Quantum and Classical Loschmidt Echoes

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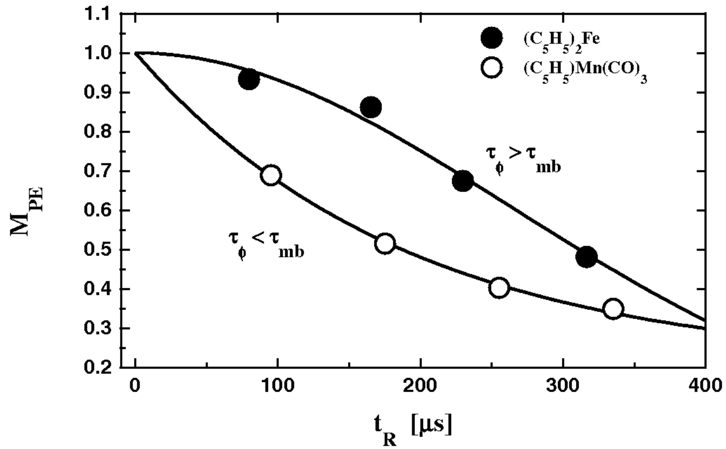
Motivation:
Why to study Loschmidt echoes?



- To understand origins of macroscopic irreversibility from reversible microscopic equations of motion...
- ...both classically and quantum mechanically.
- ...in relation to chaotic or solvable (integrable) nature of the underlying equations of motion.
- To understand, and therefore engineer roboust quantum information processing.

Outline of the Course

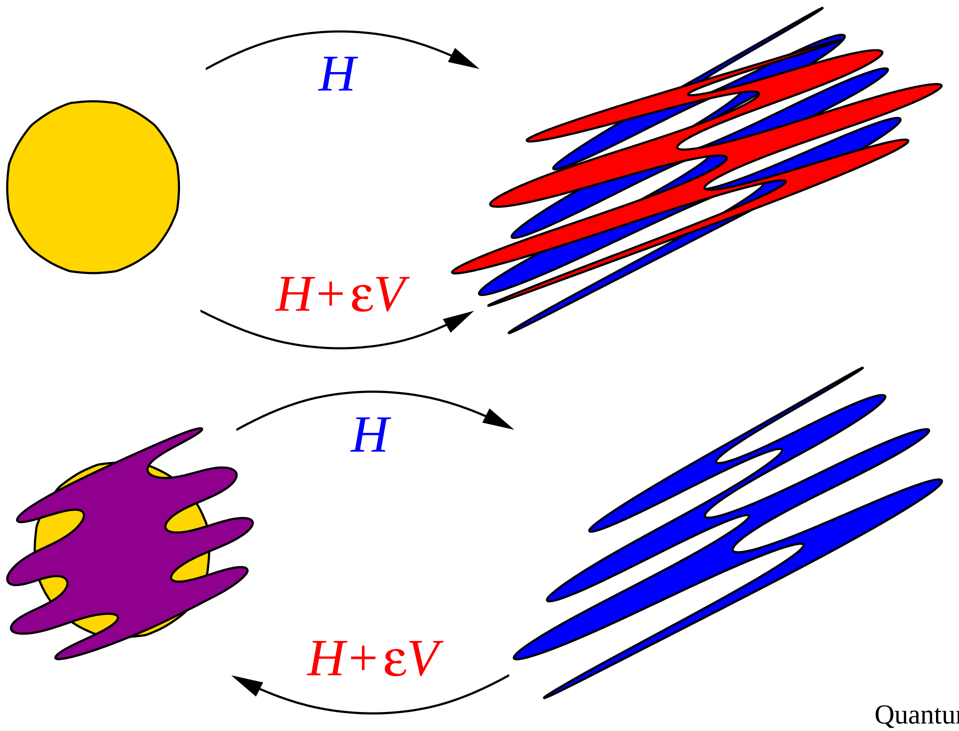
1. General theoretical framework
2. Quantum echo-dynamics: Non-integrable (chaotic) case
3. Random matrix theory of echo-dynamics
4. Quantum echo-dynamics: Integrable case
5. Classical echo-dynamics
6. Time scales and transition from regular to chaotic
7. Application to Quantum Information

Prologue: Classical Loschmidt echoes

Consider two sligtly different systems, $h(\vec{x})$ and $h(\vec{x}) + \epsilon v(\vec{x})$.

Classical fidelity or classical Loschmidt echo:

$$
\begin{aligned}
f(t) = \langle \rho(t) \rho_\epsilon(t) \rangle &= \langle \exp(\mathcal{L} t) \rho_0 \exp(\mathcal{L}_\epsilon t) \rho_0 \rangle \\
&= \langle \rho_0 \exp(-\mathcal{L} t) \exp(\mathcal{L}_\epsilon t) \rho_0 \rangle
\end{aligned}
$$



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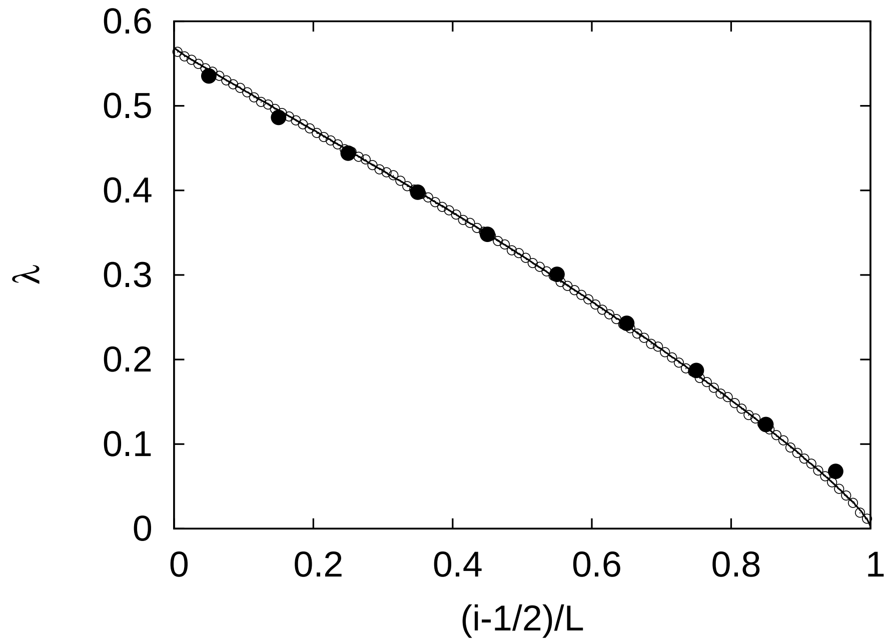

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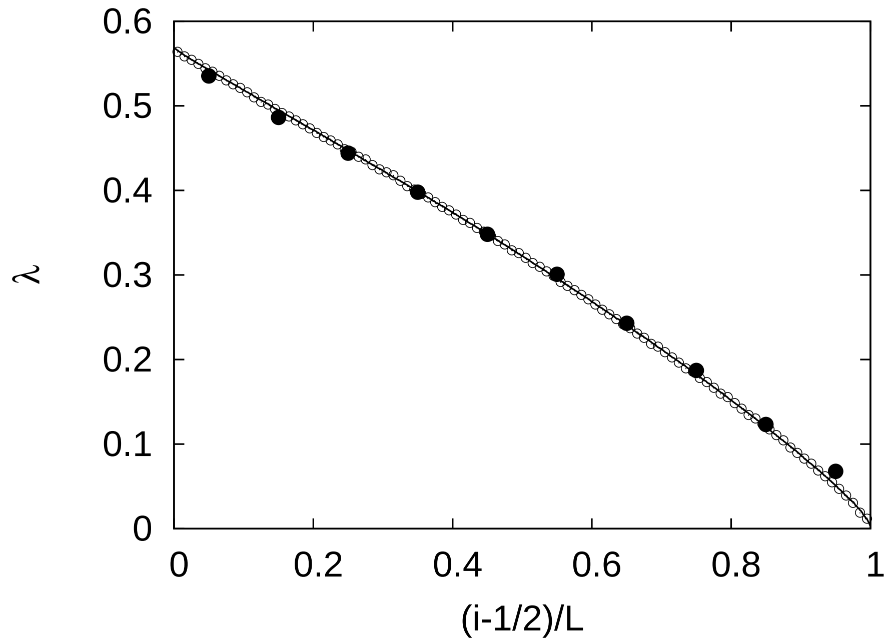

1 General theoretical framework

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H_\varepsilon(t) &= H_0 + \varepsilon \cdot V \\
U_\varepsilon(t) &= \hat{T} \exp(-\mathrm{i} \int_0^t \mathrm{d}t' H_\varepsilon(t')/\hbar)
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We study *unperturbed* and *perturbed* time-evolutions

$$|\psi_0(t)\rangle = U_0(t)|\psi\rangle, \quad \text{and} \quad |\psi_\delta(t)\rangle = U_\varepsilon(t)|\psi\rangle,$$

and define the fidelity

$$F(t) = |f(t)|^2, \qquad f(t) = \langle\psi_0(t)|\psi_\varepsilon(t)\rangle = \langle\psi|M_\varepsilon(t)|\psi\rangle$$

in terms of an expectation value of the echo-operator

$$M_\varepsilon(t) = U_0^\dagger(t)U_\varepsilon(t)$$

(1) Echo operator

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It satisfies

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with effective Hamiltonian $\varepsilon \tilde{V}(t)$,

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It is a solution of

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(1) Born expansion and linear response

The equation for the echo-operator can be (formally) solved in terms of a power series

$$M_\varepsilon(t) = \mathbf{1} + \sum_{m=1}^{\infty} \frac{(-\mathrm{i}\varepsilon)^m}{\hbar^m m!} \int_0^t \mathrm{d}t_1 \mathrm{d}t_2 \cdots \mathrm{d}t_m \hat{T} \tilde{V}(t_1) \tilde{V}(t_2) \cdots \tilde{V}(t_m).$$

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Truncating at second order $m = 2$ and putting into expression $F(t) = |\langle M_\varepsilon(t)\rangle|^2$ we obtain

$$F_\varepsilon(t) = 1 - \frac{\varepsilon^2}{\hbar^2} \int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t'' C(t', t'') + O(\varepsilon^4)$$

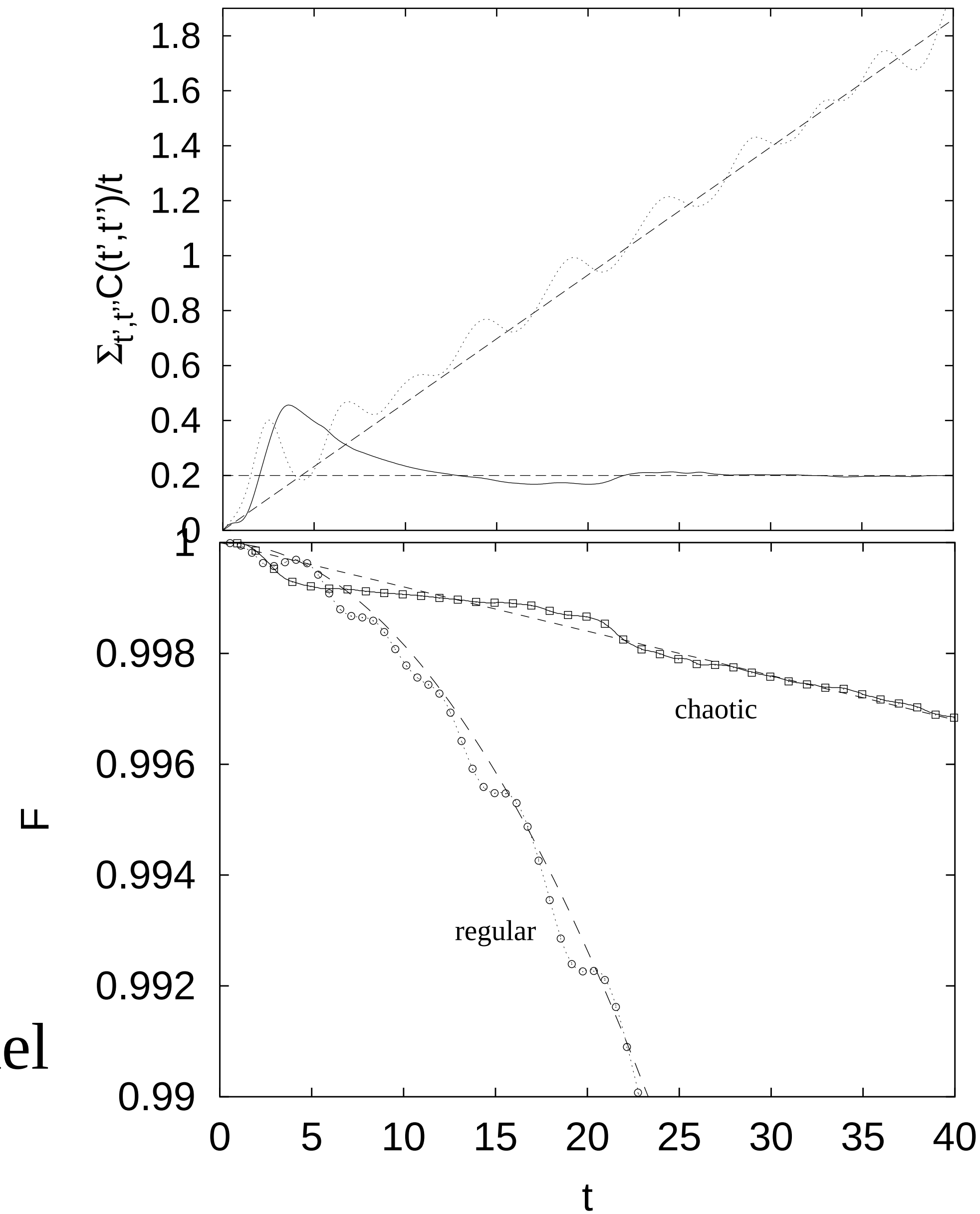
where $C(t', t'') = \langle \tilde{V}(t') \tilde{V}(t'') \rangle - \langle \tilde{V}(t') \rangle \langle \tilde{V}(t'') \rangle$,

is just 2-point time-correlations function of the perturbation.

(1) Illustration: Chaotic vs. Regular dynamics

- Chaotic dynamics,
  $C(t', t'') \to C(t'' - t')$
  after $\sim \log \hbar$ time,
  $C(t \to \infty) \to 0. \;\Rightarrow$
  $F(t) = 1 - \text{const} \times \delta^2 t.$
- Regular dynamics, typically
  $\bar{C} = \langle \bar{V}^2 \rangle - \langle \bar{V} \rangle^2 \neq 0. \;\Rightarrow$
  $F(t) = 1 - \text{const} \times \delta^2 t^2.$

Example: Jaynes-Cummings model



(1) Another formulation of linear response

Let us define an integrated perturbation operator $\Sigma(t)$

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Then, the dobly integrated temporal correlation function rewrites in terms of an *uncertainty* of operator $\Sigma(t)$:

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For quantum dynamics with fast decay of memory (correlartions) the growth is diffusive $\langle\Sigma^2\rangle - \langle\Sigma\rangle^2 \propto t$, whereas for regular dynamics one expects ballistic behaviour, $\langle\Sigma^2\rangle - \langle\Sigma\rangle^2 \propto t^2$

(1) Effect of conservation laws

Adding a constant, or conservation law to a perturbation, makes correlation integrals to increase quadratically.

$$C_{\mathrm{i}}(t) = \int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t'' C(t', t'') = \langle \Sigma^2(t) \rangle - \langle \Sigma(t) \rangle^2$$

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Let $\{Q_n, n = 1, 2 \ldots M\}$ be an orthonormalized set of conserved quantities w.r.t. initial state $|\Psi\rangle$, such that $\langle Q_n Q_m \rangle = \delta_{nm}$. Then any time-independent perturbation can be decomposed uniquely as

$$V = \sum_{m=1}^{M} c_m Q_m + V'$$

with coefficients $c_m = \langle V Q_m \rangle$ and V' being the remaining non-trivial part of the perturbation, by construction orthogonal to *all* trivial conservation laws,

$$\langle Q_m V' \rangle = 0, \quad \text{for all } m.$$

In such a case the correlation integral will always grow asimptotically as a quadratic function

$$C_{\rm i}(t) \to \left(\sum_{m=1}^{M} c_m^2 \right) t^2$$

(1) Quantum Zeno regime

For very short times, below a certain time scale t_{Z}, namely before the correlation function starts to decay, $|t'|, |t''| < t_{\mathrm{Z}}$, $C(t', t'') \approx C(0,0) = \langle V^2 \rangle$, the fideily always exhibits (universal) quadratic decay

$$F(t) = 1 - \frac{\varepsilon^2}{\hbar^2} \langle V^2 \rangle t^2$$

for

$$|t| < t_{\mathrm{Z}} = \left(\frac{C(0,0)}{\mathrm{d}^2 C(0,t)/\mathrm{d}t^2} \right)^{1/2} = \hbar \left(\frac{\langle V^2 \rangle}{\langle [H_0, V]^2 \rangle} \right)^{1/2}$$

(1) Temporally stochastic perturbations

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Then the Born series expansion of fidelity can be summed up to all orders with the result

$$F(t) = \exp\left(-\frac{\varepsilon^2}{\hbar^2}v^2 t\right).$$

Excercise for students: prove (derive) the above formula!

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For stochastic uncorrelated perturbations fidelity thus decays exponentially with the rate which only depends on the magnitude of perturbation only and not on dynamics of the unperturbed system.

(1) Baker-Campbell-Hausdorff expansion

Applying BCH expansion $e^A e^B = \exp(A + B + (1/2)[A, B] + \ldots)$:

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\begin{aligned}
M_\varepsilon(t) &= \exp\left\{ -\mathrm{i}\frac{\varepsilon}{\hbar}\int_0^t \mathrm{d}t' \tilde{V}(t') + \frac{\varepsilon^2}{2\hbar^2}\int_0^t \mathrm{d}t' \int_{t'}^t \mathrm{d}t'' [\tilde{V}(t'), \tilde{V}(t'')] + \ldots \right\} \\
&= \exp\left\{ -\frac{\mathrm{i}}{\hbar}\left(\Sigma(t)\varepsilon + \frac{1}{2}\Gamma(t)\varepsilon^2 + \ldots \right) \right\}
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- Third and fourth order of BCH expansion can be estimated by $\mathrm{const}\varepsilon^3 t$ and $\mathrm{const}\varepsilon^4 t^2$, respectively.

2nd order BCH expansion provides good approximation of the echo operator up to times $O(\varepsilon^{-1})$.

(1) Perturbations with zero time average

Def: time average of the perturbation operator

$$\bar{V} = \lim_{t\to\infty} \frac{\Sigma(t)}{t} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathrm{d}t' \tilde{V}(t').$$

Arbitrary perturbation V can be decomposed into its time average $\overline{V}$ (diagonal in eigenbasis of H_0) and the residual part V_{res} (offdiagonal in eigenbasis of H_0)

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$V = (d/dt)W = (\mathrm{i}/\hbar)[H_0, W]$.
Then $\Sigma(t) = U_0(-t) W U_0(t) - W = W(t) - W(0)$, and within linear reponse, fidelity is frozen

$$F(t) = 1 - \frac{\varepsilon^2}{\hbar^2}(\langle \Sigma^2(t) \rangle - \langle \Sigma(t) \rangle^2) \geq 1 - 4\frac{\varepsilon^2}{\hbar^2} r^2, \qquad r^2 = \sup_t \left[\langle W(t)^2 \rangle - \langle W(t) \rangle^2 \right].$$

(1) Situations of fidelity freeze

- When the perturbation V can be wrtitten as time-derivative.
- When the unperturbed system is invariant under a certain unitary symmetry operation P, say parity, $PH_0 = H_0P$, whereas the symmetry changes sign of the perturbation $PV = -VP$.
- When the unperturbed system is invariant under a certain anti-unitary symmetry operation T, say time-reversal, $TH_0 = H_0T$, whereas the symmetry changes sign of the perturbation $TV = -VT$.
- If diagonal elements of the perturbation are taken out *by hand* and put to the unperturbed part (“mean-field”).

(1) Time averaged fidelity

$$\bar{F} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathrm{d}t' F(t')$$

Let E_k and cR E_k^ε denote the energy spectra, and $P_{kl} = \langle E_k | E_l^\varepsilon \rangle$ transition matrix between perturbed and unperturbed systems.

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If ρ is a general (mixed) initial state, then fidelity amplitude can be written out

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Assuming spectra to be non-degenerate:

$$\bar{F} = \sum_{ml} |(\rho P)_{ml}|^2 |P_{ml}|^2.$$

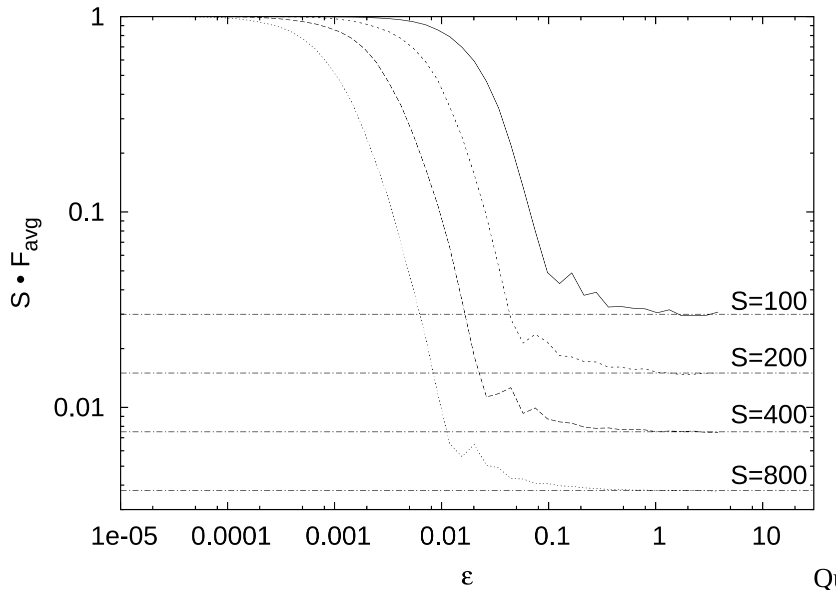
(1) Time averaged fidelity cont.

In case of weak perturbation $\varepsilon \ll \varepsilon_*$, $P \to \mathbb{1}$ for

$$\bar{F}_{\text{weak}} = \sum_l \rho_{ll}^2$$

In case of strong perturbation $\varepsilon > \varepsilon_*$ and strongly non-integrable dynamics, we may assume P to be random orthogonal $\beta = 1$ (unitary $\beta = 2$) matrix and

$$\bar{F}_{\text{strong}} = \frac{4-\beta}{N} \sum_l \rho_{ll}^2 + \frac{1}{N} \sum_{l,n}^{l \neq m} |\rho_{lm}|^2.$$



(1) State averaged fidelity

Let us for example consider averaging over an ensemble of random initial states $|\Psi\rangle = \sum_n c_n |E_k\rangle$:

$$\langle\langle\Psi|A|\Psi\rangle\rangle =: \langle\langle A\rangle\rangle = \langle\langle\sum_{ml} c_m^* \, A_{ml} \, c_l\rangle\rangle = \frac{1}{N} \operatorname{tr} A.$$

Then, assuming that in the limit $N \to \infty$ coeeficients c_n are gaussian uncorrelated and random, we obtain

$$\langle\langle F(t)\rangle\rangle = \sum_{mlpr} \langle\langle c_m^* [M_\varepsilon(t)]_{ml} \, c_l \, c_p \, [M_\varepsilon(t)]_{pr}^* \, c_r^* \rangle\rangle = |\langle\langle f(t)\rangle\rangle|^2 + \frac{1}{N}.$$

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Using the Heisenberg uncertainty relation for the operators P_ε and V,

$$\delta V(t)\delta P_\varepsilon(t) \leq \frac{1}{2}\left|\langle\Psi_0(t)|[P_\varepsilon, V]|\Psi_0(t)\rangle\right|$$

we can estimate the time-derivative of fidelity

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Separating the variables and integrating by parts we get an inequality:

$$F(t) \ge \cos^2(\phi(t)), \quad \phi(t) = \frac{\varepsilon}{\hbar}\int_0^t \mathrm{d}t'\delta V(t')$$

(1) Polarization echo

Prepare intial state $|\Psi_0\rangle$ as eigenstate of certain obserbable, say polarization of a local spin s_0^z:

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Then, perform an echo-experiment and measure the local spin:

$$m_\varepsilon(t) = \langle M_\varepsilon^\dagger(t) s_0^{\mathrm{z}} M_\varepsilon(t)\rangle$$

(1) Polarization echo

Prepare intial state $|\Psi_0\rangle$ as eigenstate of certain obserbable, say polarization of a local spin s_0^z:

$$s_0^{\mathrm{z}}|\Psi_0\rangle = m_0|\Psi_0\rangle, \qquad m_0 = \pm 1/2.$$

Then, perform an echo-experiment and measure the local spin:

$$m_\varepsilon(t) = \langle M_\varepsilon^\dagger(t) s_0^{\mathrm{z}} M_\varepsilon(t)\rangle$$

Polarization echo $P_\varepsilon(t)$ is defined as the probability that the local polarization of the spin is restored after the echo dynamics

$$P_\varepsilon(t) = \frac{1}{2} + 2m_0 m_\varepsilon(t) = \frac{1}{2} + 2\langle s_0^{\mathrm{z}} M_\varepsilon^\dagger(t) s_0^{\mathrm{z}} M_\varepsilon(t)\rangle$$

Polarization echo may have different behaviour than fidelity!

(2) Observable echo

For a general observable A, for which the initial state $|\Psi_0\rangle$ has to be an eigenstate, we define an $A-$echo as

$$P_\varepsilon^A(t) = \frac{\langle AM_\varepsilon^\dagger(t)AM_\varepsilon(t)\rangle}{A^2}$$

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Using 2nd-order echo-operator

$$M_\varepsilon(t) = \mathbf{1} - \mathrm{i}\frac{\varepsilon}{\hbar}\Sigma(t) - \frac{\varepsilon^2}{2\hbar^2}\hat{T}\Sigma^2(t) + O(\varepsilon^3)$$

we find

$$P_\varepsilon^A(t) = 1 - \frac{\varepsilon^2}{\hbar^2}\frac{\langle A^2\Sigma^2(t)\rangle - \langle A\Sigma(t)A\Sigma(t)\rangle}{\langle A^2\rangle} + O(\varepsilon^4)$$

(1) Composite systems

We consider two-partite systems: central system plus environment

$$\mathcal{H} = \mathcal{H}_{\rm c} \otimes \mathcal{H}_{\rm e}$$

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We are mainly interested in the reduced density matrix of the central system

$$\rho_{\mathrm{c}}(t) := \operatorname{tr}_{\mathrm{e}}[\rho(t)], \qquad \rho_{\mathrm{c}}^{\mathrm{M}}(t) := \operatorname{tr}_{\mathrm{e}}[\rho^{\mathrm{M}}(t)],$$

where $\rho^{\mathrm{M}}(t) = M_{\varepsilon}(t)\rho(0)M_{\varepsilon}^{\dagger}(t)$.

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where $\rho^{\mathrm{M}}(t) = M_\varepsilon(t)\rho(0)M_\varepsilon^\dagger(t)$.
We assume that initial state is a product state

$$|\psi(0)\rangle = |\psi_{\mathrm{c}}(0)\rangle \otimes |\psi_{\mathrm{e}}(0)\rangle =: |\psi_{\mathrm{c}}(0); \psi_{\mathrm{e}}(0)\rangle.$$

(1) Composite systems cont.

Fidelity of the total unitary evolution (system+environment) is

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$$F_{\mathrm{R}}(t) := \mathrm{tr}_{\,\mathrm{c}}[\rho_{\mathrm{c}}(0)\rho_{\mathrm{c}}^{\mathrm{M}}(t)].$$

As a measure of entanglement between the central system and the environment under-echo dynamics we define echo purity

$$F_{\mathrm{P}}(t) := \mathrm{tr}_{\,\mathrm{c}}[\{\rho_{\mathrm{c}}^{\mathrm{M}}(t)\}^2].$$

In case when unperturbed evolution is decoupled $U_0 = U_{\mathrm{c}} \otimes U_{\mathrm{e}}$ the echo purity is identical to purity of the forward evolution!

(1) Inequality between fidelity, reduced fidelity and echo purity

One can prove the following inequality for an arbitrary pure state $|\psi\rangle$ and an arbitrary pure product state $|\phi_{\rm c};\phi_{\rm e}\rangle$

$$|\langle\phi_{\rm c};\phi_{\rm e}|\psi\rangle|^4 \le |\langle\phi_{\rm c}|\rho_{\rm c}|\phi_{\rm c}\rangle|^2 \le \operatorname{tr}_{\rm c}[\rho_{\rm c}^2],$$

where $\rho_{\rm c} := \operatorname{tr}_{\rm e}[|\psi\rangle\langle\psi|]$.

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Specializing to the case of echo-dynamics we find a very useful estimate

$$F^2(t) \le F_{\mathrm{R}}(t)^2(t) \le F_{\mathrm{P}}(t)^2(t)$$

(1) Composite systems: linear response

Expanding the echo operator to second order we can straintforwardly derive the linear response expressions for the measures of echodynamics of composite systems

$$\begin{aligned}
1 - F(t) &= \left(\frac{\varepsilon}{\hbar}\right)^2 \langle \Sigma(t)(\mathbb{1} \otimes \mathbb{1} - \rho_{\mathrm{c}} \otimes \rho_{\mathrm{e}})\Sigma(t)\rangle \\
1 - F_{\mathrm{R}}(t) &= \left(\frac{\varepsilon}{\hbar}\right)^2 \langle \Sigma(t)(\mathbb{1} - \rho_{\mathrm{c}}) \otimes \mathbb{1}\Sigma(t)\rangle \\
1 - F_{\mathrm{P}}(t) &= 2\left(\frac{\varepsilon}{\hbar}\right)^2 \langle \Sigma(t)(\mathbb{1} - \rho_{\mathrm{c}}) \otimes (\mathbb{1} - \rho_{\mathrm{e}})\Sigma(t)\rangle .
\end{aligned}$$

writing the expectation value in the initial product state as usual, $\langle \bullet \rangle = \operatorname{tr}\left[(\rho_{\mathrm{c}} \otimes \rho_{\mathrm{e}})\bullet\right]$.

2 Quantum echo-dynamics: Non-integrable (chaotic) case

We consider here echo-dynamics of strongly non-integrable dynamical systems for which we can assume ergodicity and mixing, at least in the limit $\hbar \to 0$, or thermodynamic limit for systems of many particles.

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In other words, there exist a relaxation time scale t_{E} s.t. for typical physical observable A

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For $t \gg t_{\mathrm{E}}$ we have $C(t', t'') = C(t' - t'')$ and linear response formula for fidelity rewrites as

$$F(t) = 1 - \frac{\varepsilon^2}{\hbar^2} \left\{ tC(0) + 2 \int_0^t \mathrm{d}t' (t - t') C(t') \right\} + O(\varepsilon^4).$$

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If $C(t)$ decays faster than t^{-1} then a characteristic mixing time exist s.t. if $t \gg t_{\mathrm{mix}}$

$$F(t) = 1 - 2(\varepsilon/\hbar)^2 \sigma t, \quad \sigma = \int_0^\infty \mathrm{d}t C(t) = \lim_{t\to\infty} \frac{\langle \Sigma^2(t)\rangle - \langle \Sigma(t)\rangle^2}{2t}.$$

(2) Beyond linear response

$$F_{\mathrm{em}}(t) = 1 - \frac{\delta^2}{\hbar^2}\sigma t, \qquad \sigma = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t C(t)$$

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Further assume: $n-$point mixing, i.e.
$\langle V(t_1)V(t_2)\ldots V(t_{2n-1})V(t_{2n})\rangle \longrightarrow \langle V_{t_1}V_{t_2}\rangle\cdots\langle V_{t_{2n-1}}V_{t_{2n}}\rangle$
if $t_{2j+1} - t_{2j} \gg t_{\mathrm{mix}}$, $j = 1, 2 \ldots n-1$. Then if $t \gg n t_{\mathrm{mix}}$:

$$\hat{\mathcal{T}} \quad \int \mathrm{d}t_1 \cdots \mathrm{d}t_{2n} \langle V(t_1)V(t_2)\cdots V(t_{2n})\rangle \rightarrow$$

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Similar arguments $\Rightarrow$ terms with **odd** number of $V(t)$ vanish in leading order!

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Similar arguments $\Rightarrow$ terms with **odd** number of $V(t)$ vanish in leading order!

We can now sum-up the fidelity to all orders:

$$\boxed{F_{\rm em}(t) = \exp(-t/\tau_{\rm em}), \qquad \tau_{\rm em} = \frac{\hbar^2}{\delta^2\sigma}.}$$

Alternatively, this regime of fidelity decay is usually derived in terms of Fermi-Golden-Rule.

(2) Numerical example: Kicked top

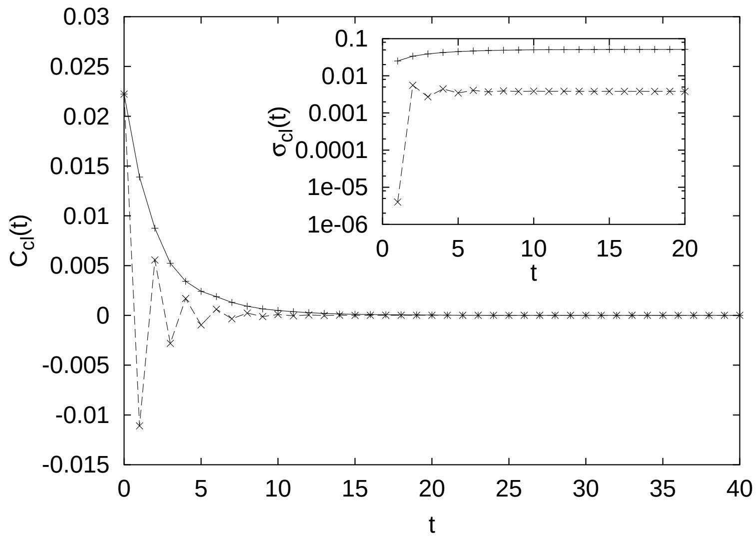
We consider quantized kicked top (Haake et al 1987):

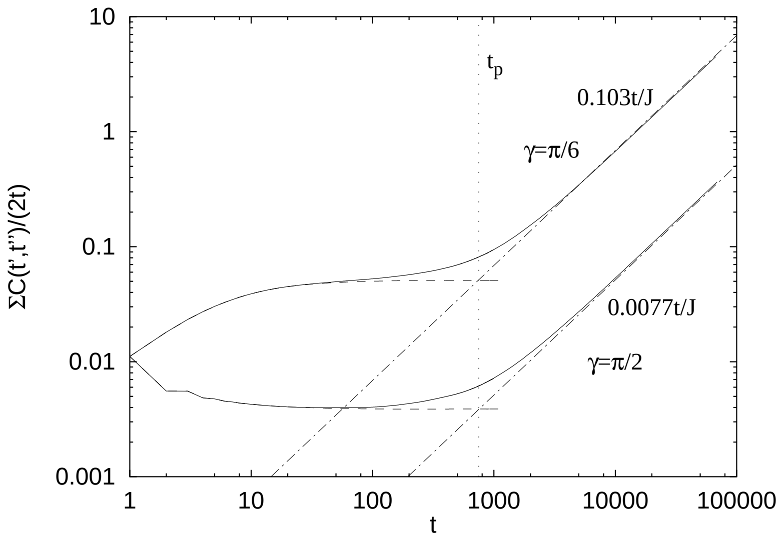
$$H(t) = \frac{1}{2}\alpha\hbar^2 J_z^2 + \sum_{m=-\infty}^{\infty} \delta(t-m)\gamma\hbar J_y,$$

$$U = \exp(-i\gamma J_y)\exp(-i\alpha J_z^2/2J).$$

Classical limit: $\hbar = 1/J \to 0$.

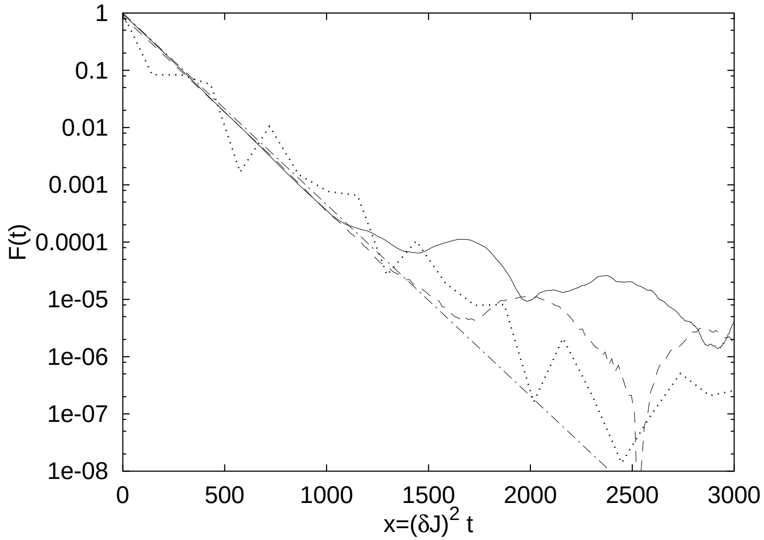
Perturbation: $V = \frac{1}{2}\left(\frac{J_z}{J}\right)^2$.



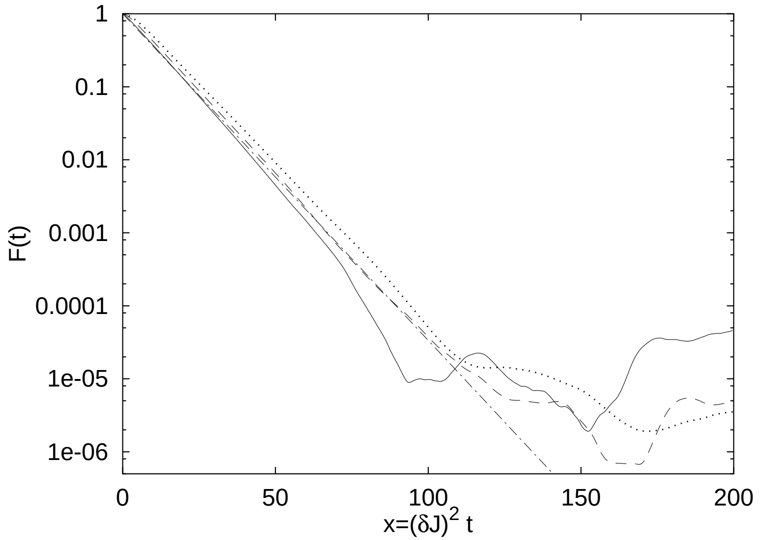


(2) Numerical example: cont.

For $\gamma = \pi/2$:



For $\gamma = \pi/6$:



(2) Beyond Heisenberg time

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Namely, for $t > t_{\mathrm{H}}$ where $t_{\mathrm{H}} = \frac{1}{2}N \propto \hbar^{-d}$ is the Heisenberg time, quantum evolution start to feel discreteness of the spectrum, $C_{\mathrm{i}}(t) = \bar{C}t^2$ with

$$\bar{C} = \frac{4\sigma_{\mathrm{cl}}}{N}.$$

So within linear response

$$F(t) = 1 - \frac{\varepsilon^2}{\hbar^2}\frac{4\sigma_{\mathrm{cl}}}{N}t^2.$$

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(2) Beyond linear response, beyond Heisenberg time...

For very small perturbations $\varepsilon \ll \mathrm{const}/t$, s.t. second term of BCH expansion can be neglected:

$$f(t) = \sum_k \exp\left(-\mathrm{i} V_{kk} \varepsilon t/\hbar\right)/N$$

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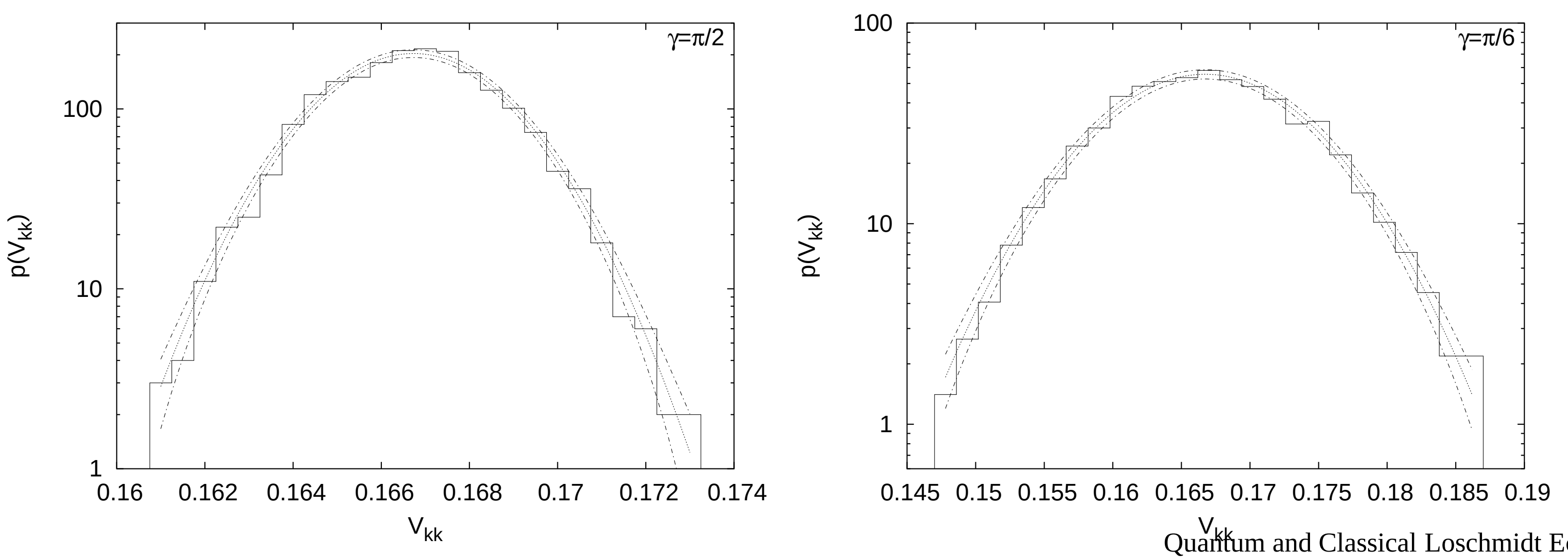
This sum can be computed statistically if V_{kk} are replaced by gaussian random variables with variance $2\sigma/N$

$$F(t) = \exp\left(-(t/\tau_\mathrm{p})^2\right), \qquad \tau_\mathrm{p} = \sqrt{\frac{N}{4\sigma_\mathrm{cl}}}$$

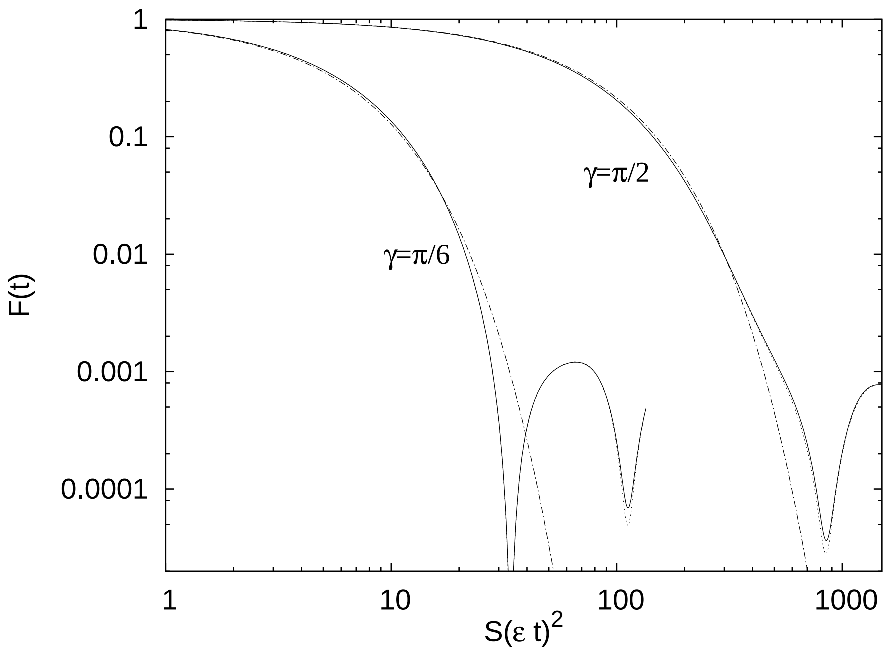


(2) ...is Gaussian (perturbative) fidelity decay

Gaussian decay starts right at the beginning **if** the fidelity decay time scale $\tau_{\rm em}$ becomes longer than $t_{\rm H}$. This gives $\varepsilon > \varepsilon_{\rm p}$ where

$$\varepsilon_{\rm p} = \frac{\hbar}{\sqrt{\sigma_{\rm cl} N}}$$

is the perturbative border.



(2) Vanishing time averaged perturbation and fidelity freeze

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Assume that $V = (d/dt)W$ is a time-derivative, or time-difference for kicked systems.
Then, up to time $t_2 \sim 1/\varepsilon$ the fidelity freezes to a plateau which is given by the first term in BCH expansion:

$$F(t) \approx F_{\text{plat}} = \left| \langle \exp\left(-\frac{\mathrm{i}\varepsilon}{\hbar} w\right) \rangle_{\text{cl}} \langle \exp\left(\frac{\mathrm{i}\varepsilon}{\hbar} W\right) \rangle \right|^2$$

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where w is a classical limit of time-integrtad perturbation W.
For longer times $t > t_2$ we find either exponential or gaussian decay, with rescaled perturbation $\varepsilon \to \varepsilon^2$

$$F(t) \approx F_{\mathrm{plat}} \exp\left(-\frac{\varepsilon^4}{2\hbar^2}\sigma_R t\right), \quad t < t_{\mathrm{H}}$$

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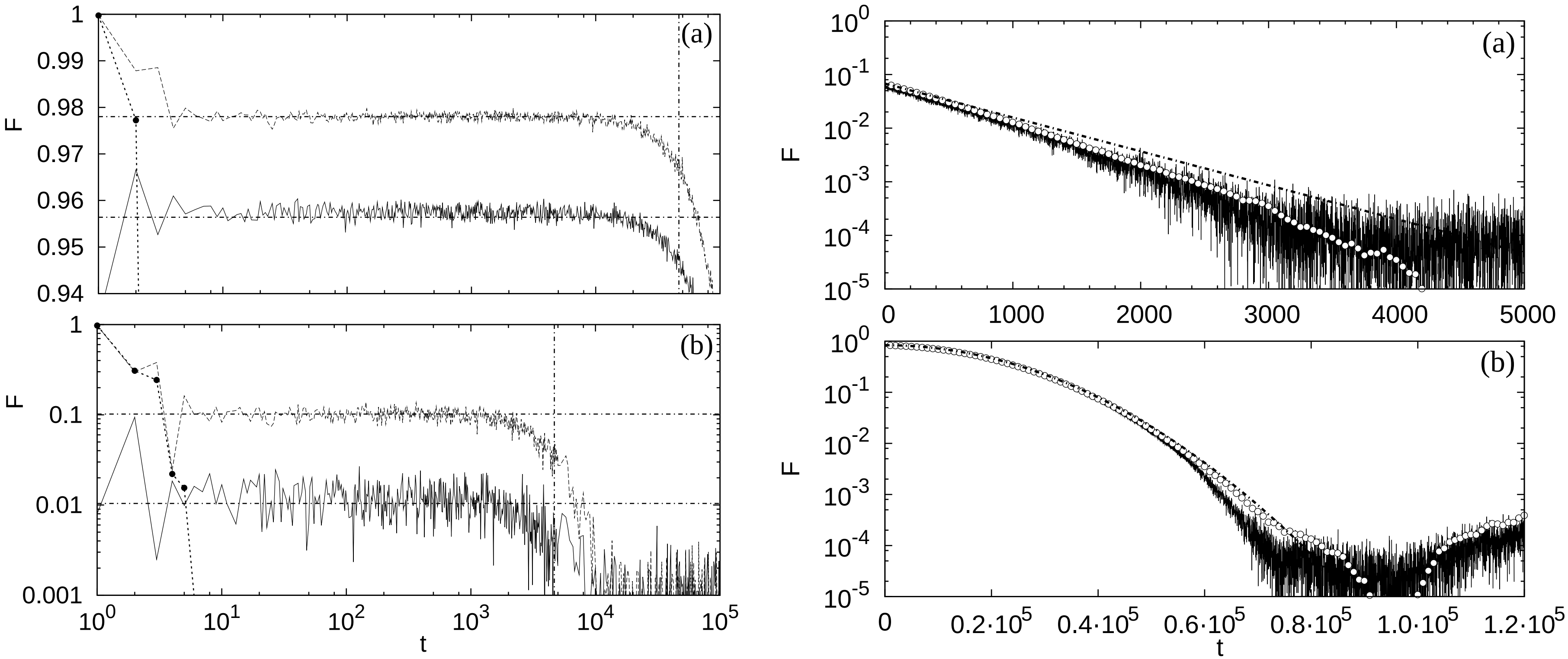
and diffusion constant σ_R computed with respect to observable $R(t) = (\mathrm{i}/\hbar)[W, (d/dt)W]$.

(2) Numerical example: fidelity freeze in kicked top

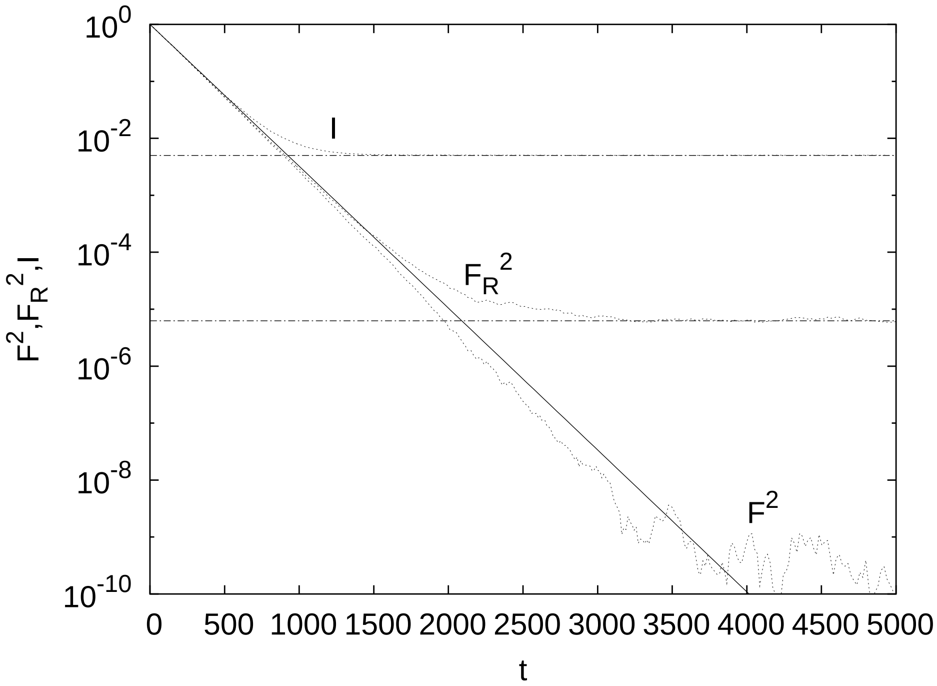
$$V = \frac{S_{\mathrm{x}}^2 - S_{\mathrm{z}}^2}{2S^2}, W = \frac{S_{\mathrm{z}}^2}{2S^2}, R = -\frac{S_{\mathrm{x}} S_{\mathrm{y}} S_{\mathrm{z}} + S_{\mathrm{z}} S_{\mathrm{y}} S_{\mathrm{x}}}{2S^3}.$$



(2) Composite systems

For chaotic dynamics the upper bound of inequalities is reached. In the FGR regime, and in the limit of large dimensions of both subsystems, we have

$$F_{\mathrm{P}}(t) \approx F_{\mathrm{R}}^{2}(t) \approx F^{2}(t) = \exp\left(-2t/\tau_{\mathrm{em}}\right)$$



(2) Semiclassical orbit evaluation of fidelity

Due to Feynman, quantum propagator can be written as

$$\langle\vec{q}|U_\delta(t)|\vec{q}'\rangle = \int_{\vec{r}(0)=\vec{q}'}^{\vec{r}(t)=\vec{q}} \mathcal{D}\vec{r}(t)\exp\left(-\frac{\mathrm{i}}{\hbar}S[\vec{r}(t)]\right),\ S[\vec{r}(t)] := \int_0^t \mathrm{d}t L_\delta(\vec{r},\dot{\vec{r}}$$

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Applying *method of stationary phase*, we obtain (Van-Vleck)

$$\langle \vec{q} | U_\delta(t) | \vec{q}' \rangle = \sum_{\text{cl.paths j}} \left| \det \frac{\partial^2 S_j}{\partial q_m \partial q'_n} \right|^{-d/2} \exp\left(\frac{\mathrm{i}}{\hbar} S_j - \mathrm{i}\frac{\pi}{2}\nu_j \right)$$

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Plugging this into expression

$$F(t) = \left|\int \mathrm{d}\vec{q}\mathrm{d}\vec{q}'\mathrm{d}\vec{q}''\psi_0^*(\vec{q})\langle\vec{q}|U_0(-t)|\vec{q}'\rangle\langle\vec{q}'|U_\delta(t)|\vec{q}''\rangle\psi_0(\vec{q}'')\right|^2$$

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Chaotic dynamics and $t < t_{\mathrm{E}}$: due QC corresp., $F(t) \propto \exp(-\lambda t)$.

(2) Uniform semiclassical formula of Vanicek

Within diagonal approximation of fidelity amplitude, justified by classical shadowing theorem, and for general initial state ρ, descriped by the Wigner function $W_\rho(\vec{q},\vec{p})$, Vanicek derived very elegant semiclassical expression of fidelity amplitude

$$f(t) = \int \mathrm{d}\vec{q}\mathrm{d}\vec{p}\, W_\rho(\vec{q},\vec{p}) \exp\left(-\frac{\mathrm{i}}{\hbar}\varepsilon \int_0^t \mathrm{d}t' v(\vec{q}(t'),\vec{p}(t'),t') \right),$$

where $v(\vec{q},\vec{p},t)$ is a classical limit of the perturbation.

Vanicek formula semiclassically repreduces all the regimes of fidelity decay below the Heisenberg time!

3 Random matrix theory of echo-dynamics

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Take for example $H_\varepsilon = H_0 + \varepsilon V$, where H_0 and V are random, complex hermitean $\beta_V = 2$ (or real symmetric $\beta_V = 1$, or quaternionic symmetric $\beta_V = 4$) matrices.
We fix units and perturbations strength such than mean level spacing of H_0 is 1 (in the center of the band), meaning $t_{\mathrm{H}} = 2\pi$, and variance of (off-diagonal) matrix elements of V is 1, $\langle V_{jk}^2 \rangle = 1$.

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Take for example $H_\varepsilon = H_0 + \varepsilon V$, where H_0 and V are random, complex hermitean $\beta_V = 2$ (or real symmetric $\beta_V = 1$, or quaternionic symmetric $\beta_V = 4$) matrices.
We fix units and perturbations strength such than mean level spacing of H_0 is 1 (in the center of the band), meaning $t_{\mathrm{H}} = 2\pi$, and variance of (off-diagonal) matrix elements of V is 1, $\langle V_{jk}^2 \rangle = 1$.

Our program now is to derive simple expressions for fidelity and other measures of echo-dynamics by averaging over H_0 and V and compare to experimental and numerical data.

(3) Linear response: Fidelity and spectral form factor

Let us recall the expression for the echo-operator to second order

$$M_\varepsilon(t) = \mathbf{1} - \mathrm{i}\,2\pi\varepsilon \int_0^t \mathrm{d}t'\ \tilde{V}(t') - (2\pi\varepsilon)^2 \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t''\ \tilde{V}(t')\,\tilde{V}(t'') + \mathcal{O}(\varepsilon_0^3)\ .$$

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Let us frist make average over an ensemble of Vs. Linear term clearly averages out. For the quadratic term wee need 2-point correlator

$$\begin{aligned} \langle [\tilde{V}(\tau)\,\tilde{V}(\tau')]_{\nu,\nu'} \rangle &= \sum_\mu \langle V_{\nu,\mu}\ V_{\mu,\nu'} \rangle\ \langle e^{2\pi\mathrm{i}[(E_\nu - E_\mu)\tau + (E_\mu - E'_\nu)\tau']} \rangle \\ &= \delta_{\nu,\nu'} \left\{ \frac{2}{\beta_V} + \delta(\tau-\tau') - b_2(\tau-\tau') \right\} \end{aligned}$$

where $b_2(\tau)$ is the 2-point spectral form factor of H_0.

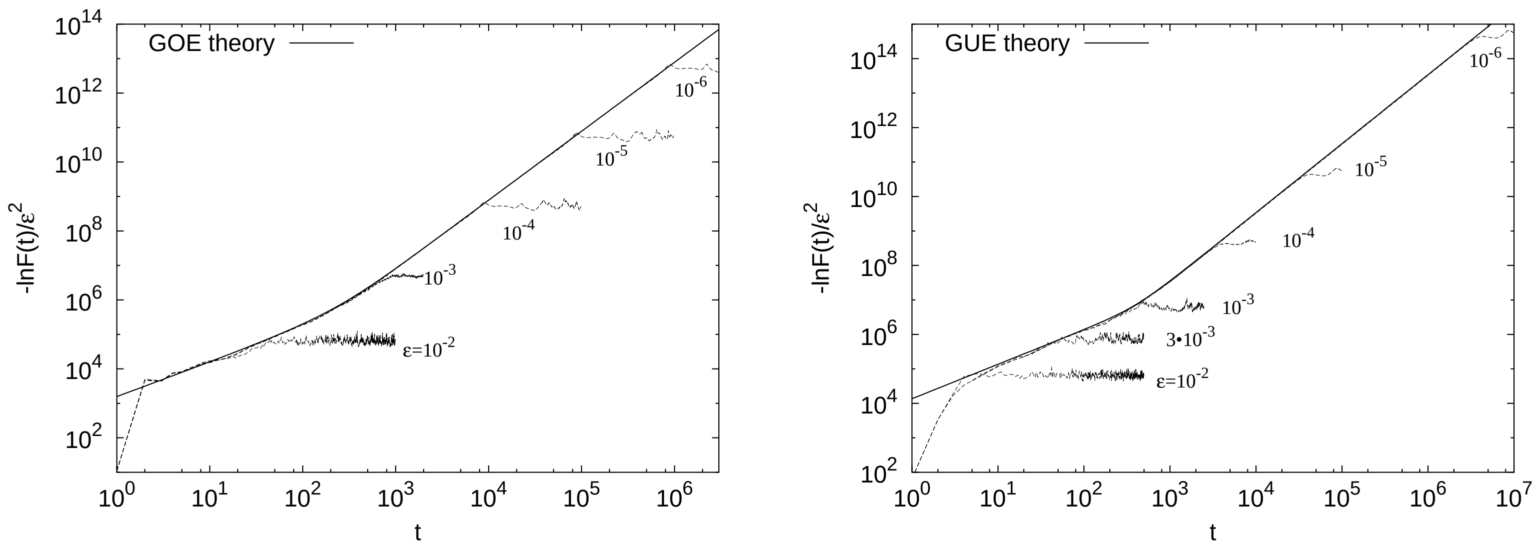
(3) Linear response cont.

Plugging the correlation function expression into the formula we obtain ensemble averaged fidelity amplitude

$$\langle f_\varepsilon(t)\rangle = 1 - (2\pi\varepsilon)^2 \left[t^2/\beta_V + t/2 - \int_0^t \mathrm{d}\tau' \int_0^{\tau'} \mathrm{d}\tau\, b_2(\tau) \right] + \mathcal{O}(\varepsilon^4)\ .$$

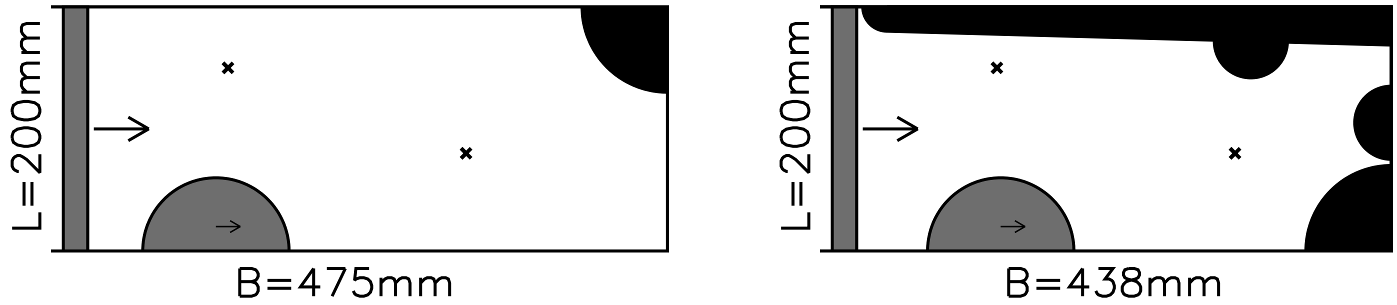
Conjecture: for *not-large* perturbations and fidelities down-to $f \sim 0.1$:

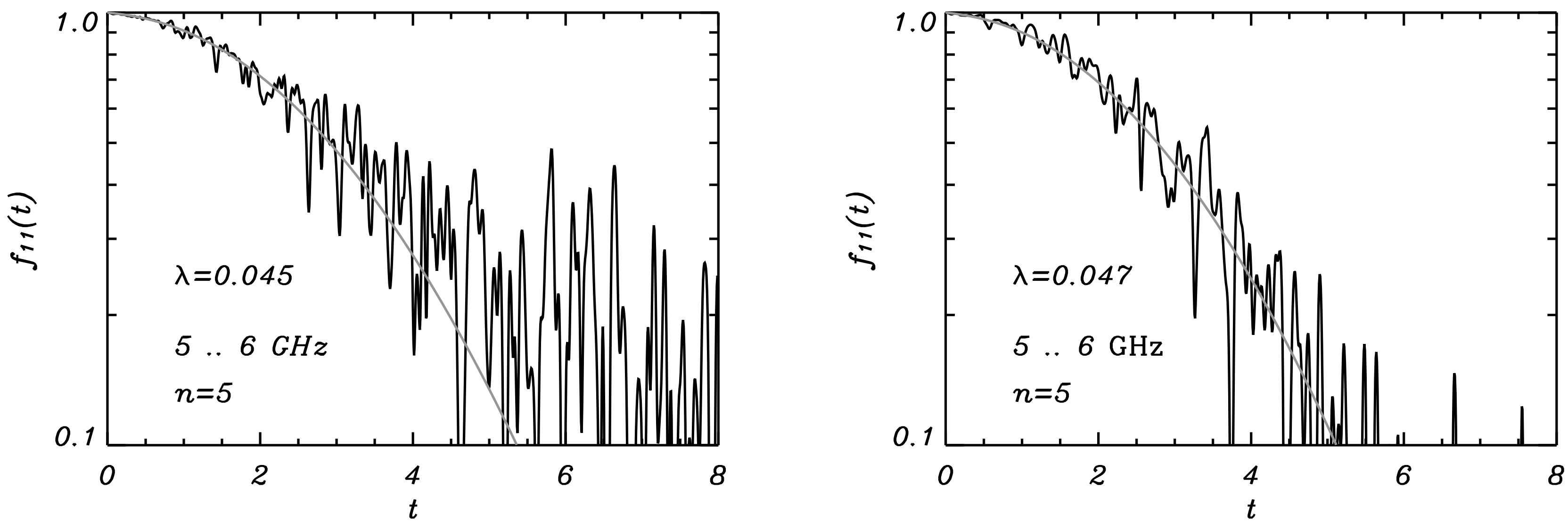
$$\langle f_\varepsilon(t)\rangle = \exp\left[-(2\pi\varepsilon)^2 \left(t^2/\beta_V + t/2 - \int_0^t \mathrm{d}\tau' \int_0^{\tau'} \mathrm{d}\tau\, b_2(\tau) \right) \right]$$



(3) RMT and experiments: microwave billiards

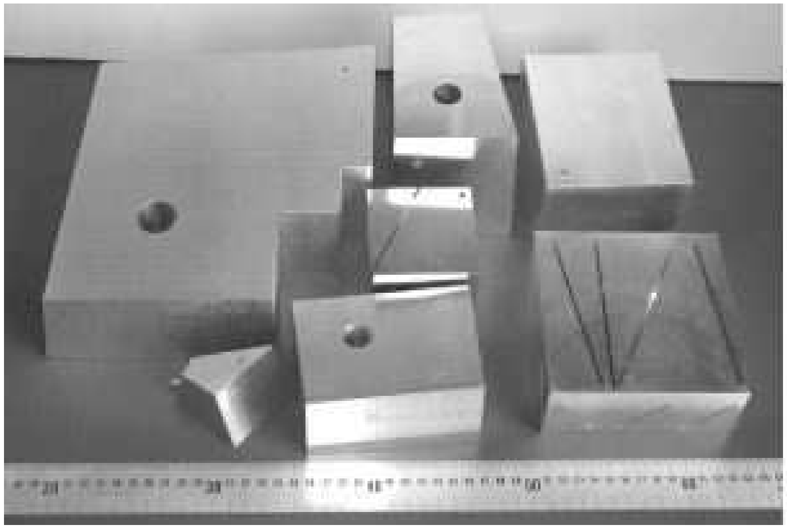
Schäffer et al (2004) measured scattering fidelity in microwave billiards.

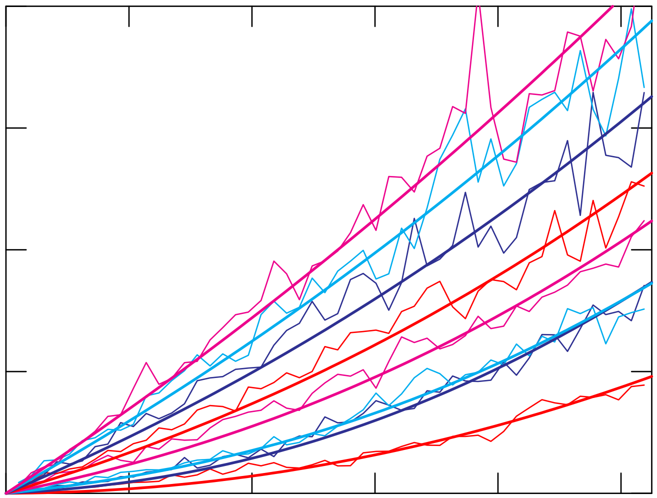




(3) RMT and experiments: acoustics

Lobkis and Weaver (2003) measuring “distortion” $D(t)$ of an acoustic response of solid aluminium blocks upon variation of temperature (dilation = perturbation).





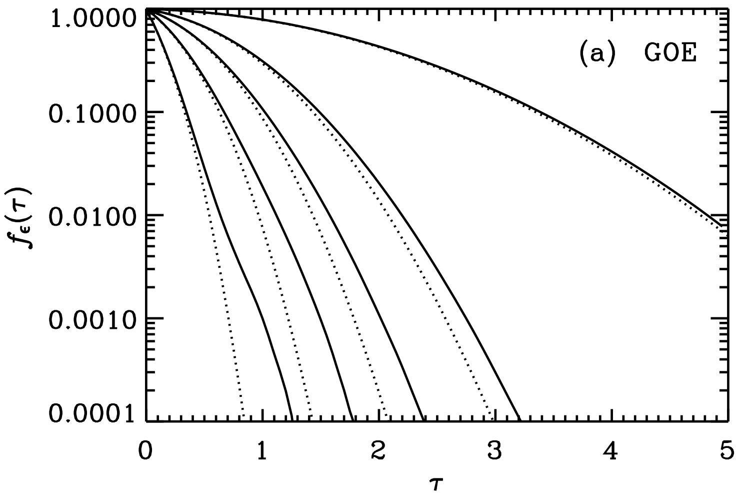
$D(t)$ can be re-interpreted as fidelity. RMT gives good fit of data with a single fitting parameter

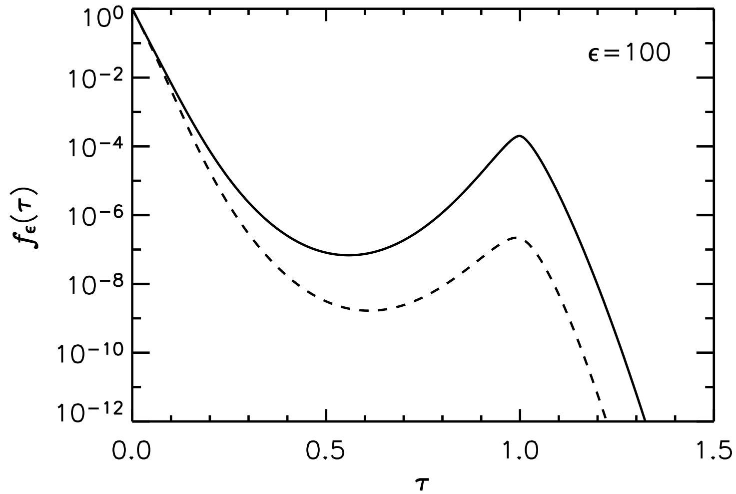
(3) SUSY non-perturbative results

Exact expressions of fidelity amplitude in terms of supersymmetric Gaussian intergals have been obtained by Stöckmann and Scäffer (2004), and solved exactly in the limit $N \to \infty$.

The result for the GUE (simpler) case reads:

$$\langle f_\varepsilon(t) \rangle = \frac{1}{t} \int_0^{\min(t,1)} \mathrm{d}u \; (1 + t - 2u) \; e^{-(2\pi\varepsilon)^2 \, (1+t-2u) \, t/2}$$





4 Quantum echo-dynamics: Integrable case

Here, time-averaged correlation is non-vanishing $\bar{C} = \lim_{t\to\infty} \frac{1}{t^2} \int_0^t \mathrm{d}t' \mathrm{d}t'' C(t', t'') \neq 0.$

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Assume: $\exists$ scale t_{ave} on which $\bar{C}$ converges.
For $t \gg t_{\mathrm{ave}}$ and small ε we have:

$$|f_{\mathrm{ne}}(t)|^2 = 1 - \frac{t^2}{\tau_{\mathrm{ne}}^2}, \qquad \tau_{\mathrm{ne}} = \frac{\hbar}{\varepsilon\sqrt{\bar{C}}}.$$

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Then: order m term in fidelity expansion for $t \gg m t_{\mathrm{ave}}$

$$\hat{\mathcal{T}} \int \mathrm{d}t_1 \cdots \mathrm{d}t_m \langle V(t_1)V(t_2)\cdots V(t_m)\rangle = t^m \langle \bar{V}^m \rangle.$$

So the fidelity can again be summed-up:

$$\boxed{f_{\mathrm{ne}}(t) = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{it\varepsilon}{\hbar}\right)^m \langle \bar{V}^m \rangle = \langle \exp(-i\varepsilon\bar{V}t/\hbar)\rangle.}$$

(4) Regular case: LDOS and semiclassics

Let v_n be the spectrum and $|v_n\rangle$ the eigenstates of $\bar{V}$.
Then, the fidelity is a Fourier transform of LDOS

$$F_{\rm ne}(t) = \langle \exp(i\varepsilon\bar{V}t/\hbar)\rangle = \int {\rm d}v\, e^{ivt\varepsilon/\hbar} d_\rho(v),$$

$$d_\rho(v) = \sum_n \delta(v - v_n)\langle v_n|\rho|v_n\rangle.$$

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If integrability $\Rightarrow \exists$ quantized classical actions $\vec{I}$, with eigenstates $|\vec{n}\rangle$ and eigenvalues $\vec{I}_{\vec{n}} = \hbar(\vec{n} + \vec{\alpha}/4)|\vec{n}\rangle$.

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$$v_{\vec{n}} = \bar{v}(\vec{I}_{\vec{n}}),$$

where $\bar{v}(\vec{I})$ is a time-average of classical limit of V.

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$$v_{\vec{n}} = \bar{v}(\vec{I}_{\vec{n}}),$$

where $\bar{v}(\vec{I})$ is a time-average of classical limit of V. Replacing $\sum_{\vec{n}} \to \hbar^{-d} \int {\rm d}^d \vec{I}$ we find

$$\boxed{F_{\rm ne}(t) \approx \hbar^{-d} \int {\rm d}^d \vec{I} \exp\{ it\bar{v}(\vec{I})\varepsilon/\hbar \} D_\rho(\vec{I})}$$

assuming $D_\rho(\vec{I}_{\vec{n}}) := \langle \vec{n} | \rho | \vec{n} \rangle$ is a smooth function of $\vec{I}$.

(4) Regular case: random initial state

Averaging over (random) initial states $\Rightarrow \rho = \mathbb{1}/\mathcal{N}$: $D_\rho(\vec{I}) = \frac{1}{\mathcal{N}} = \frac{(2\pi\hbar)^d}{\mathcal{V}}$.

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Let $\vec{I}_\eta$, $\eta = 1, 2, \ldots p$ be points of stationary phase $\frac{\partial \bar{v}(\vec{I}_p)}{\partial \vec{I}} = 0$ and $\bar{\mathbf{V}}_\eta = \left\{ \frac{\partial^2 \bar{v}(\vec{I}_\eta)}{\partial I_j \partial I_k} \right\}_{j,k=1}^{d}$:

$$\boxed{f_{\mathrm{ne}}(t) = \frac{(2\pi)^{3d/2}}{\mathcal{V}} \left| \frac{\hbar}{t\varepsilon} \right|^{d/2} \sum_{\eta=1}^{p} \frac{\exp\{it\bar{v}(\vec{I}_\eta)\varepsilon/\hbar + i\nu_\eta\}}{|\det \bar{\mathbf{V}}_\eta|^{1/2}}}$$

where $\nu_\eta = \pi(m_+ - m_-)/4$ and $m_\pm =$ numbers of positive/negative eigenvalues of $\bar{\mathbf{V}}_\eta$.

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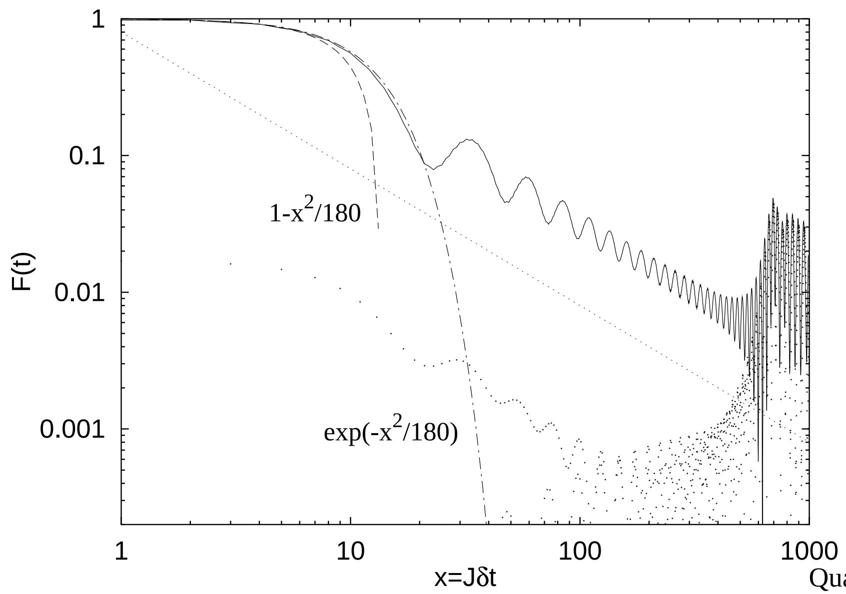
Numerical example: Kicked top in quasi-regular regime



(4) Regular case: coherent initial state

Consider d-dimensional general coherent initial state centered at $(\vec{I}^*, \vec{\theta}^*)$:

$$\langle \vec{n} | \vec{I}^*, \vec{\theta}^* \rangle = \left(\frac{\hbar}{\pi} \right)^{d/4} |\det \Lambda|^{1/4} \exp \left\{ -\frac{1}{2\hbar} (\vec{I}_{\vec{n}} - \vec{I}^*) \cdot \Lambda (\vec{I}_{\vec{n}} - \vec{I}^*) - i \vec{n} \cdot \vec{\theta}^* \right\}$$

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Method of stationary phase: $\exists$ unique stationary point

$$\vec{I}_s = \vec{I}^* - \frac{it\varepsilon}{2} \Lambda^{-1} \vec{v}' + \mathcal{O}(\varepsilon^2), \quad \text{where} \quad \vec{v}' := \frac{\partial \bar{v}(\vec{I}^*)}{\partial \vec{I}}$$

giving

$$\boxed{f_{\rm ne}(t) = \exp \left\{ -\frac{(\vec{v}' \cdot \Lambda^{-1} \vec{v}')\varepsilon^2}{4\hbar} t^2 + \frac{i\bar{v}(\vec{I}^*)\varepsilon}{\hbar} t \right\}.}$$

Note a Gaussian decay of fidelity

$$|F_{\rm ne}(t)|^2 = \exp\left(-\frac{t^2}{\tau_{\rm ne}^2} \right), \qquad \tau_{\rm ne} \sim \hbar^{1/2} \varepsilon^{-1}.$$

Numerical example: double-kicked top

Two coupled kicked tops $\vec{J}_1$ and $\vec{J}_2$ with Floquet map:

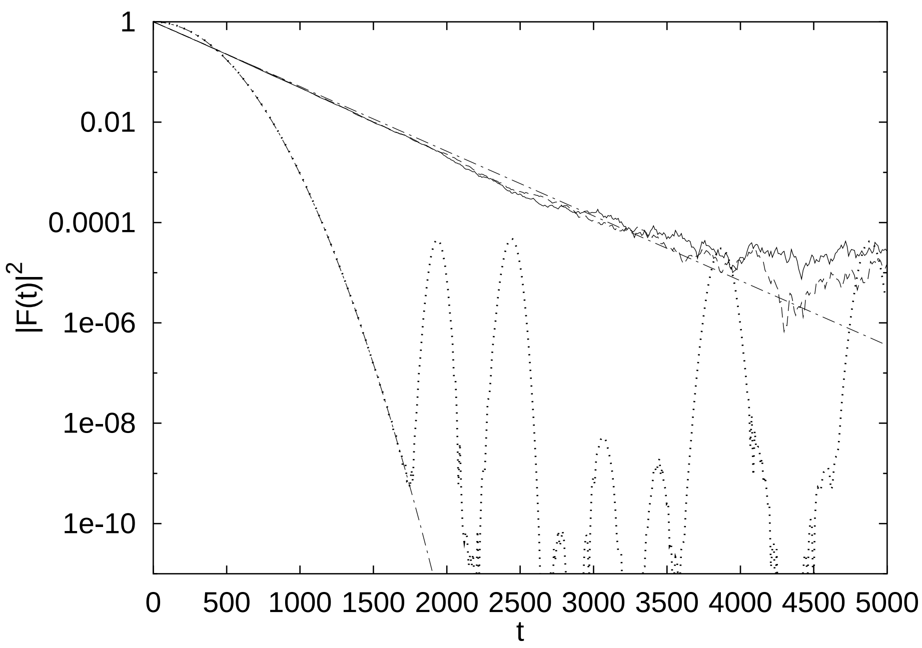
$$U(\epsilon) = \exp\left(-i\frac{\pi}{2}J_{1\mathrm{y}}\right)\exp\left(-i\frac{\pi}{2}J_{2\mathrm{y}}\right)\exp\left(-i\epsilon J_{1\mathrm{z}}J_{2\mathrm{z}}/J\right).$$

Perturbation: $V = J_{1\mathrm{z}}J_{2\mathrm{z}}/J^2$.

We take $J = 200$, $\varepsilon = 8 \cdot 10^{-4}$.

1 • quasi-regular case $\epsilon = 1$ (dotted curve)

2 • chaotic case $\epsilon = 20$ (solid curve, dashed=random state)



5 Classical fidelity

Q-C Correspondence:

Write quantum fidelity in terms of the Wigner functions:

$$F(t) = (2\pi\hbar)^d \int \mathrm{d}\vec{x} W_\rho(\vec{x},t) W_\rho^\varepsilon(\vec{x},t).$$

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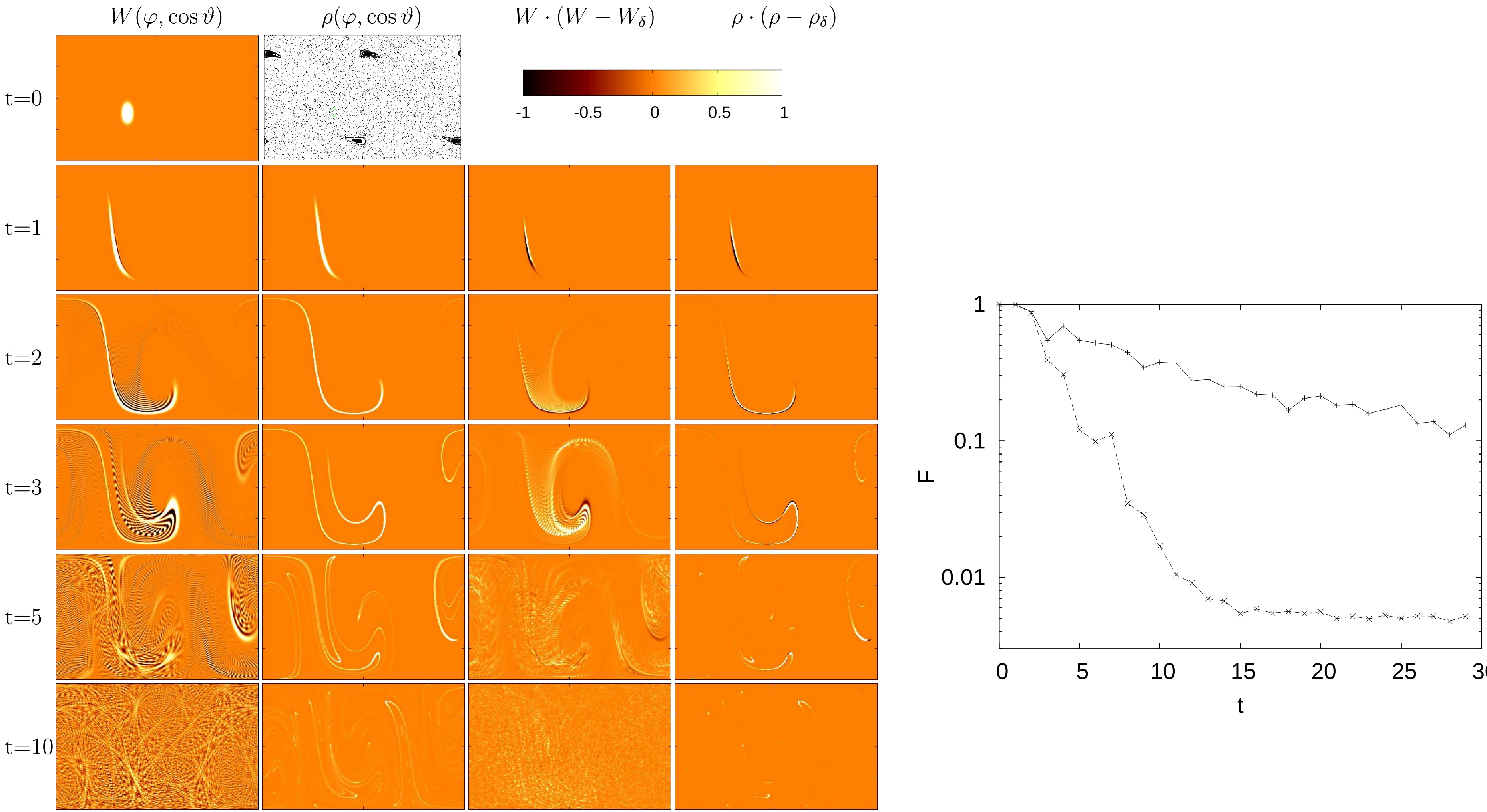
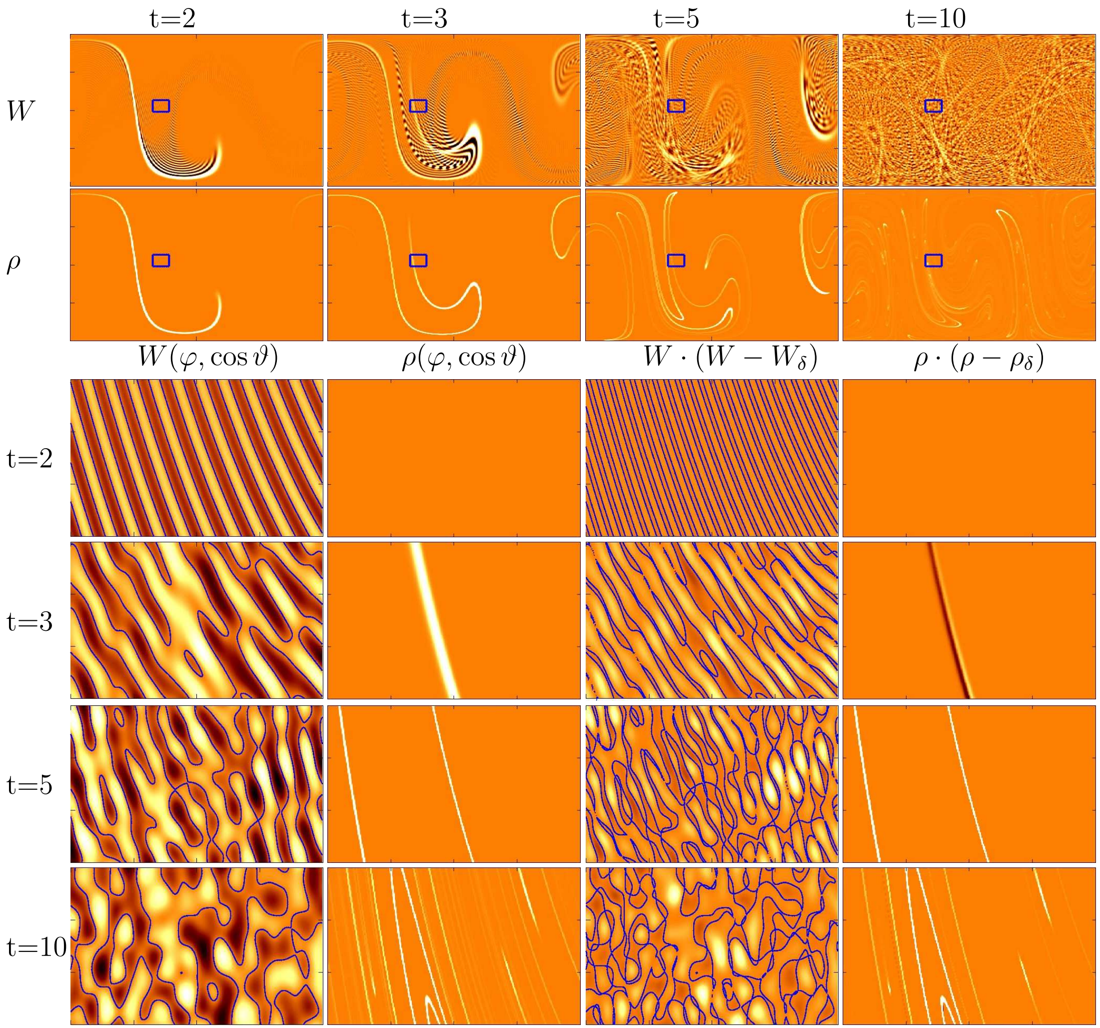
Replace Wigner function by the Liuoville density $\rho(\vec{x})$ and we have the classical fidelity

$$F_{\mathrm{cl}}(t) = \int \mathrm{d}\vec{x} \rho(\vec{x}, t) \rho_\varepsilon(\vec{x}, t).$$

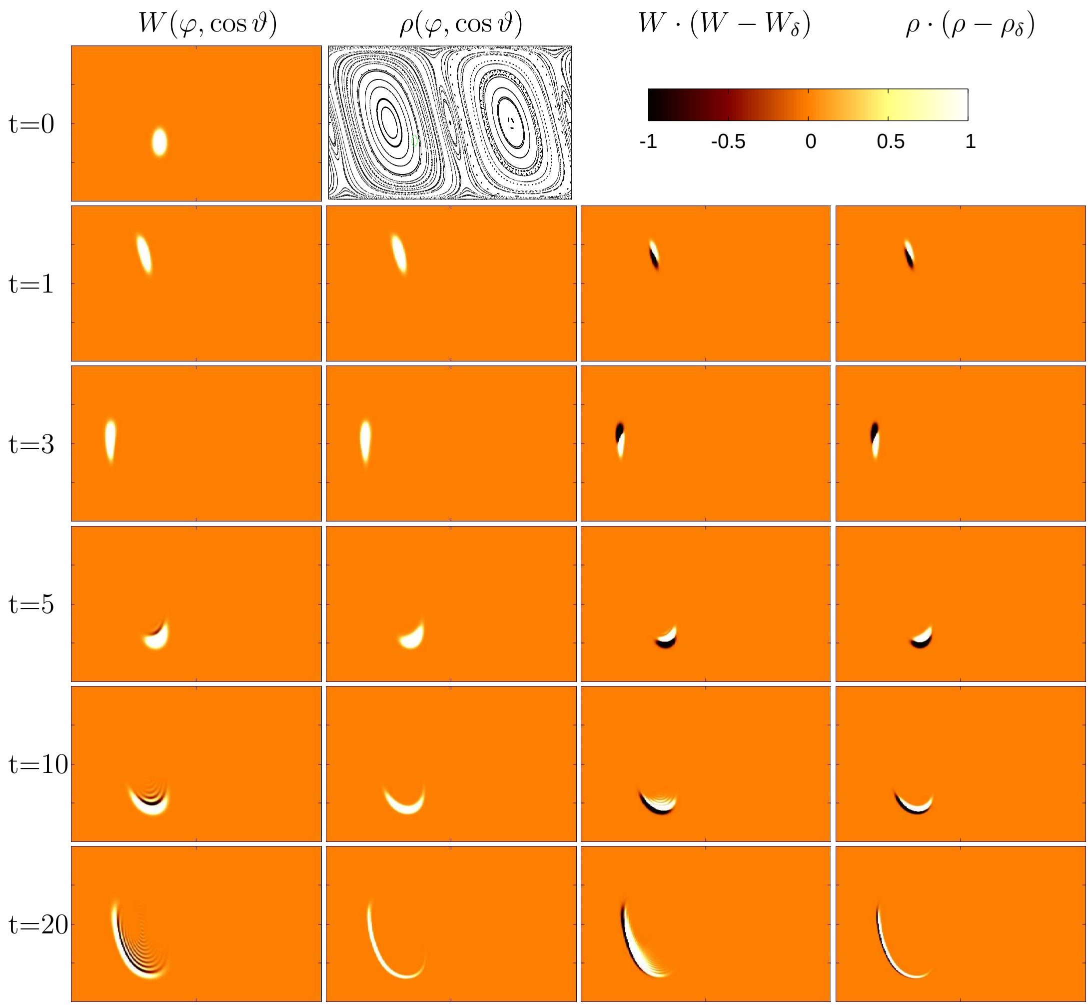
(5) Q-C correspondence: chaotic

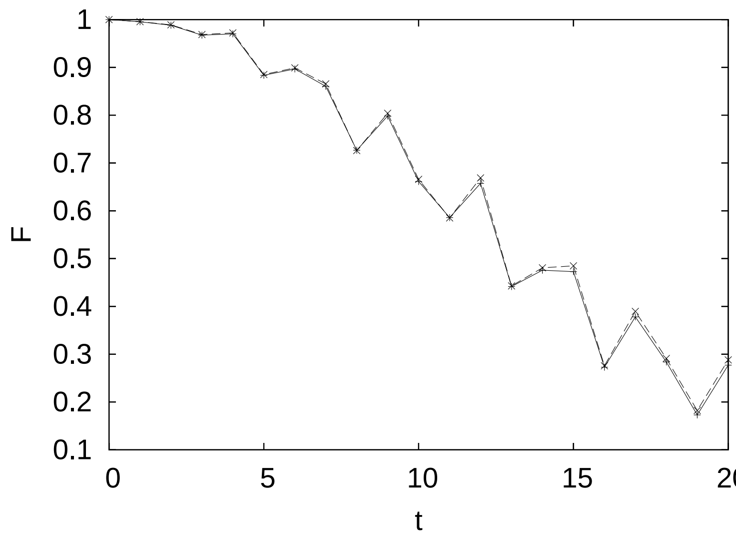


(5) Q-C correspondence: zoom-in



(5) Q-C correspondence: regular





(5) Theory of classical fidelity

Idea: Write the classical Liouvile dynamics in INTERACTION PICTURE:

$$\rho_{\mathrm{E}}(\vec{x},t) = U_{\mathrm{E}}(t)\rho_0(\vec{x},0), \quad F_{\mathrm{cl}}(t) = \int_\Omega \mathrm{d}\vec{x}\rho_{\mathrm{E}}(\vec{x},t)\rho_0(\vec{x},0).$$

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The classical echo-operator $\hat{U}_{\mathrm{E}}(t) = \hat{U}_0^{\dagger}(t)\,\hat{U}_{\varepsilon}(t)$ with Liouvilean propagators

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}_{\varepsilon}(t) = \hat{L}_{H_{\varepsilon}(\vec{x},t)}\hat{U}_{\varepsilon}(t), \qquad \hat{L}_{A(\vec{x},t)} := \left(\vec{\nabla}A(\vec{x},t)\right)\cdot J\vec{\nabla},$$

again satisfies Liouville equation with echo Hamiltonian

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}_{\mathrm{E}}(t) = \hat{L}_{H_{\mathrm{E}}(\vec{x},t)}\,\hat{U}_{\mathrm{E}}(t), \qquad H_{\mathrm{E}}(\vec{x},t) = \varepsilon V\left(\vec{\phi}_t(\vec{x}),t\right).$$

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Trajectories of the echo-flow satisfy time-dependent Hamilton's equations:

$$\dot{\vec{x}} = J\,\vec{\nabla}H_{\mathrm{E}}(\vec{x},t).$$

(5) Classical fidelity: linear response

Hamilton's equations for the echo-flow can be solved perturbatively for small ε with the solutions

$$F_{\mathrm{cl}}^{\mathrm{ch}}(t) = 1 - \varepsilon^2 C[\rho]^2 \exp(2\lambda_{\max} t) + O(\varepsilon^4), \quad t \gg 1/\lambda_{\max}$$

for chaotic dynamics with maximal Lyapunov exponent $\lambda_{\max}$, and

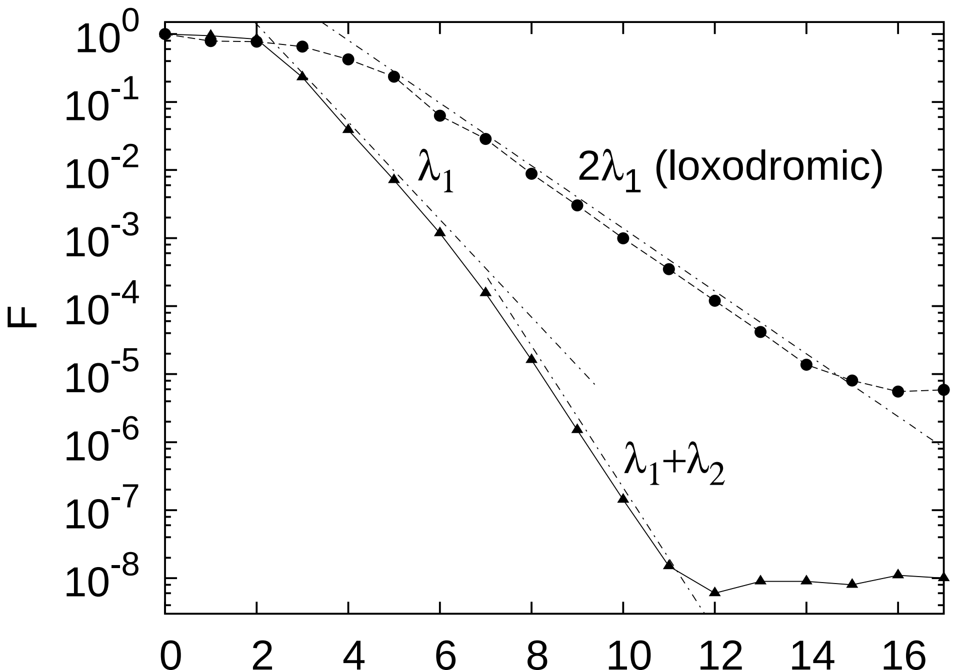
$$F_{\mathrm{cl}}^{\mathrm{reg}}(t) = 1 - \varepsilon^2 C'[\rho]^2 t^2 + O(\varepsilon^4)$$

for regular dynamics, where $C[\rho], C'[\rho]$ are some constants which depend on initial density ρ only.

(5) Classical fidelity: chaotic few body systems

Elaborating on classical fidelity in interaction picture one can derive a cascade of Lyapunov decays for ergodic and strongly chaotic few body systems:

$$F_{\rm cl}(t) \approx \prod_{j;\ t_j<t} \exp\left[-\lambda_j(t-t_j)\right], \quad t_j = (1/\lambda_j)\log(\nu_j/(\varepsilon\gamma_j))$$

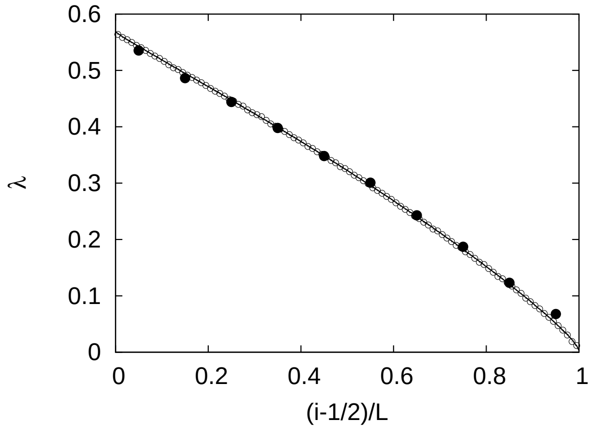


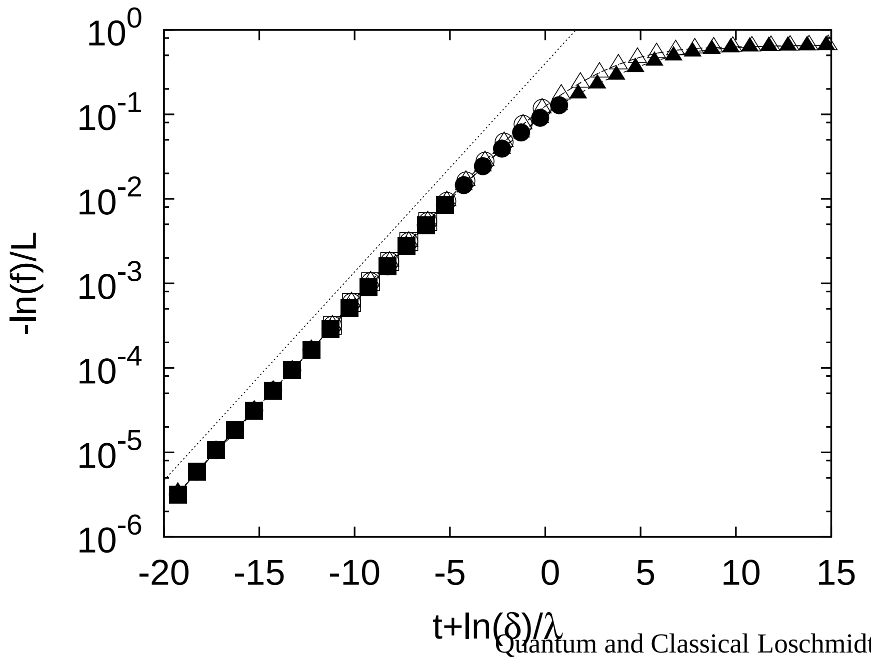
(5) Classical fidelity: chaotic many-body systems

For chaotic many-body systems, one can use similar thinking to derive doubly-exponential decay

$$f(t) = \exp\left(-\alpha N \delta^{\beta} \exp(\beta \lambda_{\max} t)\right),$$

where $\beta = 1$ or 2, depending on whether intitial density is (Lipshitz) continuous or not.





(5) Classical fidelity: linearly unstable systems

For systems which have no exponential instability ($\lambda_{\max} = 0$) but are linearly unstable, and ergodic and mixing, we find universal scaling of classical fidelity

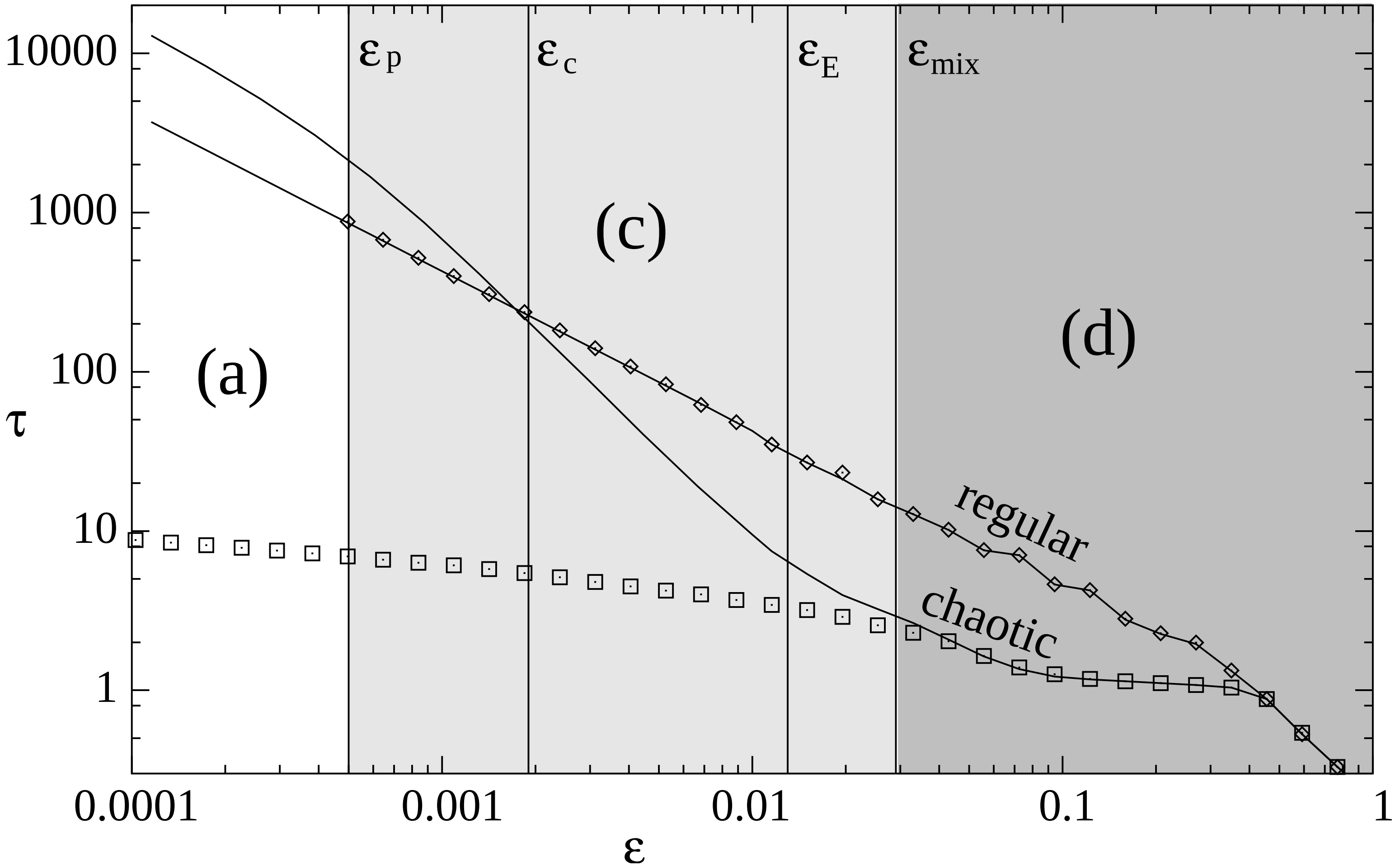
$$F_{\mathrm{cl}}(t) = \phi(|\varepsilon|^{2/5} t)$$

For small t, $F_{\mathrm{cl}} = 1 - C|\varepsilon| t^{5/2}$,
for long t, $F_{\mathrm{cl}} = \exp(-C'|\varepsilon|^{2/5}|t|)$.

The prominent example of such systems is hard-point gas of unequal particles in one-dimension, or any polygonal billiard with at least two irrational angles.

6 Scaling of fidelity decay time scales

Fidelity decay time against perturbation parameter:



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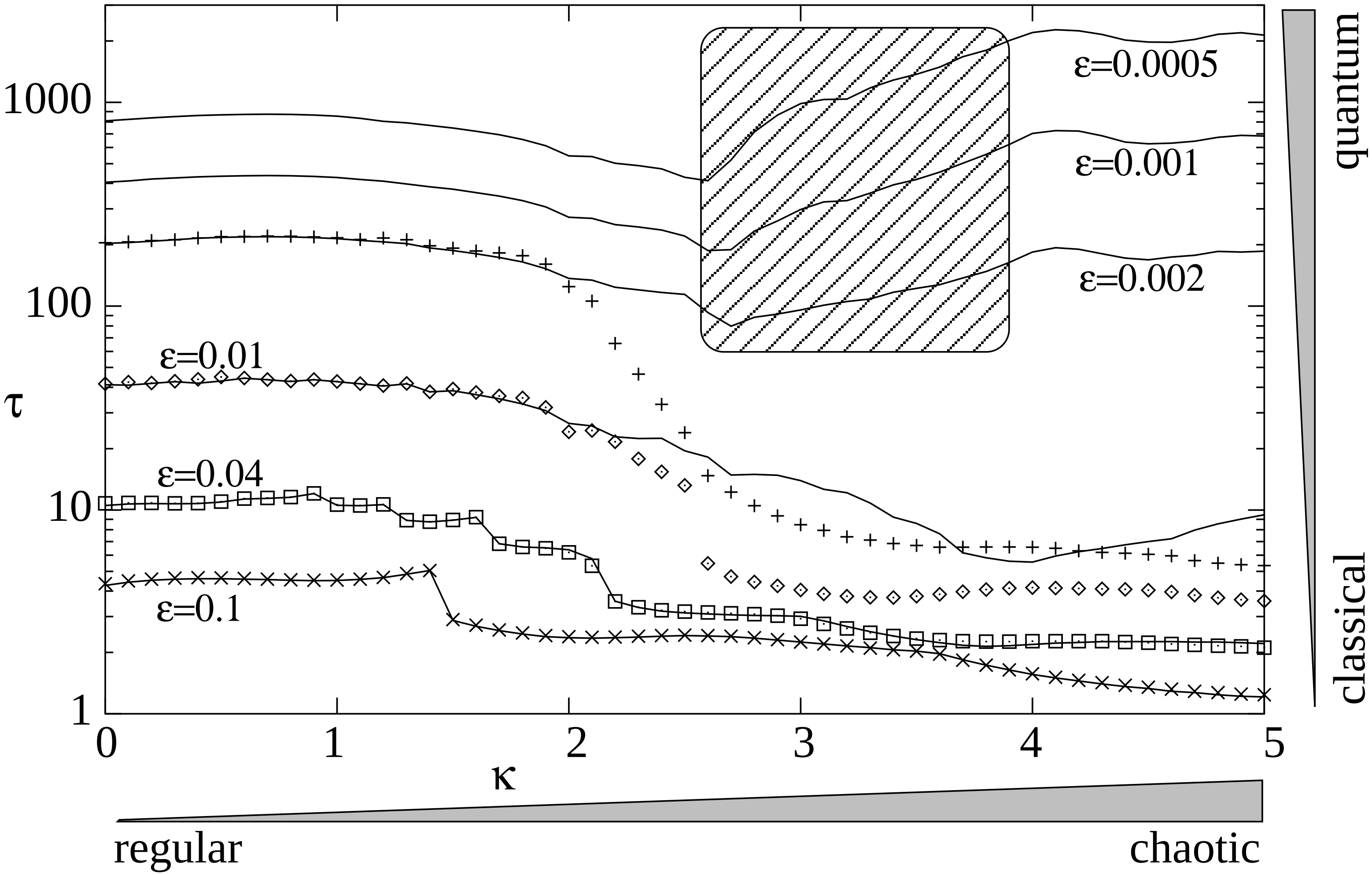

7 Application: Quantum information

$$U = U(T) \cdots U(2)U(1).$$

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Propagator

$U(t, t') = U(t)U(t-1) \cdots U(t'+2)U(t'+1),$

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7 Application: Quantum information

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Perturbation $V(t)$, $U_\delta(t) = U(t)\exp(-i\delta V(t))$.
Fidelity (linear reposnse):

$$F = 1 - \delta^2 \sum_{t,t'=1}^{T} C(t,t')$$

where $C(t,t') = \langle\psi|U(0,t)V(t)U(t,t')V(t')U(t',0)|\psi\rangle$ is temporal correlator of the generator of perturbation.

Optmization of quantum algorithms

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Let us assume:

- Random initial state $|\psi\rangle$
- Random static perturbation $\langle V_{jk} V_{lm} \rangle = 2^{-n} \delta_{jm} \delta_{kl}$:

$$C(t, t') = \left| 2^{-n} \operatorname{tr} U(t, t') \right|^2 .$$

Quantum Fourier transformation

Write the matrix

$$U_{jk} = \frac{1}{\sqrt{N}} \exp(2\pi i jk/N),$$

$N = 2^n$, in terms of $T = n(n+1)/2$ 1-qubit and 2-qubit gates

$$\mathrm{A}_j = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}_j, \qquad \mathrm{B}_{jk} = \mathrm{diag}\{1, 1, 1, e^{i\pi/2^{|k-j|}}\}_{jk}.$$

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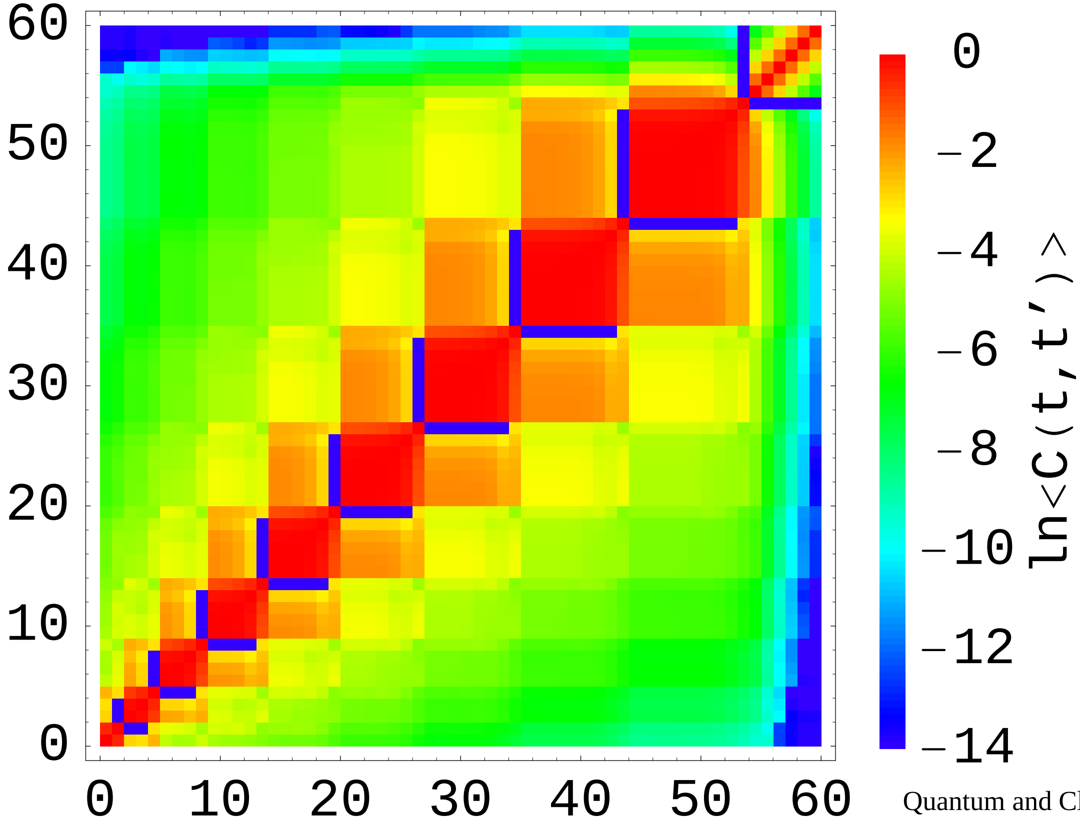
E.g., for $n = 4$:

$$U = \mathrm{T}_{03}\mathrm{T}_{12}\mathrm{A}_0\mathrm{B}_{01}\mathrm{B}_{02}\mathrm{B}_{03}\mathrm{A}_1\mathrm{B}_{12}\mathrm{B}_{13}\mathrm{A}_2\mathrm{B}_{23}\mathrm{A}_3.$$

The correlator

Blocks of B-gates result in long-tails of the correlator, and consequently, fast decay of fidelity, $\sum_{t,t'} C(t,t') \propto n^3$.

Example for $n = 10$:



Improved QFT

Replace almost diagonal B-gates in terms of a pair of new gaites

$$B_{jk} = R_{jk} G_{jk}.$$

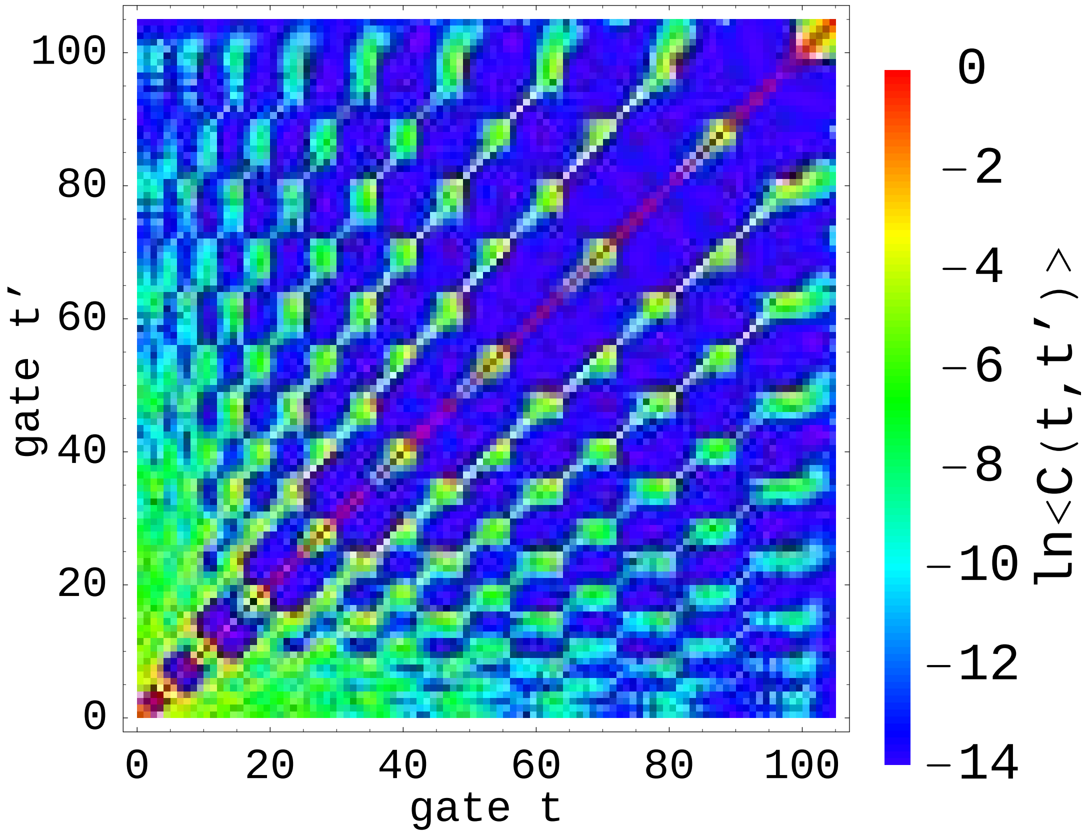
Then, redistribute the gates which commute.

We now have $T \approx n^2$ elementary gates, e.g. for $n = 4$:

$$U = \mathrm{T}_{03}\mathrm{T}_{12}\mathrm{A}_0\mathrm{R}_{01}\mathrm{R}_{02}\mathrm{R}_{03}\mathrm{G}_{01}\mathrm{G}_{02}\mathrm{G}_{03}\mathrm{A}_1\mathrm{R}_{12}\mathrm{R}_{13}\mathrm{G}_{12}\mathrm{G}_{13}\mathrm{A}_2\mathrm{R}_{23}\mathrm{G}_{23}\mathrm{A}_3$$

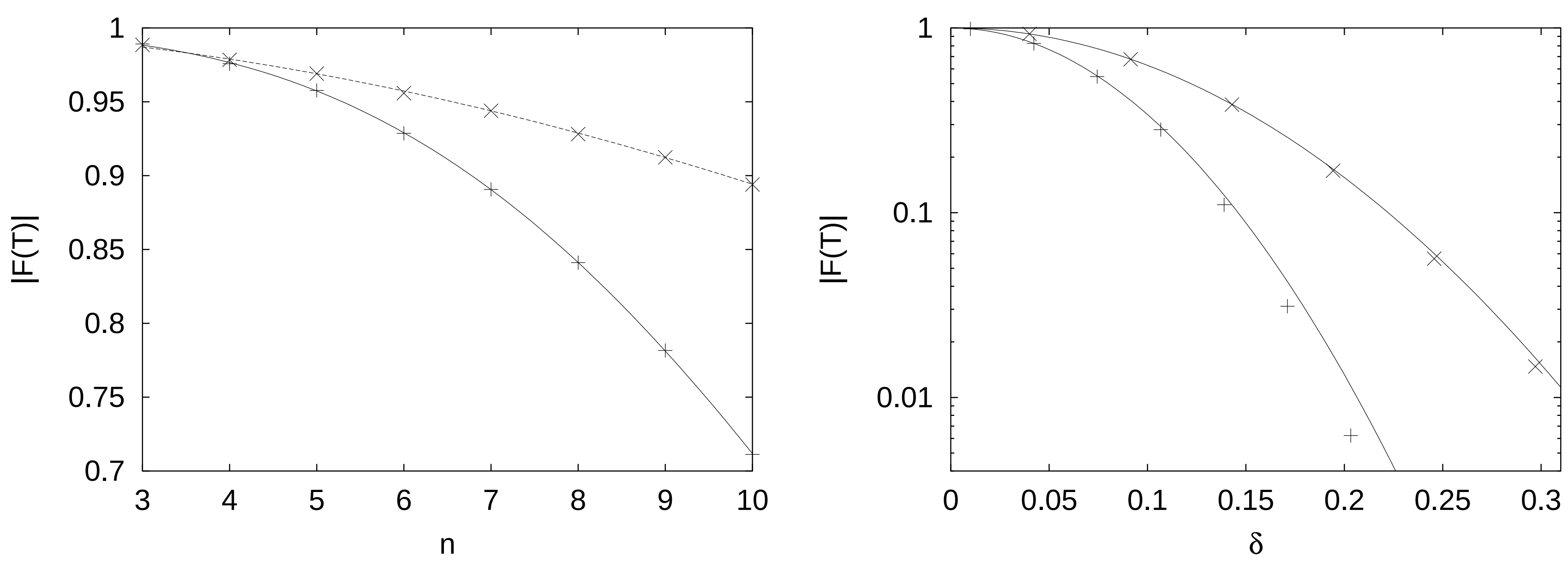
The correlator

Improved QFT exhibits much faster decay of correlations, $\sum_{t,t'} C(t,t') \propto n^2$.
Example for $n = 10$:



Improvement of quantum fidelity

Dependence on the number of qubits (for $\delta = 0.04$) and on the strength of perturbation (for $n = 8$):



Thanks!

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