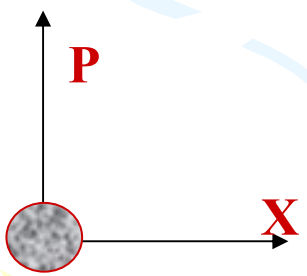
Triggered by the success of continuous-variable quantum teleportation (theoretical concept in 1997, experimental demonstration in 1998)

Light field: $X \cos(\omega t) + P \sin(\omega t)$

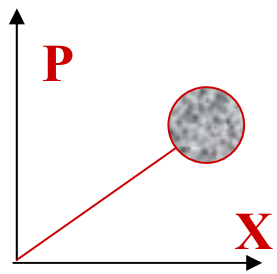
Classical optics: X and P are quadrature components
Quantum optics: X and P are non-commuting variables

$V(X)\, V(P) \geq N_0^2$ (N_0 = shot-noise variance unit)

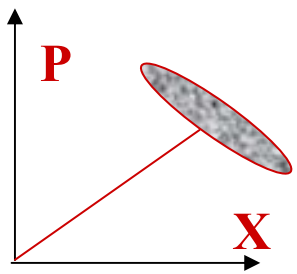
Convenient representation : phase space



Vacuum state

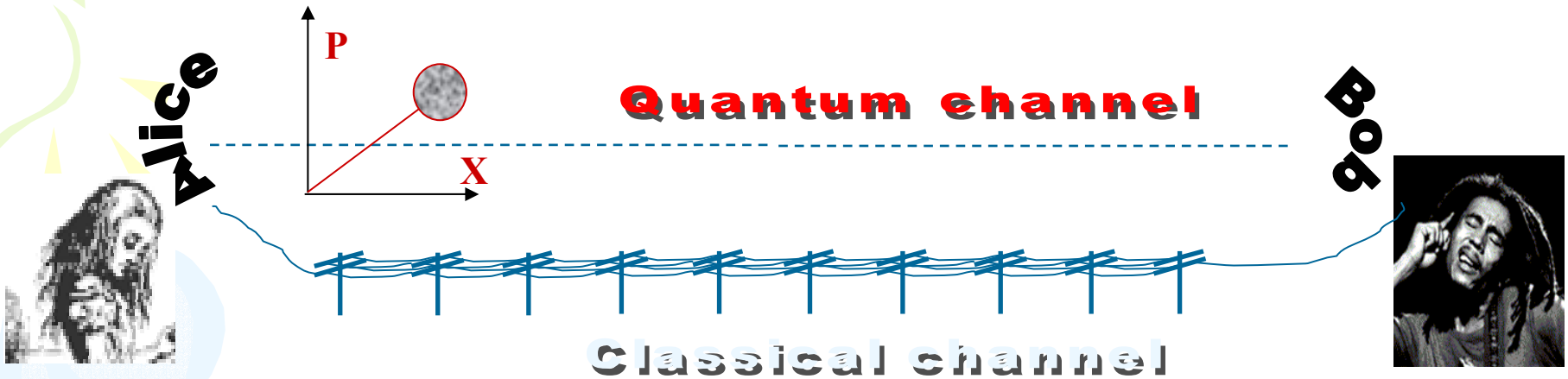


Coherent state



Squeezed state

Continuous-variable quantum key distribution



Principle of quantum key distribution :

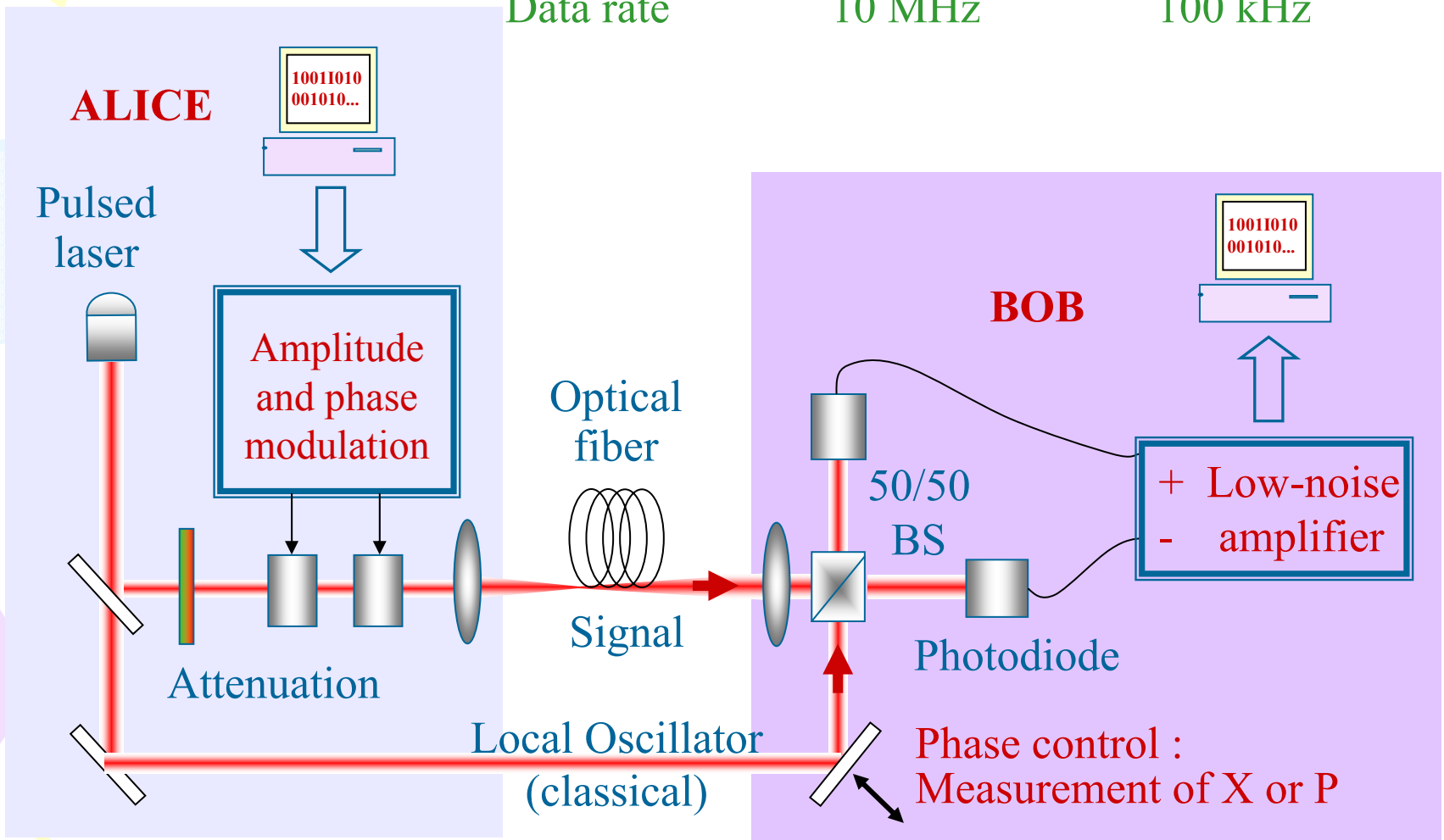
* The eavesdropper (Eve) has unlimited computational power and technological resources; she is only limited by the laws of quantum physics!

Distinct features of our protocol :

* The non-commuting variables are the quadrature components X and P
* The transmitted light made of weak coherent pulses (about 100 photons) with a gaussian modulation (of amplitude and phase)
* The detection is made using shot-noise limited homodyne detection

Homodyne detection

	Homodyne	Counting (APD)
Efficiency	> 90%	10-50 %
Dark rate	negligible	problem
Data rate	10 MHz	100 kHz



Quantum cloning w. continuous variables

- Cloning of a pair of canonical continuous variables (x,p)
  - Limits on optimal cloning (minimum noise)
  - Gaussian cloning transformation
  - Optical implementation
- Gaussian cloners with multiple inputs and outputs
  - Bound on the fidelity based on concatenation
- Non-Gaussian cloning of continuous variables
  - General displacement- and rotation-covariant cloner
  - Optical implementation of a non-Gaussian cloner
  - Maximum cloning fidelity for Gaussian states
- Application to continuous-variable QKD

Cloning 2 canonically conjugate variables

(x, p) = two quadrature components of a light mode

$[x, p] = i \Rightarrow x$ and p cannot be both measured perfectly

$\Rightarrow$ states $|x\rangle$ and $|p\rangle$ cannot be cloned perfectly

Look for an optimal *imperfect* cloner...

Symmetric

Covariant with respect to Weyl-Heisenberg displacements

Same error variance for x and p quadrature

Minimize the cloning noise variance

Limits on optimal cloning

Cloning and mesuring x on one clone and p on the other clone cannot beat the optimal joint measurement of x and p.

$$X_1 = x + \tilde{x}_1 \qquad X_2 = x + \tilde{x}_2$$
$$P_1 = p + \tilde{p}_1 \qquad P_2 = p + \tilde{p}_2$$

$$[X_1, P_2] = \underbrace{[x, p]}_{=i} + \underbrace{[x, \tilde{p}_2]}_{=0} + \underbrace{[\tilde{x}_1, p]}_{=0} + [\tilde{x}_1, \tilde{p}_2] = 0$$

$$\Delta\tilde{x}_1 \, \Delta\tilde{p}_2 \geq \tfrac{1}{2} |\langle [\tilde{x}_1, \tilde{p}_2] \rangle| = 1/2$$
$$\Delta\tilde{p}_1 \, \Delta\tilde{x}_2 \geq \tfrac{1}{2} |\langle [\tilde{p}_1, \tilde{x}_2] \rangle| = 1/2$$

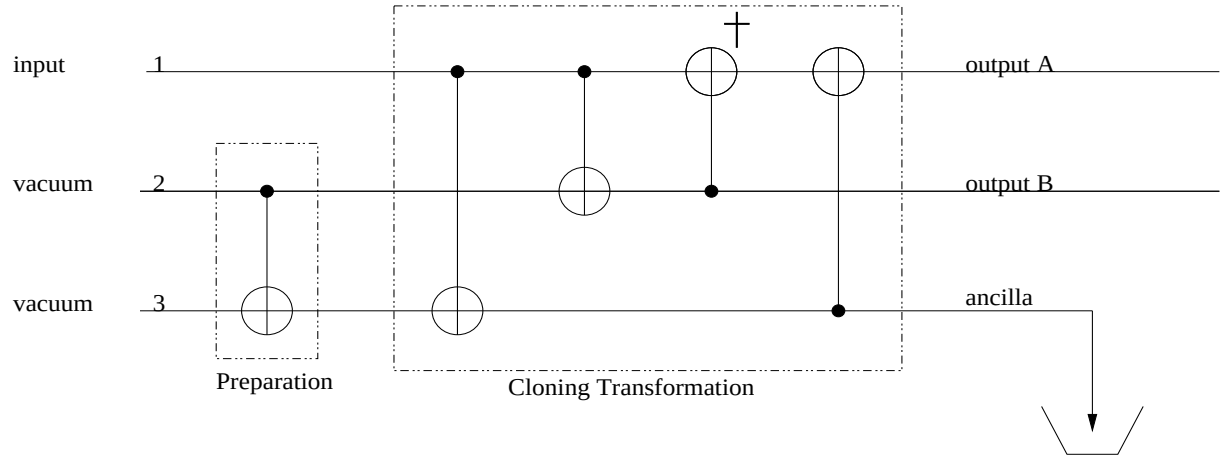
Symmetric Gaussian cloner:

$\Delta\tilde{x}_1^2 = \Delta\tilde{p}_1^2 = \Delta\tilde{x}_2^2 = \Delta\tilde{p}_2^2 = 1/2$ (minimum added noise)

This cloning transformation is

$$\hat{U}_{1,2,3} = e^{-i(\hat{x}_3 - \hat{x}_2)\hat{p}_1} e^{-i\hat{x}_1(\hat{p}_2 + \hat{p}_3)}$$

with $|\psi_{\text{prep}}\rangle = e^{-i\hat{x}_2\hat{p}_3}|0_{\text{vac}}\rangle|0_{\text{vac}}\rangle$



where $\text{C}-\text{NOT}\ |a\rangle\ |b\rangle = |a\rangle\ |a+b\rangle$.

Transformation in Schrödinger picture:

$$|0_{\text{vac}}\rangle = \frac{1}{\pi^{1/4}} \int dz\ e^{-z^2/2}\ |z\rangle$$

$$|\psi_{\text{prep}}\rangle = \frac{1}{\sqrt{\pi}} \int dy\ dz\ e^{-(y^2+z^2)/2}\ |z\rangle\ |y+z\rangle$$

$$|x\rangle|\psi_{\text{prep}}\rangle \longrightarrow \frac{1}{\sqrt{\pi}} \int dy\ dz\ e^{-(y^2+z^2)/2} \times |x+y\rangle\ |x+z\rangle\ |x+y+z\rangle$$

This machine effects Gaussian-distributed x-errors and p-errors on the input state $|\psi\rangle$:

$$\rho_{A,B} = \frac{1}{\pi}\int dx\ dp\ e^{-(x^2+p^2)}\ \hat{D}(x,p)|\psi\rangle\langle\psi|\hat{D}^\dagger(x,p)$$

with $\hat{D}(x,p) = e^{-ix\hat{p}}e^{ip\hat{x}}$ displacement operator

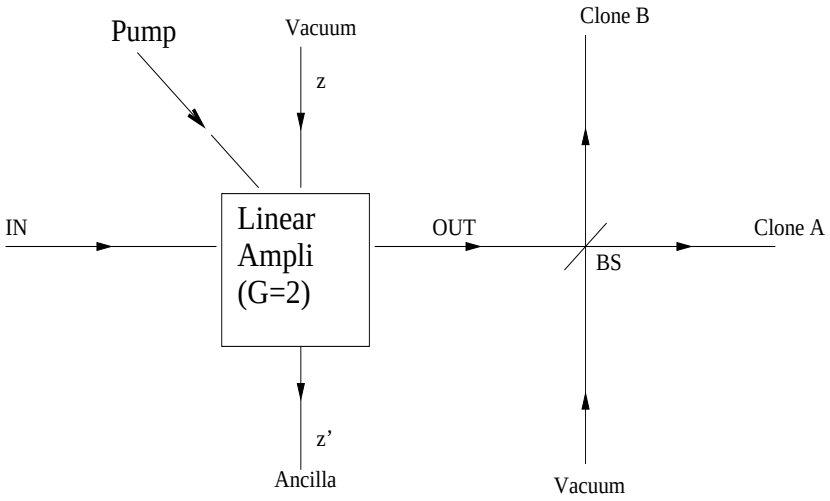
The error variances in x and p are the same:

$$\sigma^2 = 1/2 \qquad (optimal\ Gaussian\ cloner)$$

The fidelity when cloning a coherent state $|\alpha\rangle$ is calculated by using $\langle\alpha|\alpha+\beta\rangle = e^{-|\beta|^2}$

$$\begin{aligned} f &= \langle\alpha|\rho_{A,B}|\alpha\rangle \\ &= \frac{2}{\pi}\int d^2\beta\ e^{-2|\beta|^2}|\langle\alpha|\alpha+\beta\rangle|^2 \\ &= \frac{1}{1+\sigma^2} \\ &= 2/3 \end{aligned}$$

Optical implementation



N.B. Cloning-induced noise originates from vacuum fluctuations of the ancilla modes

$$\hat{a}_{\mathsf{out}} = \sqrt{2}\,\hat{a}_{\mathsf{in}} + \hat{a}_z^\dagger$$
$$\hat{a}_{z'} = \sqrt{2}\,\hat{a}_z + \hat{a}_{\mathsf{in}}^\dagger$$

$$\hat{a}_A = (\hat{a}_{\mathsf{out}} + \hat{a}_{\mathsf{vac}})/\sqrt{2} = \hat{a}_{\mathsf{in}} + (\hat{a}_z^\dagger + \hat{a}_{\mathsf{vac}})/\sqrt{2}$$
$$\hat{a}_B = (\hat{a}_{\mathsf{out}} - \hat{a}_{\mathsf{vac}})/\sqrt{2} = \hat{a}_{\mathsf{in}} + (\hat{a}_z^\dagger - \hat{a}_{\mathsf{vac}})/\sqrt{2}$$

$$\langle \hat{a}_A \rangle = \langle \hat{a}_B \rangle = \langle \hat{a}_{\mathsf{in}} \rangle$$
$$\langle \hat{a}_{z'} \rangle = \langle \hat{a}_{\mathsf{in}} \rangle^*$$

$$\Delta x_A{}^2 = \Delta p_A{}^2 = \Delta x_B{}^2 = \Delta p_B{}^2 = \tfrac{1}{2} + \tfrac{1}{2}$$
$$\Delta x_{z'}{}^2 = \Delta p_{z'}{}^2 = \tfrac{1}{2} + 1$$

Gaussian cloner with multiple in/outputs

Cloner which produces M approximate clones with the same Gaussian-distributed errors in x and p when provided with N replicas of $|\psi\rangle$.

The minimum error variance is

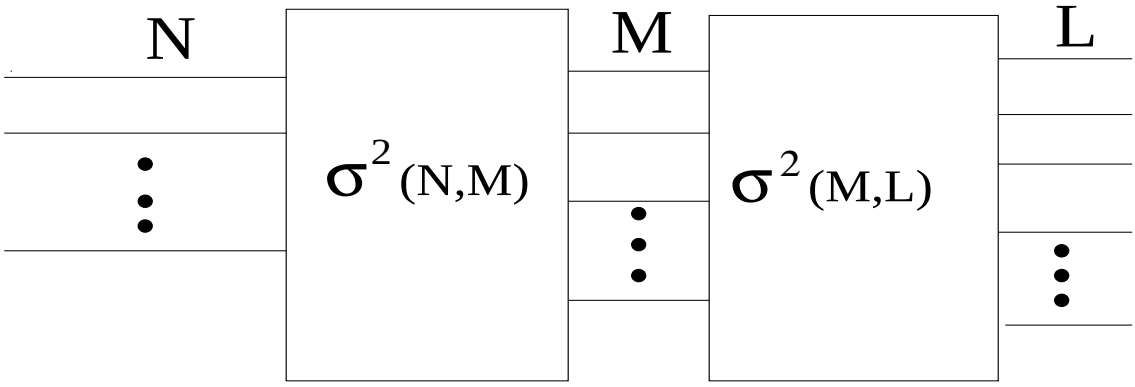
$$\sigma^2(N, M) = \frac{1}{N} - \frac{1}{M}$$

The cloning fidelity for coherent states is

$$f_{N,M} = \frac{MN}{MN + M - N}$$

N	M	$f_{N,M}$
1	2	2/3
1	3	3/5
1	∞	1/2
N	N	1
N	$2N$	$2N/(2N+1)$
N	∞	$N/(N+1)$

Proof: Consider two concatenated cloners



We have $\sigma^2(N,L) = \sigma^2(N,M) + \sigma^2(M,L)$.

For $L = \infty$ and considering optimal cloners

$$\underbrace{\sigma^2_{opt}(N,\infty)}_{1/N} \leq \sigma^2(N,M) + \underbrace{\sigma^2_{opt}(M,\infty)}_{1/M}$$

$$\Rightarrow \quad \sigma^2(N,M) \geq 1/N - 1/M$$

This bound is saturated by the Gaussian cloner

High-rate continuous-variable QKD

using Gaussian-modulated coherent light pulses

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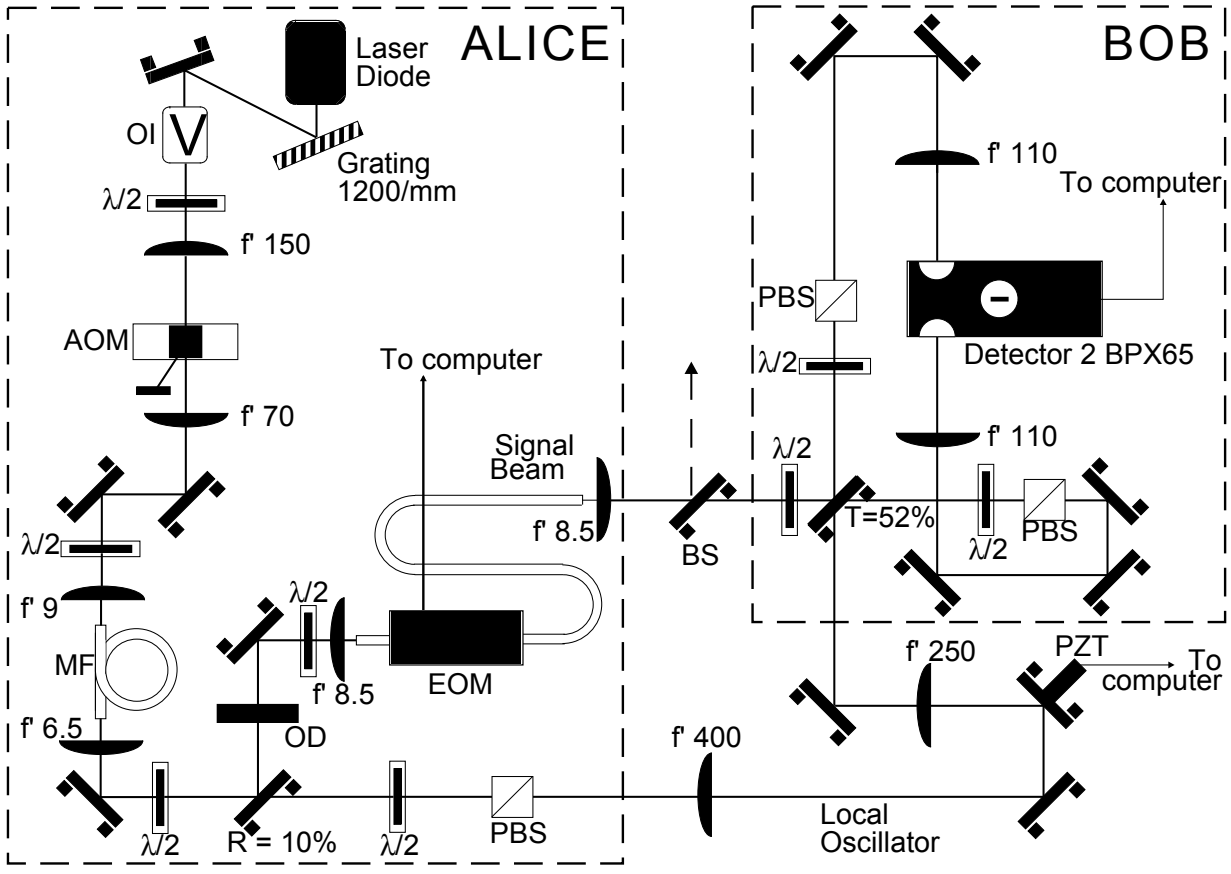
Some “key” ideas

- Key encoded in e-m field amplitudes (x or p quadrature)
  $\Rightarrow$ “gaussian key elements”
- Coherent light pulses (non-orthogonal states)
  $\Rightarrow$ no need for squeezing
- Continuous raw data converted into a usable binary key
  $\Rightarrow$ “sliced” reconciliation algorithm
- Reverse reconciliation (Bob’s data serve as the key)
  $\Rightarrow$ no limit on the attainable quantum channel loss

QUANTUM PROTOCOL IS FULLY CONTINUOUS

CLASSICAL POST-PROCESSING IS DISCRETE

Tested experimental setup



Laser diode: SDL 5412 (780nm); OI: optical isolator; AOM: acousto-optic modulator; MF: polarization maintaining fiber; OD: optical density; EOM: electro-optic amplitude modulator; PBS: polarizing beam splitter; BS: beam splitter; PZT: piezoelectric transducer

Protocol

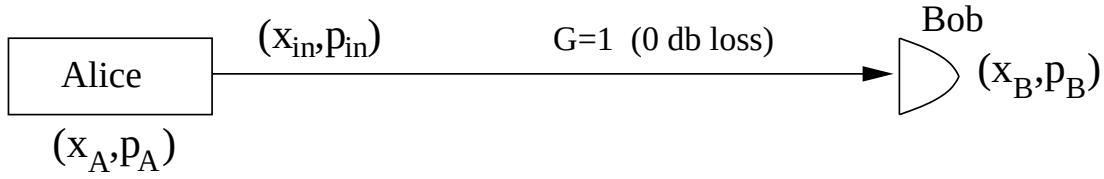
— QUANTUM —

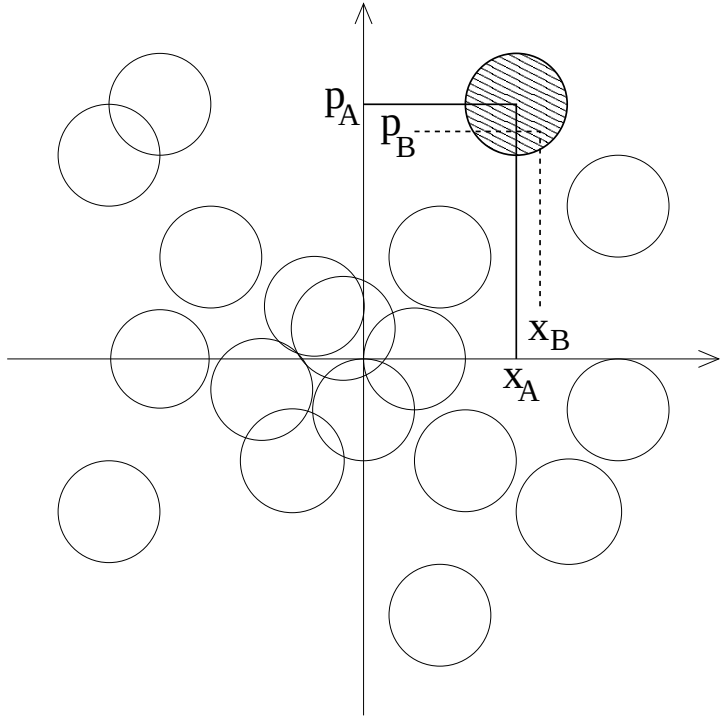
1. Alice sends $|x_A + ip_A\rangle$ with random x_A, $p_A \sim \mathcal{N}(0, V_A)$
2. Bob randomly measures either x_B or p_B
3. Bob discloses x/p bit, so Alice keeps good quadrature

— CLASSICAL —

4. Alice & Bob estimate channel loss and added noise
5. Alice & Bob reconciliate their gaussian data
   - DIRECT RECONCILIATION (DR)*: Alice sends extra bits to Bob for him to correct transmission errors*
   - REVERSE RECONCILIATION (RR)*: Bob sends extra bits to Alice for her to incorporate transmission errors*
6. Alice & Bob apply privacy amplification to wipe out Eve's estimated information about Alice (DR) or Bob (RR)

Protocol (continued)



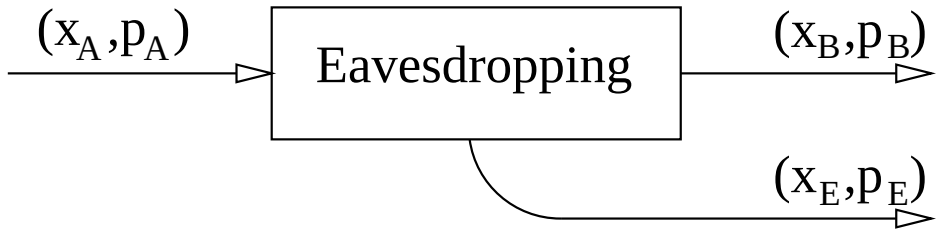


$$x_{in} = x_A + x_{vac}$$
$$p_{in} = p_A + p_{vac}$$

$$x_B = \sqrt{G}\,(x_{in} + \delta x) \qquad \text{with} \quad \langle \delta x^2 \rangle = \chi N_0$$
$$p_B = \sqrt{G}\,(p_{in} + \delta p) \qquad \text{with} \quad \langle \delta p^2 \rangle = \chi N_0$$

(G=channel gain, χ= equivalent input noise, N_0 = shot-noise variance)

Heisenberg-limited eavesdropping



Alice and Eve cannot jointly know more about (x_B, p_B) than allowed by the Heisenberg principle:

$$V(x_B|x_A)\,V(p_B|p_E) \geq N_0^2 \qquad V(p_B|p_A)\,V(x_B|x_E) \geq N_0^2$$

(N_0 = shot-noise variance)

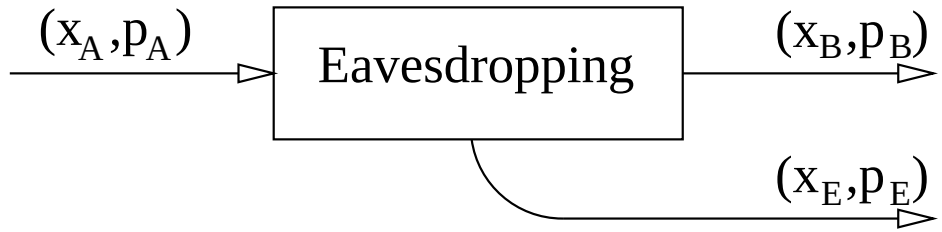
The channel parameters define the conditional variances:

$$x_B = \sqrt{G}\,(x_A + x_{vac} + \delta x) \qquad p_B = \sqrt{G}\,(p_A + p_{vac} + \delta p)$$

$$V^{\mathrm{coh}}(x_B|x_A) = G(\chi+1)N_0 \qquad V^{\mathrm{coh}}(p_B|p_A) = G(\chi+1)N_0$$

(G=channel gain, χ= equivalent input noise, N_0 = shot-noise variance)

Virtual entanglement limit



The Heisenberg relation holds regardless the way Alice prepares her ensemble (gaussian mixture of coherent states $\Leftrightarrow$ one of two entangled beams).

The channel parameters imply the limitation:

$$V(x_B|x_A) \geq G(\chi + 1/V)N_0 \qquad V(p_B|p_A) \geq G(\chi + 1/V)N_0$$

($V = 1 + V_A$= Alice's quadrature variance; χ= equivalent input noise)

Resulting bound on Eve's knowledge:

$$V(x_B|x_E) \geq \frac{N_0}{G(\chi+1/V)} \qquad V(p_B|p_E) \geq \frac{N_0}{G(\chi+1/V)}$$

Security analysis (RR)

Restricted to gaussian individual (non-collective) attacks

$$\Delta I \geq \max(\underbrace{I_{AB} - I_{AE}}_{\mathrm{DR}}, \underbrace{I_{BA} - I_{BE}}_{\mathrm{RR}})$$

$$I_{BA} = \frac{1}{2} \log_2 \left(\frac{V_B}{V_{B|A}^{\mathrm{coh}}} \right)$$

$$I_{BE} = \frac{1}{2} \log_2 \left(\frac{V_B}{V_{B|E}^{\mathrm{min}}} \right)$$

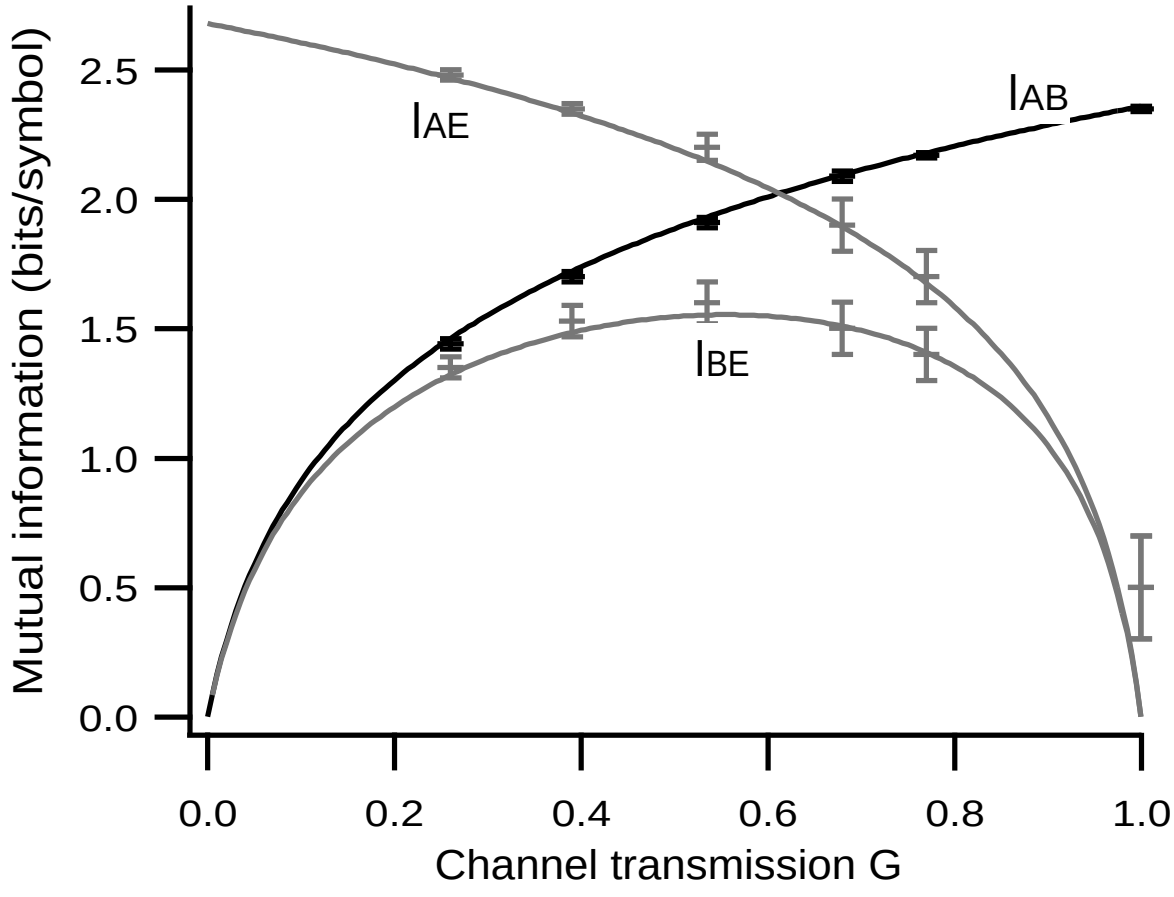
with $V_B = G(V + \chi)N_0 =$ Bob's quadrature variance

$$\Delta I_{\mathrm{RR}} = \frac{1}{2} \log_2 \left(\frac{V_{B|E}^{\mathrm{min}}}{V_{B|A}^{\mathrm{coh}}} \right) = -\frac{1}{2} \log_2 [G^2(\chi + 1)(\chi + 1/V)]$$

with $\chi = \underbrace{(1 - G)/G}_{\text{vacuum noise}} + \epsilon$ ($\epsilon =$ excess noise)

High-loss limit ($G \ll 1$): $\underline{\Delta I_{\mathrm{RR}} > 0}$ if $\epsilon < \frac{V-1}{2V} \simeq 1/2$

Information curves



◇ THE RECONCILIATION DOES NOT REACH SHANNON'S LIMIT, BUT HAS AN EFFICIENCY IN THE 70-90 % RANGE $\longrightarrow$ BOUND ON ATTAINABLE G

Practical net key rates

G	I_{AB} (bits/symbol)	RR key rate (kbps)	DR key rate (kbps)
1.00 (0 dB)	2.4	1,690 [1,920]	1,660 [1,910]
0.79 (1.0 dB)	2.2	470 [730]	270 [540]
0.68 (1.7 dB)	1.9	185 [510]	— [190]
0.49 (3.1 dB)	1.7	75 [370]	— [0]
0.26 (5.9 dB)	1.5	— [85]	— [0]

[ ] $\equiv$ IDEAL SECRET KEY RATES (ASSUMING PERFECT RECONCILIATION)

REPETITION RATE $\simeq$ 800 kHz (120 ns PULSES)

MODULATION VARIANCE $\simeq$ 30-40 $\times$ SHOT-NOISE VARIANCE

Advantage over conventional QKD

- No single-photon source needed ($\sim$ 100 photons/pulse)
- Fast homodyne detection (repetition rate $\sim$ 10 MHz)
  - *Repetition rate of single-photon detectors $\sim$ 100 kHz*
- High-dimensionality of the phase space
  - *Amplitude/phase modulation with large dynamics so that several key bits are encoded per coherent pulse*
- High secret key bit rates
  - *Practical rates $\sim 10^2 - 10^3$ kbps for low losses ($\sim$3 dB) that is, for a moderate range (~ 15 km)*

SECURE REGARDLESS THE LINE LOSS (IN PRINCIPLE!)

Eve controls Bob's errors not as well as she reads Alice's modulation

$\rightarrow$ *reverse reconciliation is better for lossy channels*

References

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QUANTUM OPTICAL ASPECTS:

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CLASSICAL PROCESSING ASPECTS:

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SECURITY ASPECTS: . . . . .