Lecture by B. Georgeot 3

Institut Henri Poincaré

Trimester at Centre Émile Borel

Quantum Information, Computation and Complexity

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Quantum computation and complex dynamics

Lectures by B. Georgeot 3: overview

1) Interference in quantum computation

2) Errors and imperfections in quantum computer: random noise in the gates

3) Static errors in quantum computers: apparition of a quantum chaos regime

Measuring interference

D. Braun and B. Georgeot, Phys. Rev. A 73, 022314 (2006)

For a process: $\rho'_{ij} = \sum_{k,l} P_{ij,kl}\rho_{kl}$, let us define an interference measure by:

$\mathcal{I}(P) = \sum_{i,k,l} |P_{ii,kl}|^2 - \sum_{i,k} |P_{ii,kk}|^2$

For unitary processes it becomes: $\mathcal{I}(P(U)) = \left(N - \sum_{i,k} |U_{ik}|^4\right)$

One has $0 \leq \mathcal{I}(P(U)) \leq N-1$

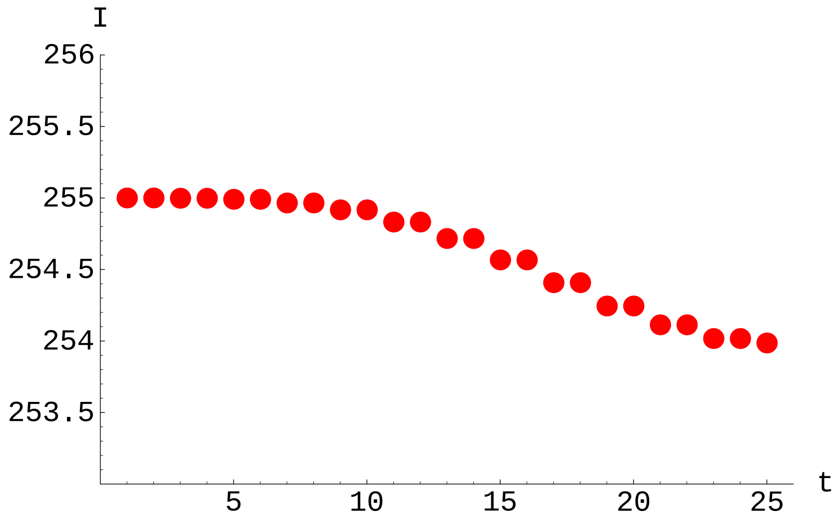
Number of "i-bits" $n_I = \log_2(\mathcal{I}(P)+1)$

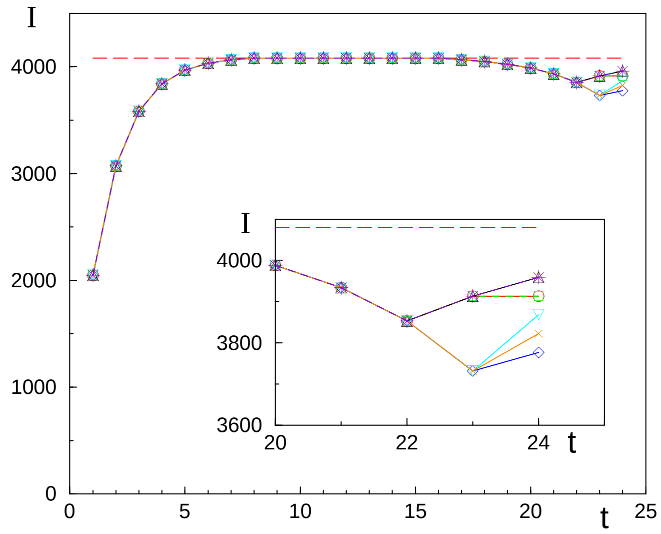
Permutation matrix $\Rightarrow \mathcal{I}(P(U)) = 0$

Fourier and Walsh-Hadamard transforms $\Rightarrow \mathcal{I}(P(U)) = N-1$

Potentially available interference

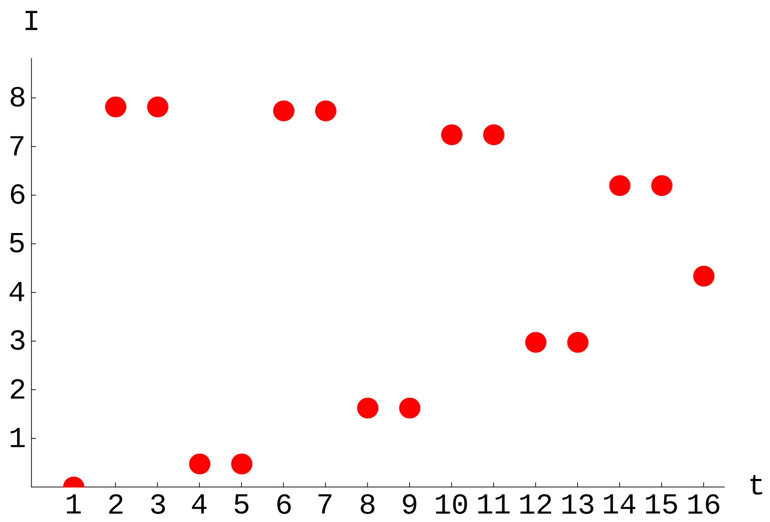
Interference up to time t for Grover (left, 8 qubits) and Shor (right, 12 qubits, factorization of 15) algorithms.

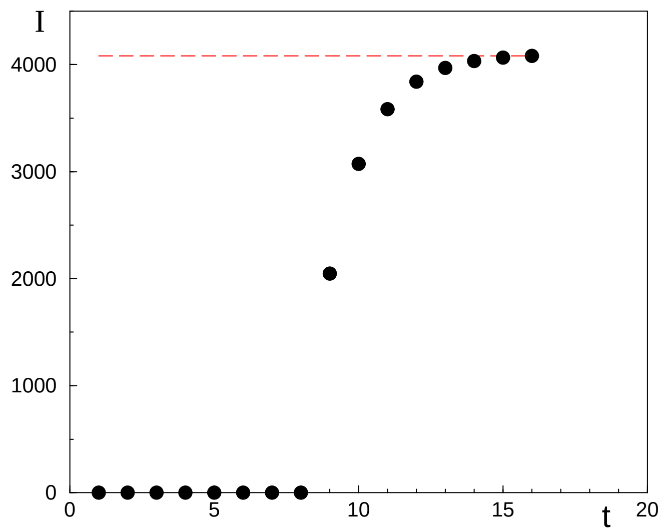




Actually used interference

Interference of the process up to time t for Grover (left, 7 qubits) and Shor (right, 12 qubits, factorization of 15) algorithms. The initial Hadamard gates creating the uniform state $1/\sqrt{N}\sum_{i=1}^{N}|i\rangle$ are now omitted.





Unitary noise in the gates

P.-H. Song and D. Shepelyansky, Phys. Rev. Lett. (2001); B. Lévi, B.Georgeot and D. Shepelyansky, Phys. Rev. E (2003).

Numerical simulations of the kicked rotator model on quantum computer

Each gate is replaced by random gate with noise parameter ϵ i. e. each random gate is rotated by ϵ from exact gate

Expectation: each gate transfers probability of order ϵ^2 from the exact wave function.

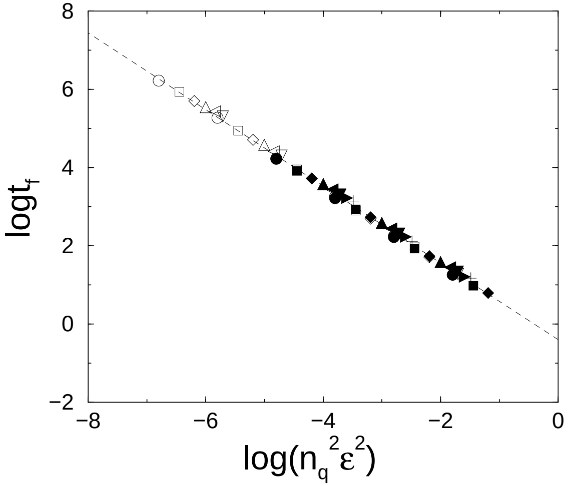
Problem: depending on the observable, this can lead to vastly different time scales.

Fidelity and second moment

Hilbert space dimension $N = 2^{n_q}$, T and k constant

Fidelity time scale

$f(t) = |<\psi_\varepsilon(t)|\psi_0(t)>|^2$

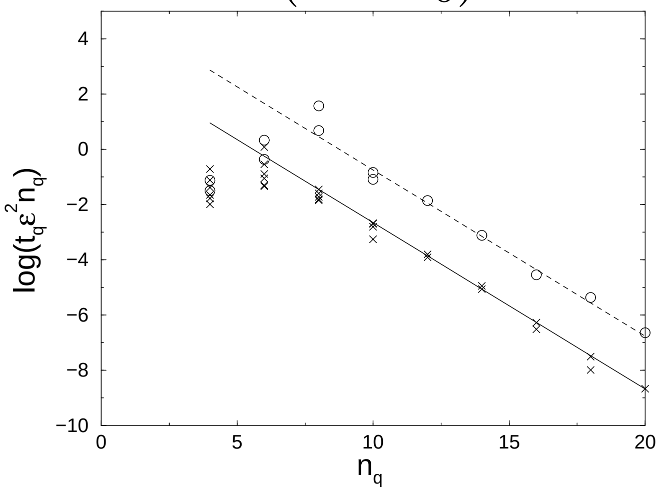


$t_f \approx C_f/(\varepsilon^2 n_q^2)$

polynomial in ε and n_q

Second moment time scale

$< n^2 >=< (n - n_0)^2 >$



$t_q \approx C_q k^4/(\epsilon^2 n_q 2^{2n_q})$

polynomial in ε exponential in n_q

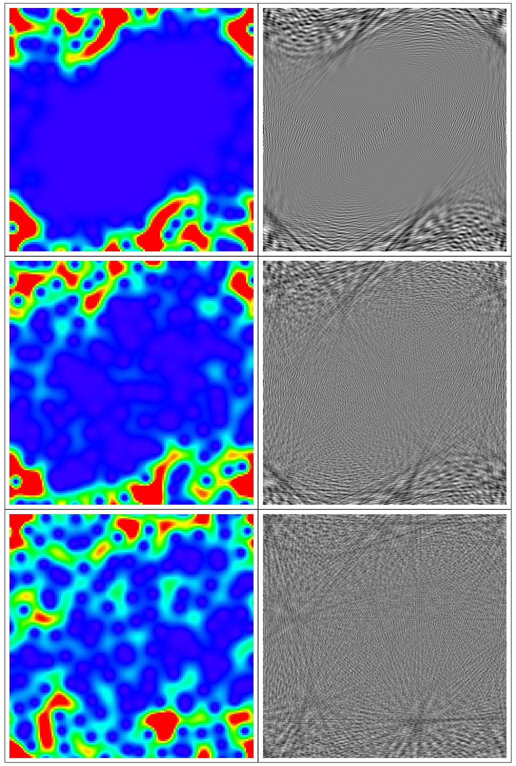
Husimi and Wigner distributions

Plot of Husimi (left) and Wigner (right) distributions

Hilbert space has dimension $N = 2^{n_q}$ with $n_q = 7$

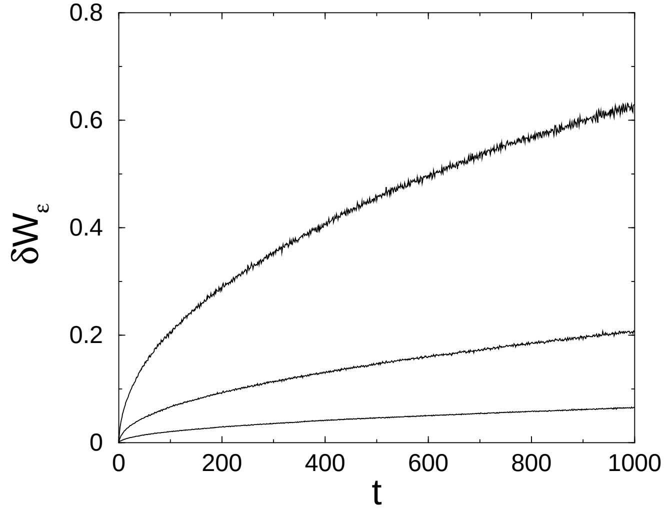
$K = 1.3 > K_g$ and $T = 2\pi/N$ (phase space has one cell only)

Top: $\epsilon = 0$; middle: $\epsilon = 0.002$; bottom: $\epsilon = 0.004$.



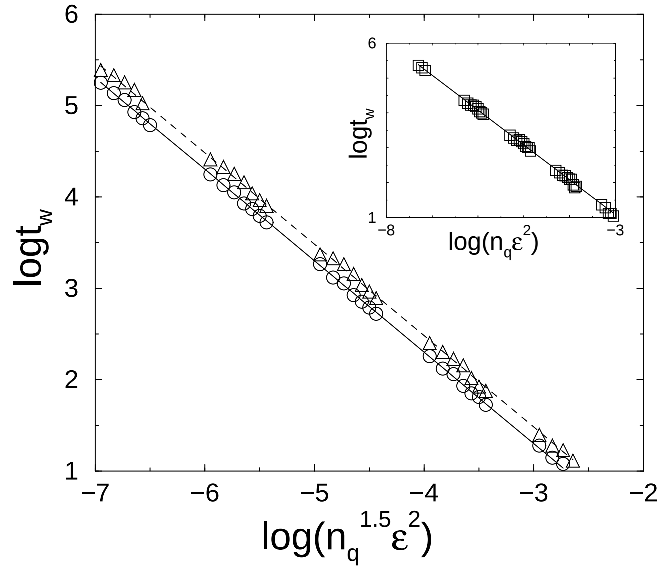
Wigner function

Relative error on the Wigner function



$\delta W = \langle |W - W_\epsilon| \rangle / \langle |W| \rangle$ for $K = K_g$, $T = 2\pi/N$, $N = 2^{n_q}$ and $n_q = 10$ in the chaotic zone. From bottom to top $\epsilon = 10^{-4}$, $\epsilon = 10^{-3.5}$, $\epsilon = 10^{-3}$.

Time scale



$t_W \approx C_W / (n_q^\alpha \epsilon^2)$

Quantum chaos in many-body systems

- Disordered interacting many-body systems: Very common around us (nuclei, atoms, quantum dots,etc...)
- When interaction is sufficiently large, wavefunctions of the system (eigenfunctions of the Hamiltonian) are often ergodic and their statistical properties are described by Random Matrix Theory $\Rightarrow$ “Quantum chaos regime”
- Applicability of Random Matrices? Transition to quantum chaos? Critical interaction strength for apparition of this regime?
- Many works $\rightarrow$ answers for interacting fermions, atoms, spin glass shards, etc..
- In presence of static imperfections, a quantum computer corresponds to this type of systems

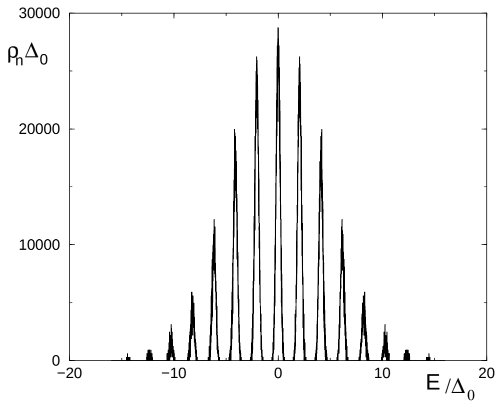
Static imperfections in a quantum computer

B.Georgeot and D. Shepelyansky, Phys. Rev. E 62, 3504 (2000); *ibid.* 62, 6366 (2000).

- energy between the two states of qubits may fluctuate
- interaction between qubits is necessary to perform two-qubit gates; $\Longrightarrow$ residual random couplings

Quantum Computer Model:
$H = \sum_i \Gamma_i \sigma_i^z + \sum_{i<j} J_{ij} \sigma_i^x \sigma_j^x$

2D lattice; J_{ij} **nearest-neighbour** coupling random uniform in $[-J, J]$; Γ_i random in $[\Delta_0 - \delta/2, \Delta_0 + \delta/2]$; σ_i Pauli matrices

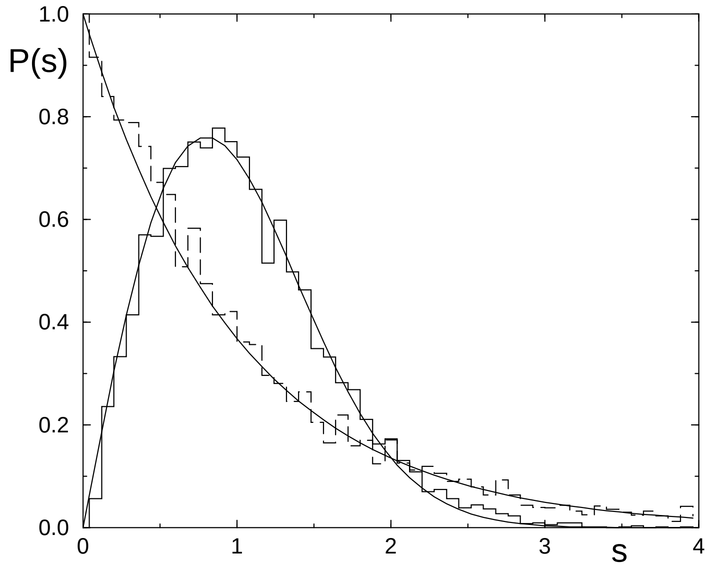


density of states
disordered quantum computer
$(\delta \ll \Delta_0)$

Transition to quantum chaos

n qubits $\Rightarrow$ $N = 2^n$ multi-qubit states ("quantum register states")

Spectral statistics $\Rightarrow$ transition from an integrable regime (Poissonian statistics) at $J/\delta = 0$ to a quantum chaos regime (Random Matrix statistics) for larger J/δ



Level statistics for $n = 16$

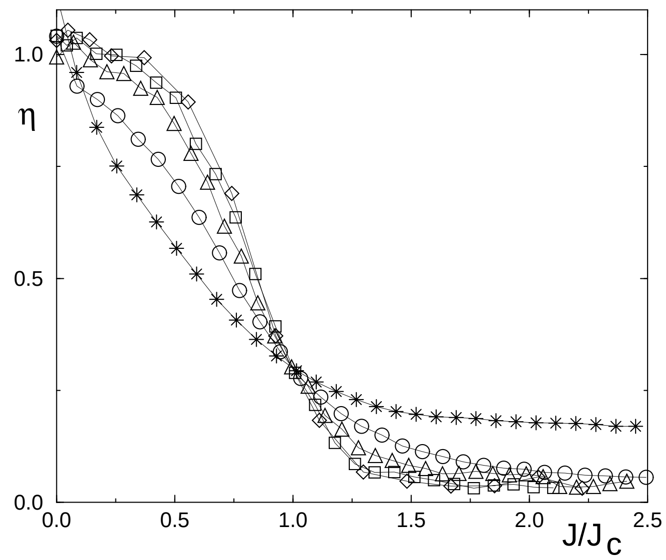
Monitoring the transition

Parameter η: varies continuously from $\eta = 1$ (Poisson) to $\eta = 0$ (Wigner)

$$\eta = \frac{\int_0^{s_0}(P(s)-P_W(s))ds}{\int_0^{s_0}(P_P(s)-P_W(s))ds}.$$

where $s_0 = 0.4729...$ is the intersection point of $P_P(s)$ and $P_{WD}(s)$

Results: sharp transition (can be smooth in other systems)

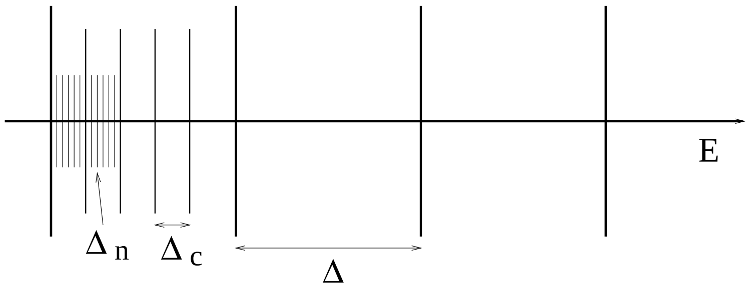


Transition for $n = 6, 9, 12, 15, 16$

Energy scales

Hamiltonian: sparse random matrix

two-body interaction $\Rightarrow$ three energy scales:



$\Delta_0 =$ one-particle level spacing

$\Delta_c =$ level spacing between directly coupled multi-particle states

$\Delta_n =$ level spacing between multi-particle states: $\ll \Delta_c$

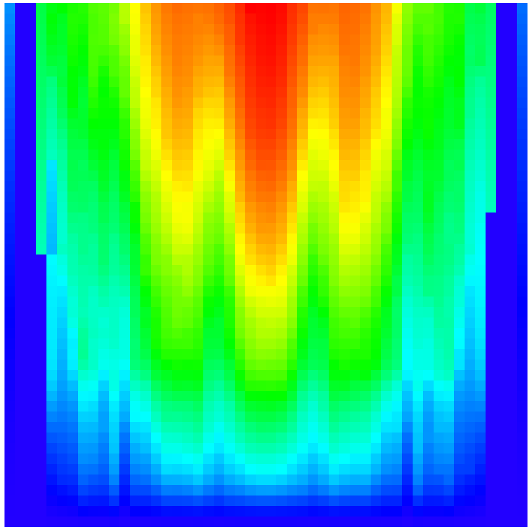
Quantum chaos border

Quantum chaos sets in for $J > J_c$.

Chaos $\Rightarrow$ mixing of exponentially many multi-qubit states, ergodicity.
$\Rightarrow$ "melting" of the quantum computer.
$\Rightarrow$ **destruction** of the computer
without coupling to the environment

sharp transition

Energy scales:
Example: nuclear spins, 1000 qubits;
level spacing $\Delta_n \sim 10^{-298}$ K
$\Delta_c \sim 10^{-3}$ K
residual interaction $J \sim 10^{-5}$ K$\gg \Delta_n$.



Quantum computer melting

Aberg criterion

S. Aberg, Phys. Rev. Lett. 64, 3119 (1990); P. Jacquod and D. Shepelyansky, Phys. Rev. Lett. 79, 1837 (1997).

Chaos $\Rightarrow$ Mixing of many-particle states

Higher orders of perturbation theory can be written in terms of two-particle terms $\Rightarrow$ mixing of many-particle states should happen when two-particle states are mixed

$\Rightarrow$ one can expect critical interaction strength to be $J_c \approx \Delta_c$

Confirmed by numerical simulations in many systems with two-body interactions

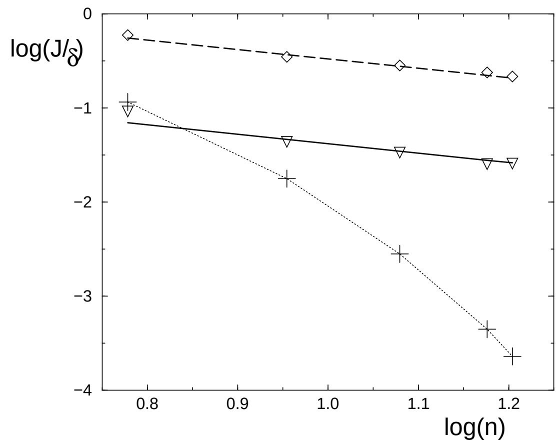
Note that $\Delta_c \gg \Delta_n$

Critical coupling

Theory:

- Aberg criterion $J_c \approx \Delta_c$
- Spectrum for $J = 0$ ($\delta \ll \Delta_0$): n bands with interband distance $2\Delta_0$ and width $\sqrt{n}\delta$.
- In a central band, one multi-qubit state is coupled to around n states in an energy interval 2δ.
- $\Rightarrow$ $J_c \approx \Delta_c \sim \delta/n \gg \Delta_n \sim n^{3/2}2^{-n}\delta$.

Numerical results:



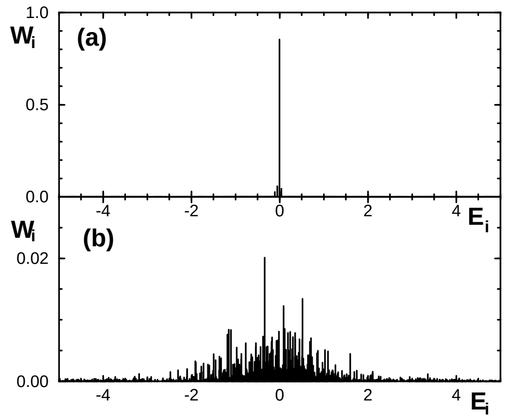
Dashed: J_c: critical interaction measured by spectral statistics
Full: J_{cs}: critical interaction measured by entropy
Dotted: Δ_n mean level spacing

Spreading of eigenstates

- Inverse participation ratio (IPR) $\xi = 1/\sum_i |W_{im}|^4$
- Quantum eigenstate entropy $S_q = -\sum_i W_{im} \log_2 W_{im}$

W_{im} =quantum probability to find the quantum register state $|\psi_i\rangle$ in the eigenstate $|\phi_m\rangle$ of the Hamiltonian ($W_{im} = |\langle\psi_i|\phi_m\rangle|^2$).

- $S_q = 0$ and $\xi = 1$ if $|\phi_m\rangle$ is one quantum register state ($J = 0$)
- $S_q = 1$ and $\xi = 2$ if $|\phi_m\rangle$ is equally composed of two $|\psi_i\rangle$
- Maximal value is $S_q = n$ and $\xi = 2^n$ if all 2^n states contribute equally to $|\phi_m\rangle$.

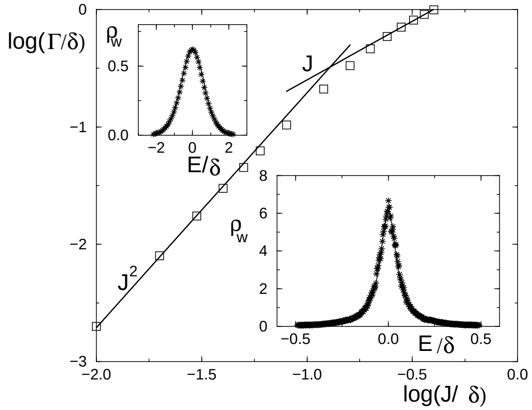


W_{im} for nonchaotic eigenstate (top) and chaotic eigenstate (bottom)

Local density of states (LDOS)

$\rho_W(E - E_i) = \sum_m W_{im} \delta(E - E_m)$

- **Breit-Wigner form:**
  $\rho_{BW}(E - E_i) = \frac{\Gamma}{2\pi((E-E_i)^2+\Gamma^2/4)}$
  valid when Γ is smaller than the bandwidth ($\Gamma < \sqrt{n}\delta$) and many levels are contained inside this width. In this regime, the Breit-Wigner width Γ is given by the Fermi golden rule: $\Gamma = 2\pi U_s^2/\Delta_c$, where U_s is the root mean square of the transition matrix element and $1/\Delta_c$ is the density of directly coupled states. In our case $U_s \sim J$ and $\Delta_c \sim \delta/n$, so that $\Gamma \sim \frac{J^2 n}{\delta}$.
- **Gaussian form:** for large J, when $\Gamma > \sqrt{n}\delta$, ρ_W becomes close to a Gaussian, whose width grows like $\Gamma \sim J$. The change from one dependence to the other takes place for $J > \delta/n^{1/4}$.



ρ_W for quantum computer

For $J > J_c$:

IPR $\xi \approx \Gamma/\Delta_n \sim J^2/(\Delta_c \Delta_n)$

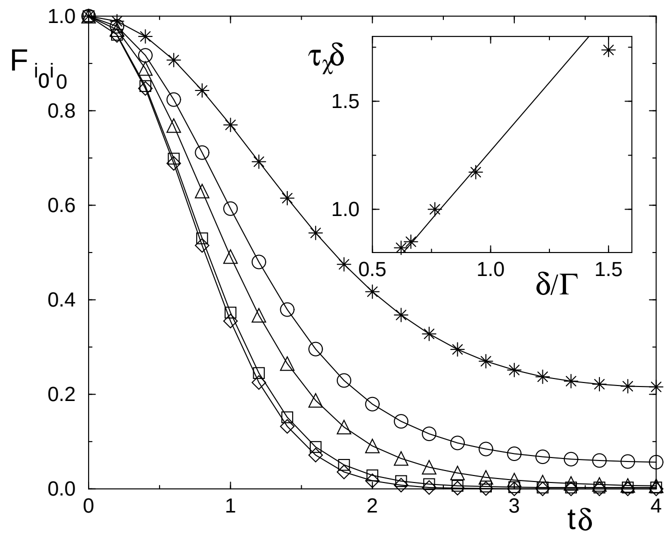
Time scales

Start with a quantum register state
$|\psi(0)\rangle = |\psi_{i_0}(0)\rangle$
Fidelity$= F(t) = |\langle\psi(t)|\psi_{i_0}(t)\rangle|^2$
Fidelity decay=Fourier transform of local density of states
For $J > J_c$**:**

- Breit-Wigner form width $\Gamma \Rightarrow$ exponential decay $\sim e^{-\Gamma t}$
- Gaussian shape width $\Gamma \Rightarrow$ gaussian decay $\sim e^{-\Gamma^2 t^2}$
- $\Rightarrow$ time scale $\tau \sim 1/\Gamma$

see also V. Flambaum, Aust. J. Phys. **53**, N4 (2000).

Additional presence of **phase errors**



Fidelity decay for
$J/\delta = 0.4 \gg J_c/\delta$
from left to right
$n = 16, 15, 12, 9, 6.$
Inset: chaotic time scale τ
as a function of $1/\Gamma$.

Static errors while an algorithm is performed

$$H = \sum_i \Gamma_i \sigma_i^z + \sum_{i<j} J_{ij} \sigma_i^x \sigma_j^x$$

2D lattice; J_{ij} **nearest-neighbour** coupling random uniform in $[-J, J]$; Γ_i random in $[\Delta_0 - \delta/2, \Delta_0 + \delta/2]$; σ_i Pauli matrices

We make the approximation that this Hamiltonian acts during a time τ_g between each gate which is taken as instantaneous.

One single rescaled parameter ε describes the amplitude of these static errors, with $\varepsilon = \delta\tau_g = J\tau_g$.

Kicked Harper model

B. Lévi and B. Georgeot Phys. Rev. E 70, 056218 (2004)

$\bar{n} = n + K \sin\theta \ , \ \bar{\theta} = \theta - L \sin\bar{n}$

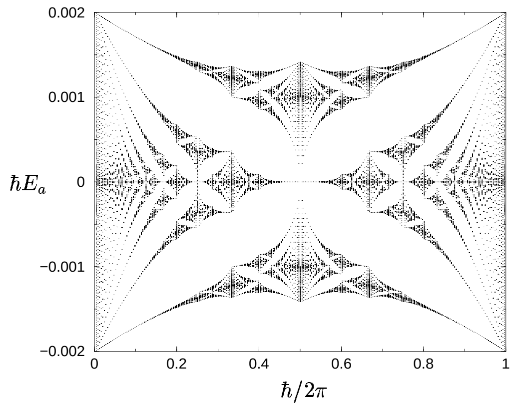
Transition to chaos as K, L increase

Quantization: $\bar{\psi} = e^{-iL\cos(\hbar\hat{n})/\hbar} e^{-iK\cos(\hat{\theta})/\hbar} \psi$

$K = L \to 0$ gives Harper model with **fractal spectrum**

dynamical localization $\longrightarrow$ similar to Anderson localization of electrons in solids

transition to a **partially delocalized regime**, with coexistence of localized and delocalized states



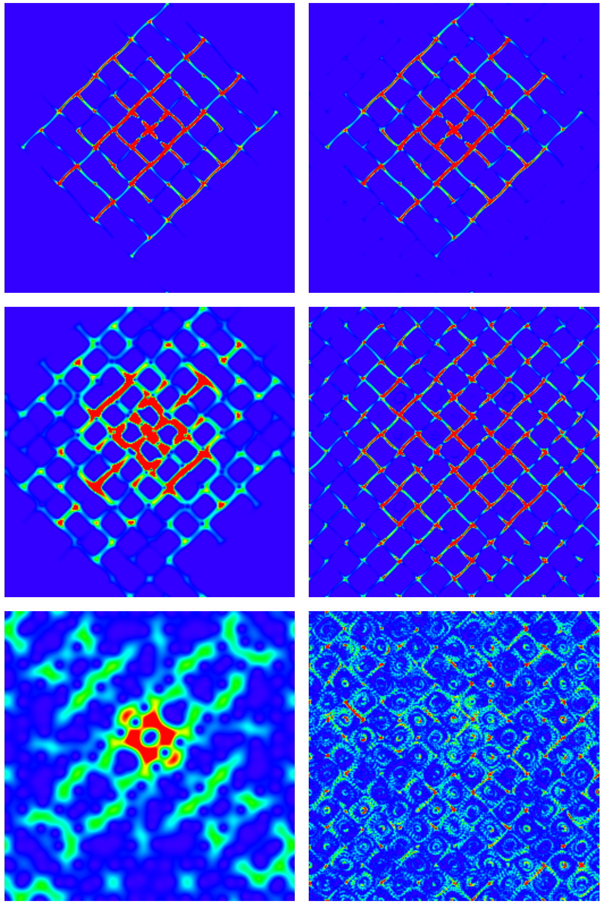
spectrum for $K, L \Rightarrow 0$

Quantum stochastic web: Husimi distribution

K, L very small $\Rightarrow$ small chaotic layer surrounding large integrable islands **“stochastic web”**
$\Rightarrow$ transport=diffusion through layer+tunneling
$\Rightarrow$much faster than classical

Figure: quantum stochastic web; Left: $\varepsilon = 0$ and from top to bottom $n_q = 14$, $n_q = 11$, $n_q = 8$; right: $n_q = 14$ and from top to bottom $\varepsilon = 10^{-6}$, $\varepsilon = 10^{-5}$, $\varepsilon = 10^{-4}$.

Effect of static errors: relative error is 1/2 for $t_h \approx C_h/(\varepsilon n_q^{1.23})$

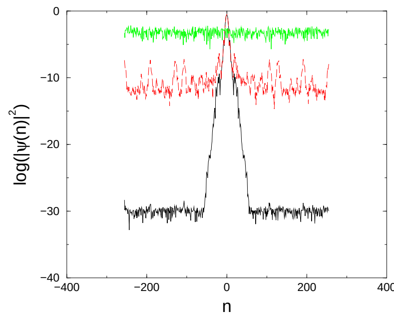


Localized regime: effect of static imperfections

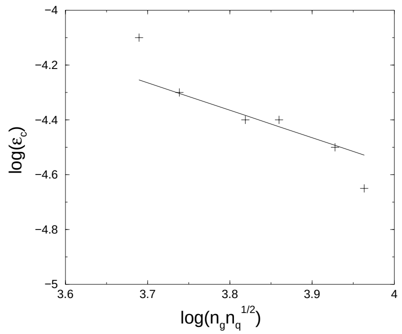
One should consider the eigenstates of the evolution operator $\hat{U}$ of the **unperturbed system** (kicked Harper) instead of quantum register states.
All states localized $\Rightarrow$ an eigenstate is **coupled to only** $\sim l$ **neighbouring states** with typical matrix element $V_{typ} \sim \varepsilon n_g \sqrt{n_q}/\sqrt{l}$

$\Rightarrow$ Quantum chaos border $\varepsilon_c \approx C_1/(n_g\sqrt{n_q}\sqrt{l})$

$\varepsilon \ll \varepsilon_c \Rightarrow l$ can be measured for very long times
$\varepsilon \gg \varepsilon_c \Rightarrow l$ can be measured up to $t \sim 1/\sigma \sim 1/(\varepsilon n_g \sqrt{n_q})$



example of wave function, for $\varepsilon = 0, 10^{-7}, 10^{-3}$



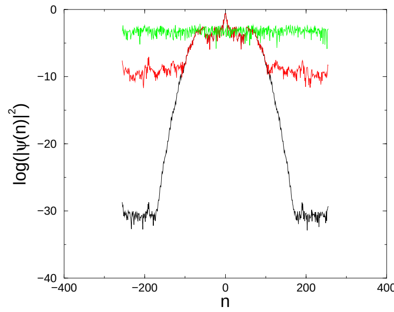
critical ϵ

Partially delocalized regime: effect of static imperfections

A certain fraction β of eigenstates of $\hat{U}$ (unperturbed) are **delocalized**.
In this case, matrix element between states with **at least one delocalized** is $\Rightarrow V_{typ} \sim \varepsilon n_g \sqrt{n_q}/\sqrt{N}$

$\Rightarrow$ Quantum chaos border $\varepsilon_c \approx C_2/(n_g \sqrt{n_q}\sqrt{N})$ exponentially small: $N = 2^{n_q}$! (cf also G. Benenti et al, Eur. Phys. J. D **20**, 293 (2002))

$\varepsilon \gg \varepsilon_c \Rightarrow$ observables measurable up to time $t \sim 1/\sigma \sim 1/(\varepsilon n_g \sqrt{n_q})$



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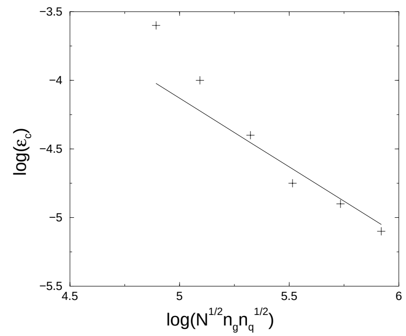

critical ϵ

Concluding remarks

- Quantum chaos tools allow to analyze effects of static imperfections in quantum computer
- Random Matrix Theory can be applied to give the fidelity decay in presence of imperfections. Different regimes identified (K. M. Frahm, R. Fleckinger and D. L. Shepelyansky, Eur. Phys. J. **D 29**, 139 (2004)), in general static imperfections parametrically more dangerous than random noise in the gates.
- PAREC method: **error correction of static errors** without **extra qubits** (O.Kern, G.Alber and D.Shepelyansky, EPJD 2005): destroys coherence of static errors to bring them on par with random noise.