QIPC & Quantum Optics

(quantum information processing with cold atoms)

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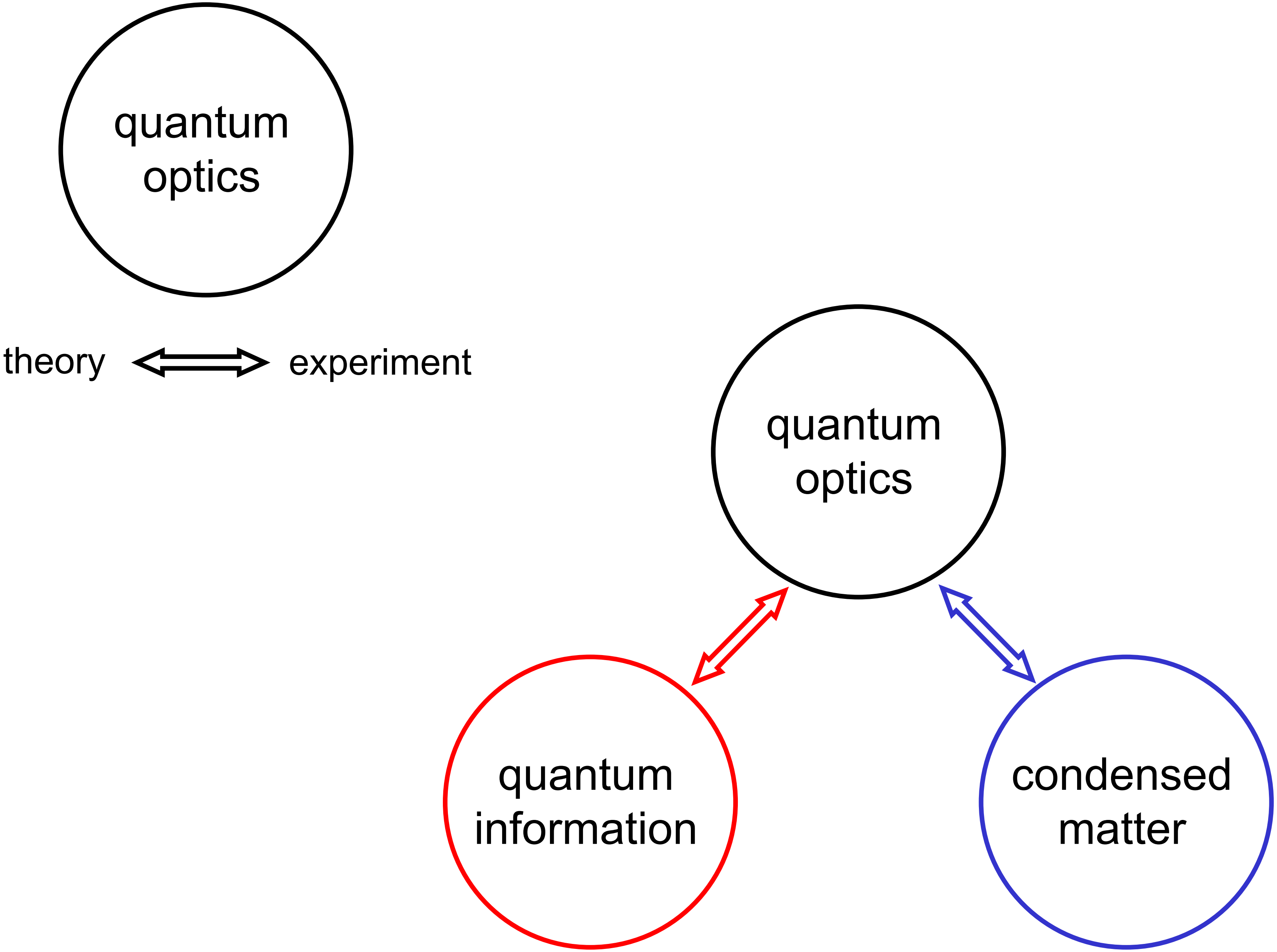
SFB
Coherent Control of Quantum Systems

€U networks



quantum optics

theory ⟺ experiment

quantum optics

quantum information

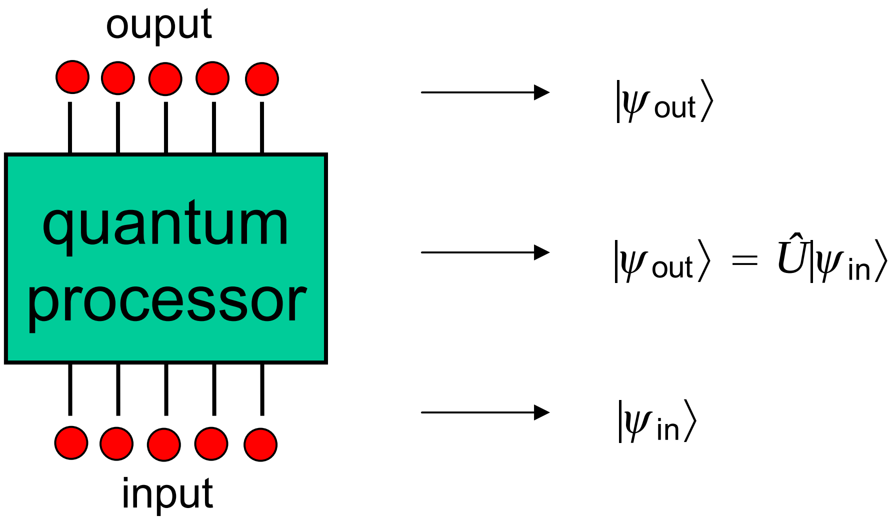
condensed matter

Introduction / Motivation / Overview

- Quantum information
  - quantum computing, quantum communication etc.
- Zoo of quantum optical systems
  - ions, neutral atoms, CQED, atomic ensembles
- Theoretical Tools of Quantum Optics
  - quantum optical systems as open quantum systems

1.1 Quantum information processing

- quantum computing

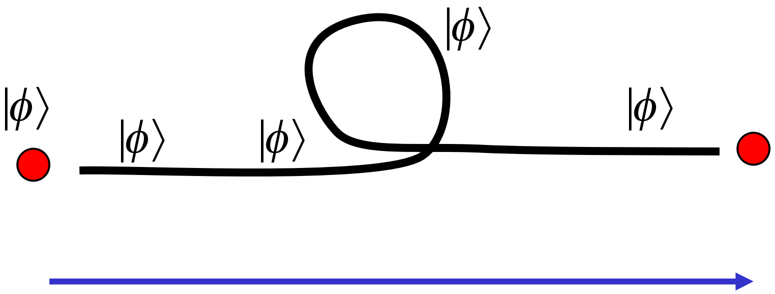


$$|\psi_{\text{out}}\rangle$$

$$|\psi_{\text{out}}\rangle = \hat{U}|\psi_{\text{in}}\rangle$$

$$|\psi_{\text{in}}\rangle$$

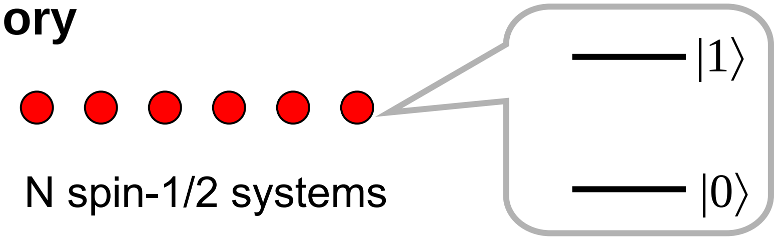
- quantum communication



transmission of a quantum state

Quantum computing

- **quantum memory**



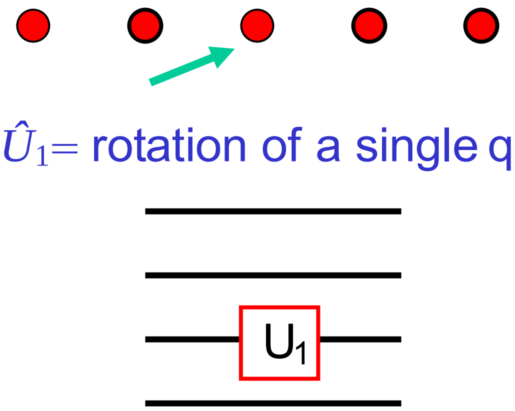
N spin-1/2 systems

quantum register

qubit

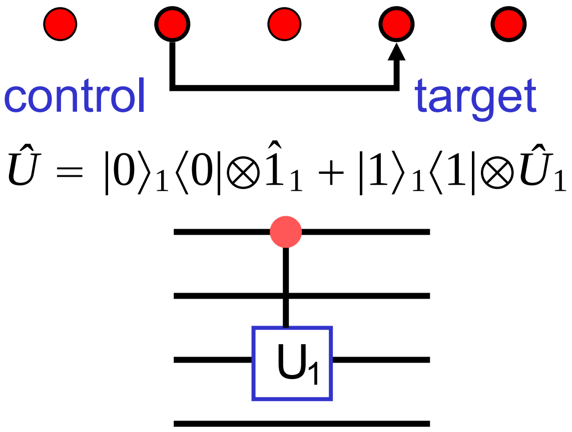
- **quantum gates**

single qubit gate:



$\hat{U}_1$= rotation of a single qubit

two-qubit gate:



control

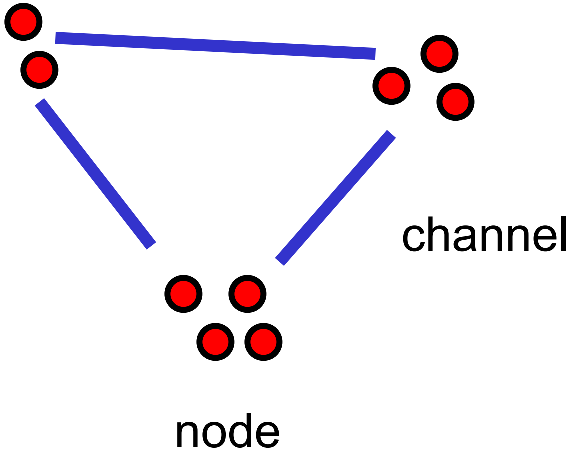
target

$$\hat{U} = |0\rangle_1\langle 0|\otimes\hat{1}_1 + |1\rangle_1\langle 1|\otimes\hat{U}_1$$

- **read out**
- **[no decoherence]**

Our goal ... implement quantum networks

- quantum network



- Nodes: local quantum computing
  - store quantum information
  - local quantum processing
  - measurement
- Channels: quantum communication
  - transmit quantum information
  - local / distant

Goals:

- map to physical (quantum optical) system
- map quantum information protocols to physical processes

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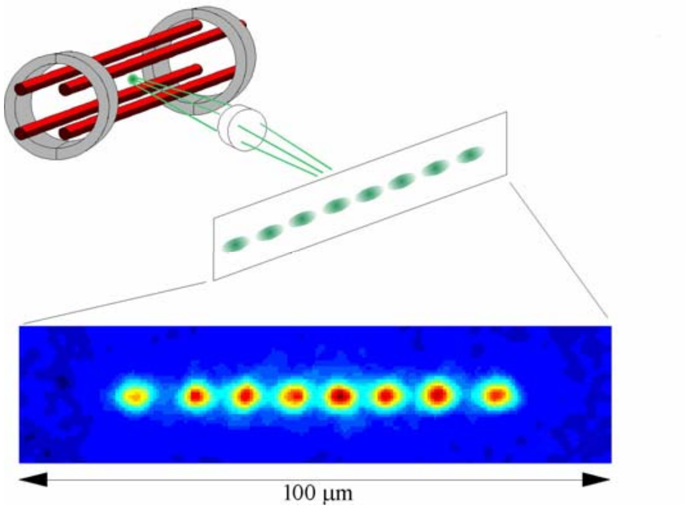
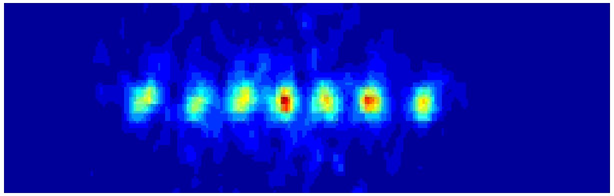
1.2 Zoo of quantum optical systems

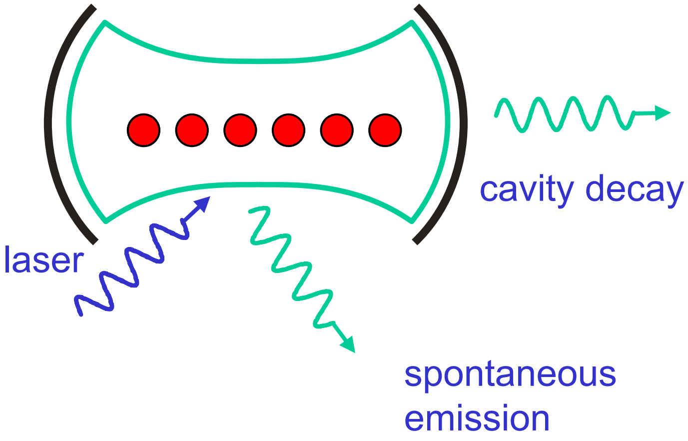
- trapped ions





collective modes

- CQED



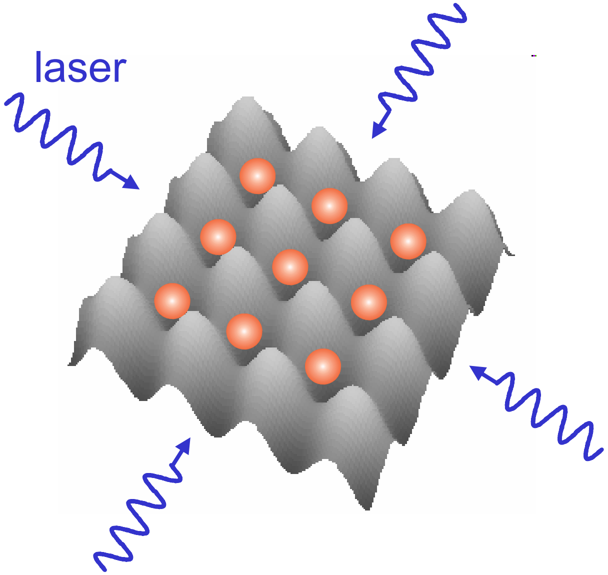
Few particle system with complete quantum control:

spin-1/2s coupled to harmonic oscillator(s)

- quantum state engineering:
  quantum computing
- state preparation & measurement

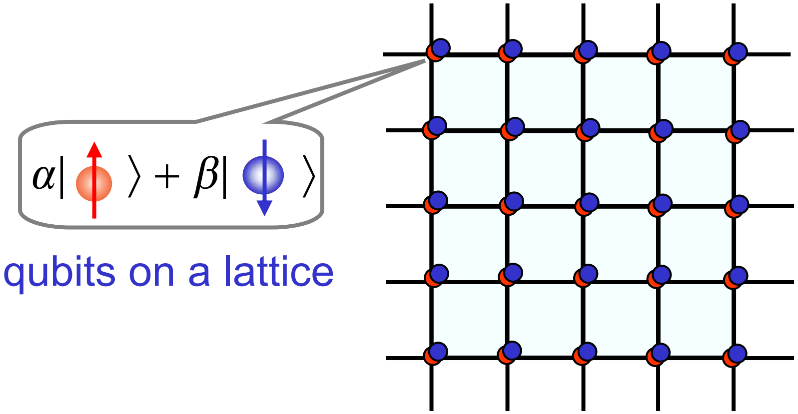
- **from BEC to Hubbard models**
  - strongly correlated systems
  - time dependent, e.g. quantum phase transitions
  - …
  - exotic quantum phases (?)

optical lattice as a regular array of microtraps for atoms



- **quantum information processing**
  - new quantum computing scenarios, e.g. "one way quantum computer"

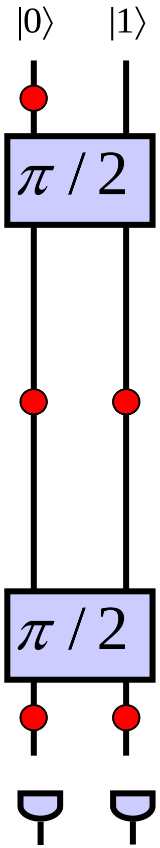
"quantum simulator"



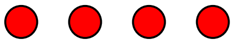
qubits on a lattice

entangling qubits via "Ising" (cluster state)

- **… measurements beyond standard quantum limit**



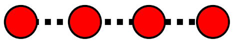
- N independent atoms



$$\Delta\omega_{\mathrm{SQL}} = \frac{1}{T\sqrt{n_{\mathrm{rep}}}}\frac{1}{\sqrt{N}}$$

standard quantum noise limit

- N entangled atoms

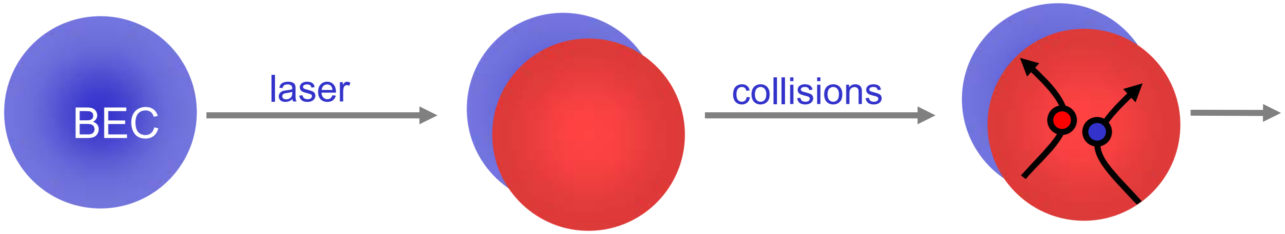


$$\Delta\omega_{\mathrm{ent}} = \frac{1}{T\sqrt{n_{\mathrm{rep}}}}\frac{1}{f(N)} \geq \frac{1}{T\sqrt{n_{\mathrm{rep}}}}\frac{1}{N}$$

Heisenberg limit:
maximally entangled state

$$|0000\rangle + |1111\rangle$$

- Entanglement via collisions: spin squeezing



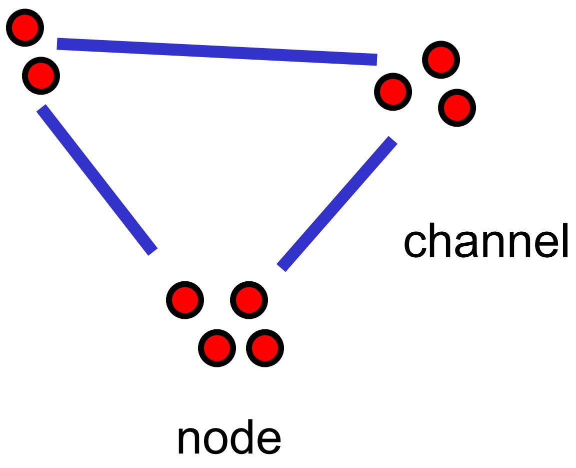
$|0\rangle^{\otimes N}$

product state

$[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)]^{\otimes N}$

product state

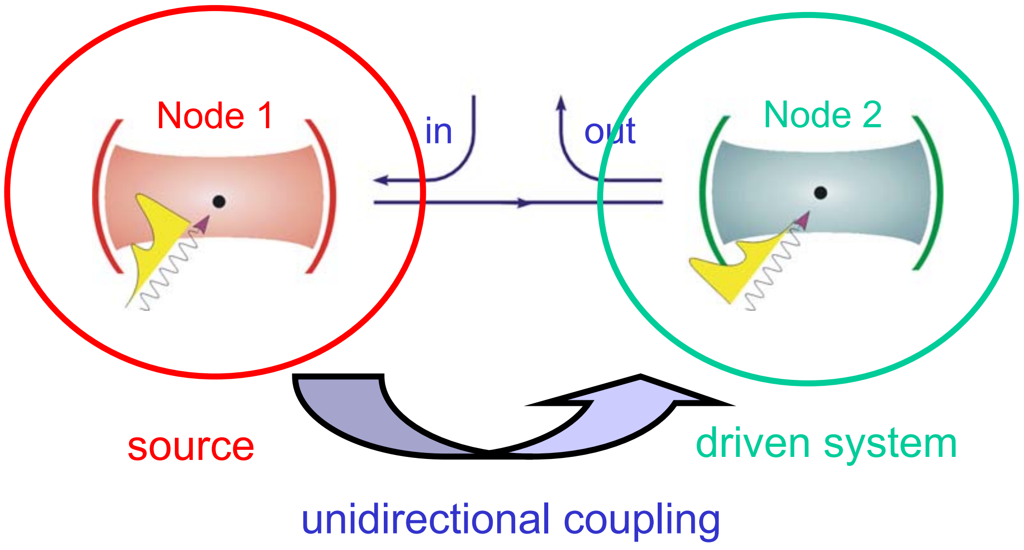
$\sum c_n |0\rangle^n |1\rangle^{N-n}$



channel

node

- **cascaded quantum system: transmission in a quantum network**



Node 1

in

out

Node 2

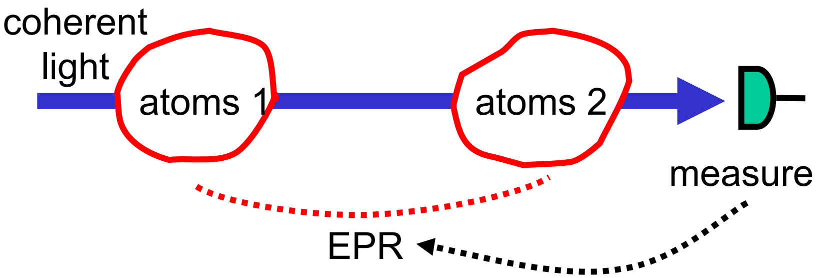
source

driven system

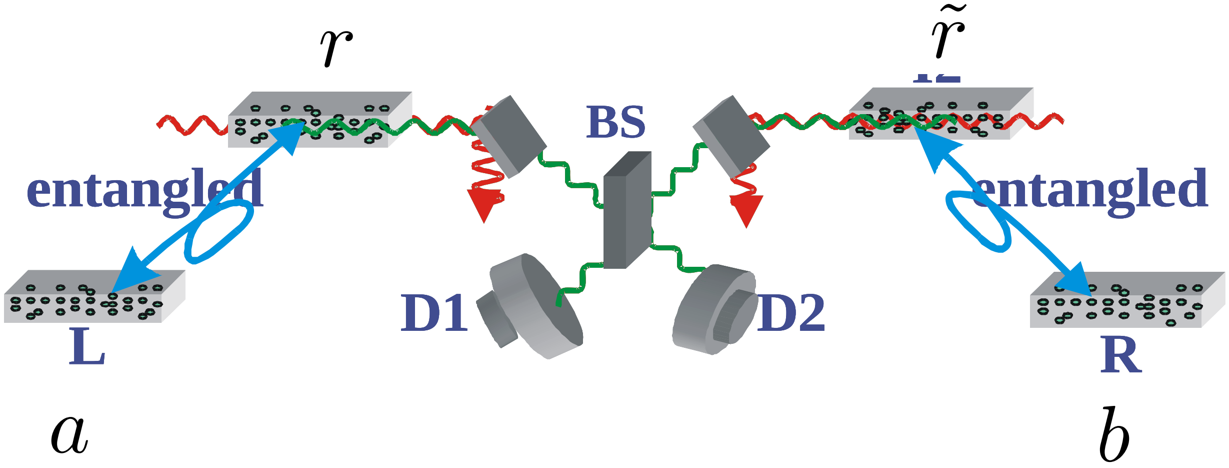
unidirectional coupling

- **atomic ensembles**

atomic / spin squeezing; quantum memory for light;
continuous variable quantum states



quantum repeater: establishing long distance EPR pairs
for quantum cryptography and teleportation



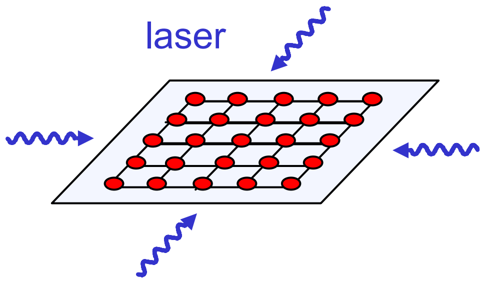
… and what we are working on at the moment

polar molecules …

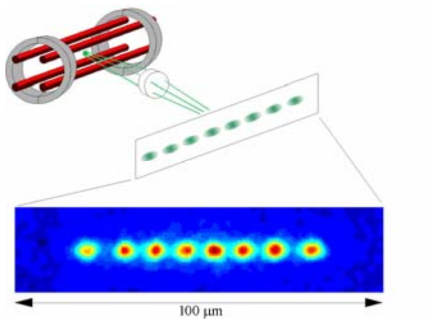
- hybrid quantum optics – solid state processors
  - coupling polar molecules to strip line cavities
- spin lattice models (of interest in topological quantum computing)
  - Kitaev xx-yy-zz on honeycomb lattices
  - Ioffe, Feigelman et al., xx-zz on square lattice

Quantum Optics with Atoms & Ions

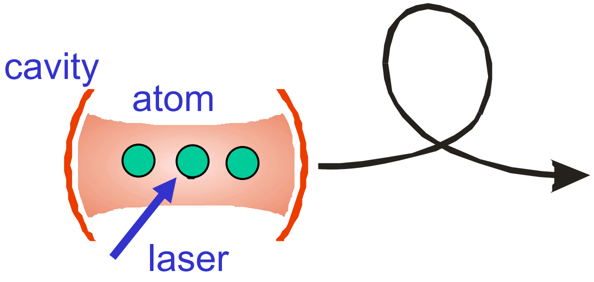
- cold atoms in optical lattices



- trapped ions / crystals of …

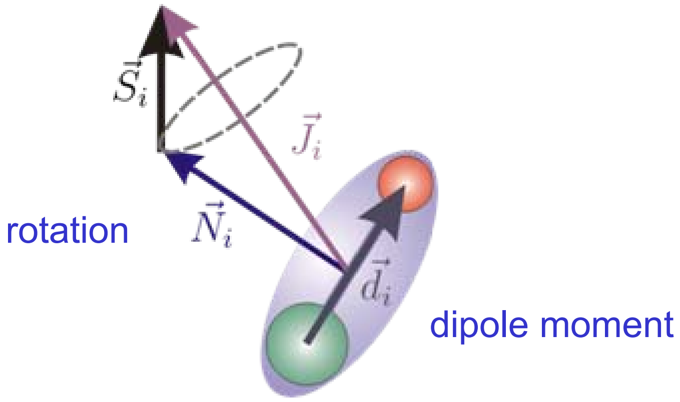


- CQED



- atomic ensembles

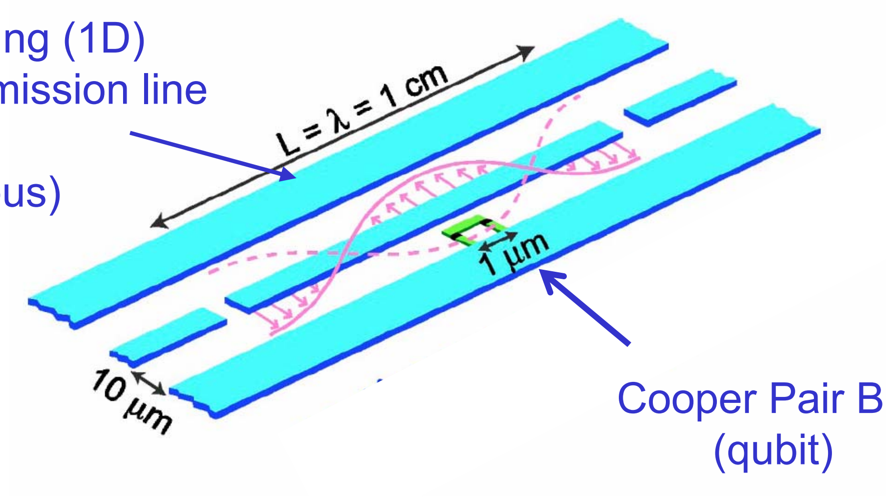
Polar Molecules



- single molecules / molecular ensembles
- coupling to optical & microwave fields
  - trapping / cooling
  - CQED (strong coupling)
  - spontaneous emission / engineered dissipation
- interfacing solid state / AMO & microwave / optical
  - strong coupling / dissipation
- collisional interactions
  - quantum deg gases / Wigner (?) crystals
  - dephasing

Hybrid Device:
solid state processor
& molecular memory
+ optical interface

R. Schoelkopf, S. Girvin et al. (Yale)



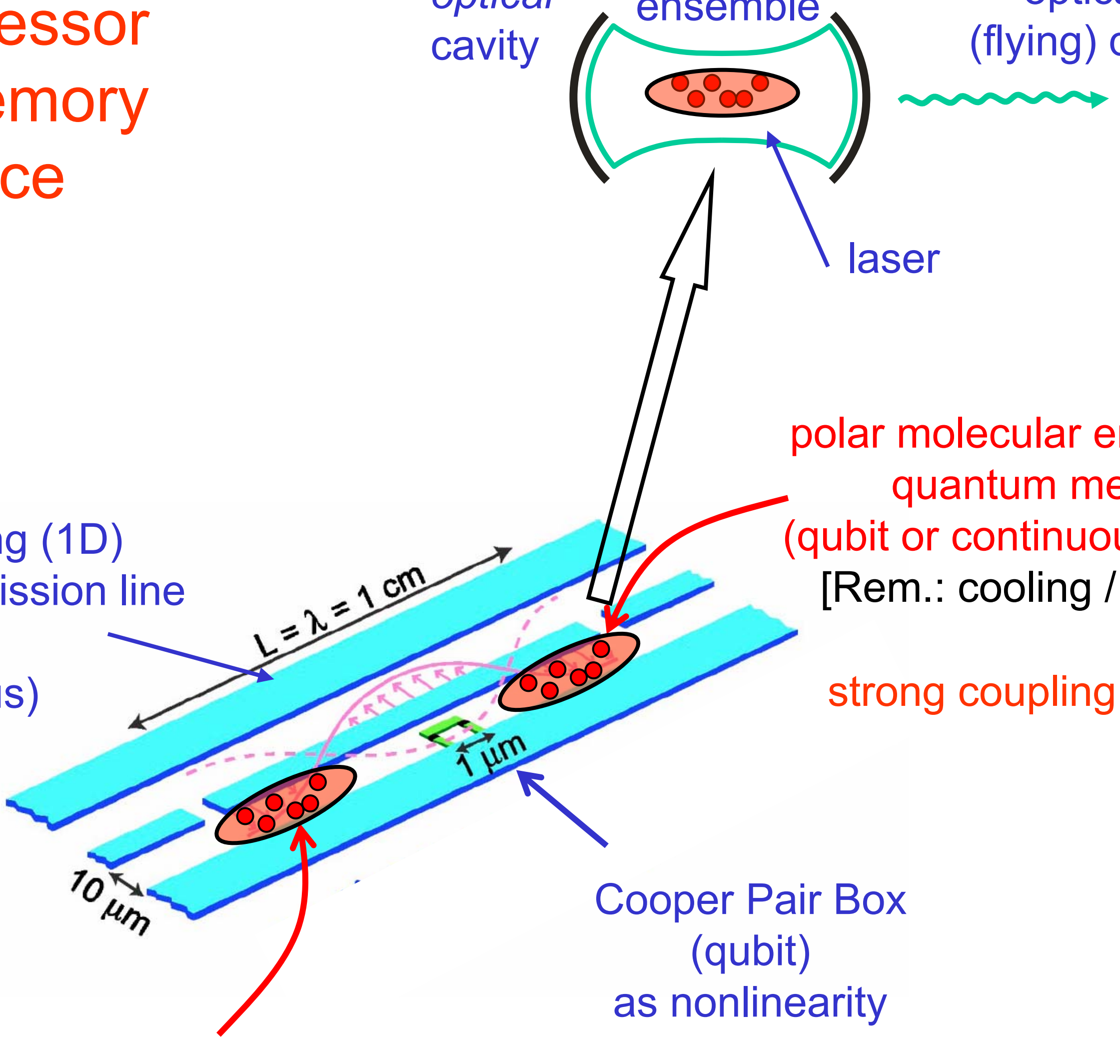
superconducting (1D)
microwave transmission line
cavity
(photon bus)

strong coupling
circuit CQED

Cooper Pair Box
(qubit)

P. Rabl, R. Schoelkopf, D. DeMille, M. Lukin …

Hybrid Device: solid state processor & molecular memory + optical interface



optical cavity

molecular ensemble

optical (flying) qubit

laser

polar molecular ensemble 1:
quantum memory
(qubit or continuous variable)
[Rem.: cooling / trapping]

strong coupling CQED

superconducting (1D)
microwave transmission line
cavity
(photon bus)

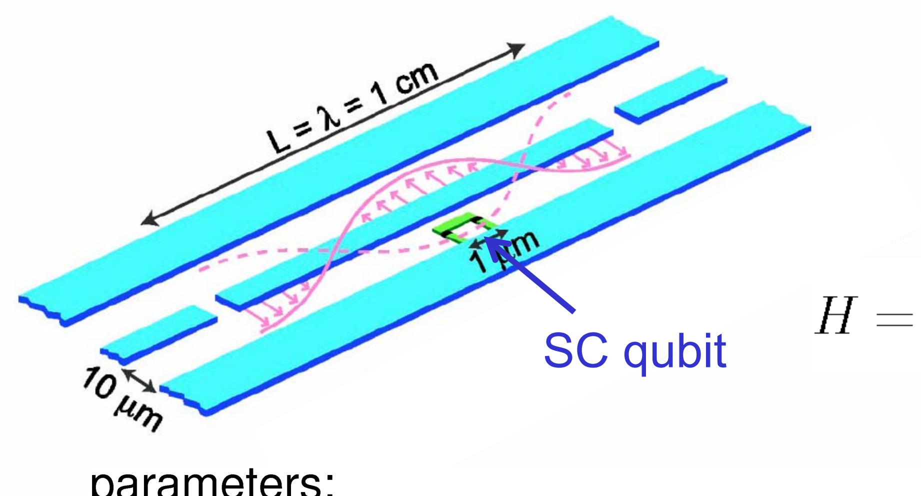
Cooper Pair Box
(qubit)
as nonlinearity

polar molecular ensemble 2:
quantum memory
(qubit or continuous variable)

R. Schoelkopf, M. Devoret,
S. Girvin (Yale)

1. strong CQED with superconducting circuits

- Cavity QED



SC qubit

$$H = \omega_c a^\dagger a + \frac{1}{2}\omega_q(t)\sigma_z + g(a\sigma_+ + \mathrm{h.c.})$$

Jaynes-Cummings

parameters:

cavity frequency $\omega_c \sim 2\pi \times 10$ GHz

cavity damping $\kappa \sim 2\pi \times 1 \ldots 0.01$ MHz

SC qubit - cavity coupling $g \sim 2\pi \times 30$ MHz

SC qubit damping $\Gamma \sim 2\pi \times 1$ MHz

good cavity

strong coupling!
(mode volume $V/\lambda^3 \approx 10^{-5}$)

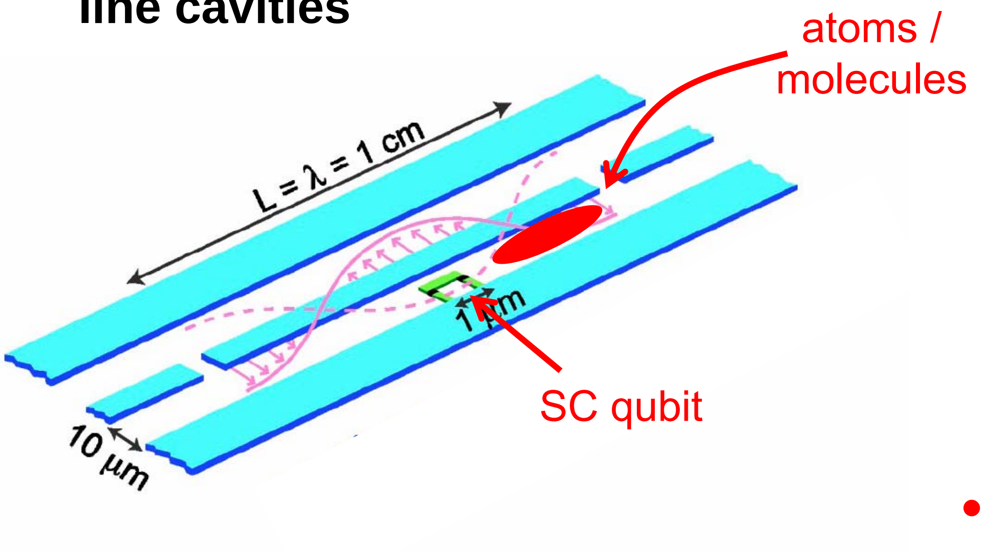
“not so great” qubits

- [... similar results expected for coupling to quantum dots (Delft)]
- [compare with CQED with atoms in optical and microwave regime]

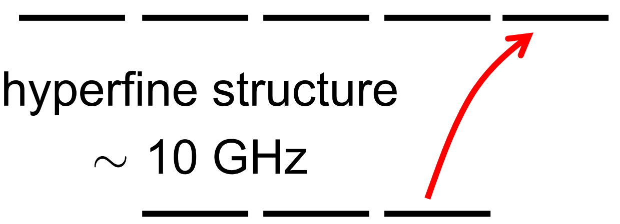
… with Yale/Harvard

2. ... coupling atoms or molecules

- **superconducting transmission line cavities**



- **hyperfine excitation of BEC / atomic ensemble**


hyperfine structure

$\sim$ 10 GHz

$$g \sim 2\pi \times 80\ \text{Hz}\ \sqrt{\#\text{atoms}}$$

- **rotational excitation of polar molecule(s)**


N=1

rotational excitations

$\sim$ 10 GHz

μ large!

N=0

$$g \sim 2\pi \times 10\ \text{KHz}\ \sqrt{\#\text{molecules}}$$

$$\sim 2\pi \times 1 \ldots 10\text{MHz}$$ ensemble ☺

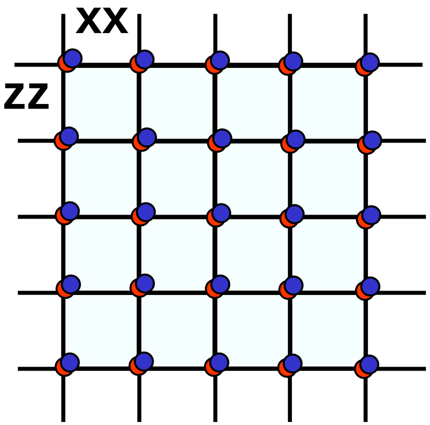
A. Micheli, G. Brennen, PZ, preprint, Dec 2005

Polar Molecules in an Optical Lattice: Lattice Spin Models

- polar molecules on optical lattices provide a complete toolbox to realize general lattice spin models *in a natural way*

Examples:

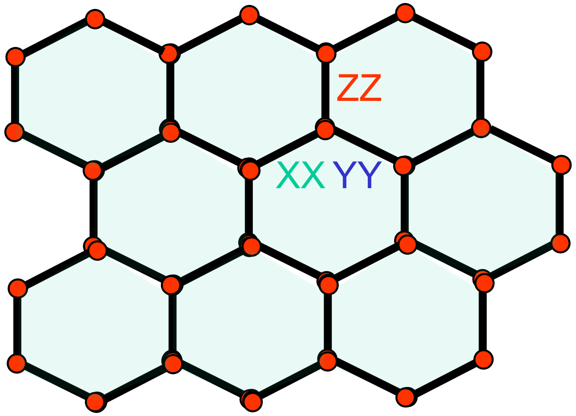
Duocot, Feigelman, Ioffe et al.



$$H^{(\mathrm{I})}_{\mathrm{spin}} = \sum_{i=1}^{\ell-1}\sum_{j=1}^{\ell-1} J(\sigma^z_{i,j}\sigma^z_{i,j+1} + \cos\zeta\,\sigma^x_{i,j}\sigma^x_{i+1,j})$$

protected quantum memory:
degenerate ground states as qubits

Kitaev

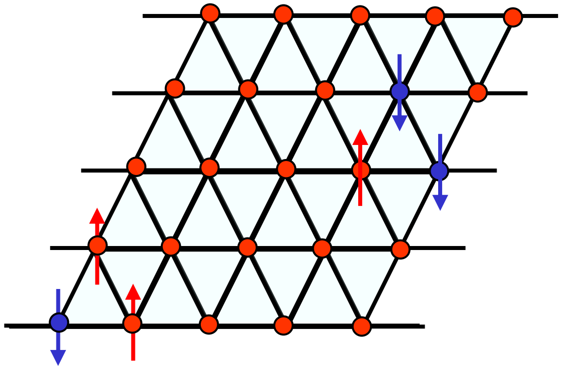


$$H^{(\mathrm{II})}_{\mathrm{spin}} = J_\perp \sum_{x-\mathrm{links}} \sigma^x_j\sigma^x_k + J_\perp \sum_{y-\mathrm{links}} \sigma^y_j\sigma^y_k + J_z \sum_{z-\mathrm{links}} \sigma^z_j\sigma^z_k$$

(Wigner-) Crystals with Polar Molecules

H.P. Büchler
V. Steixner
G. Pupillo
M. Lukin
…

- “Wigner crystals“ in 1D and 2D ($1/R^3$ repulsion – for $R > R_0$)



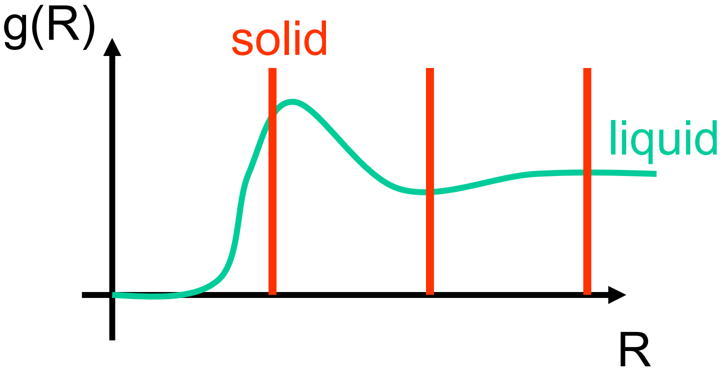
2D triangular lattice
(Abrikosov lattice)

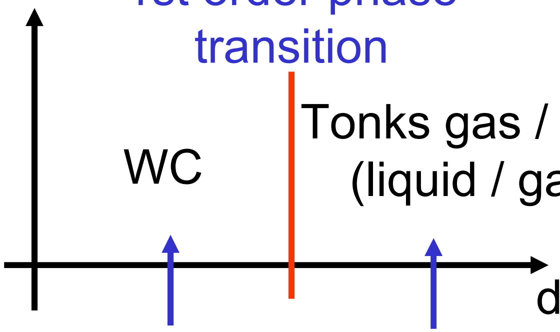
$$\gamma = \frac{\text{potential energy}}{\text{kinetic energy}} = \frac{d^2/R^3}{\hbar^2/2MR^2} \sim \frac{1}{R} \sim n^{1/3}$$

dipole-dipole: crystal for *high* density

$$\gamma = \frac{e^2/R}{\hbar^2/2MR^2} \sim R$$

Coulomb: WC for *low* density (ions)





1st order phase transition

WC

Tonks gas / BEC
(liquid / gas)

mean distance

~ 100 nm

quantum statistics

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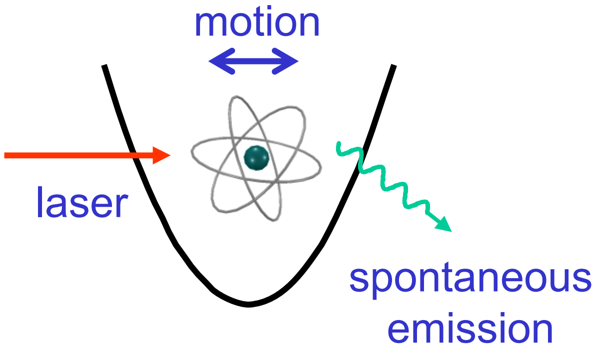
1.3 Quantum optical systems as *open* quantum systems

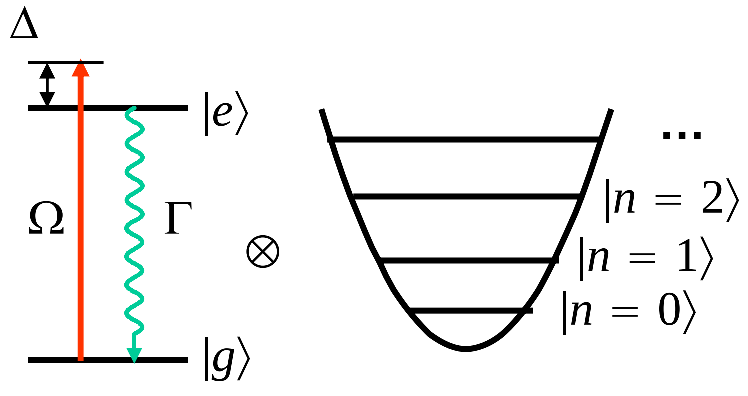
- example: trapped ion



motion

laser

spontaneous emission



Δ

Ω

Γ

$|e\rangle$

$|g\rangle$

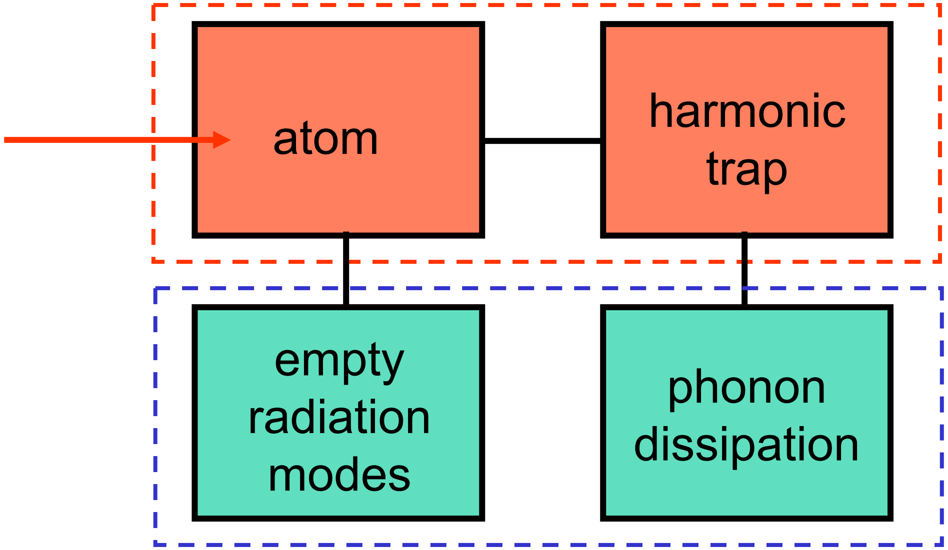
$\otimes$

...

$|n=2\rangle$

$|n=1\rangle$

$|n=0\rangle$



atom

harmonic trap

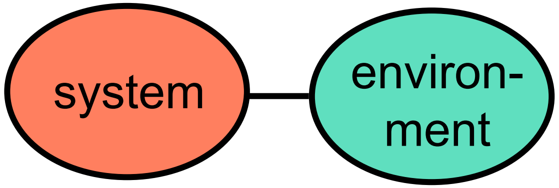
empty radiation modes

phonon dissipation

Our approach ...

Quantum Optics ⟷ Quantum Information

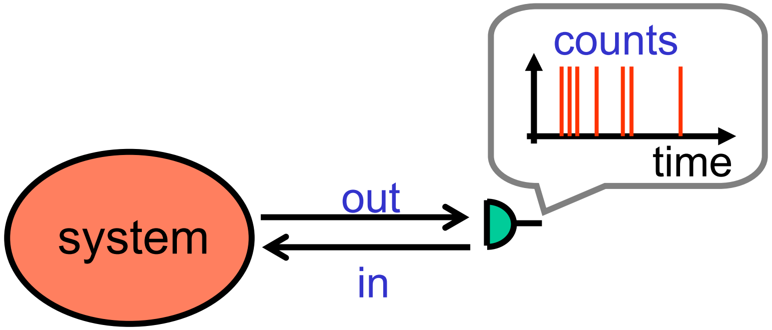
- Open quantum system



harmonic oscillators

✓ master equation

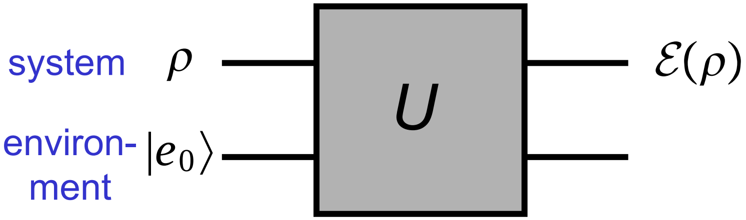
- Continuous observation



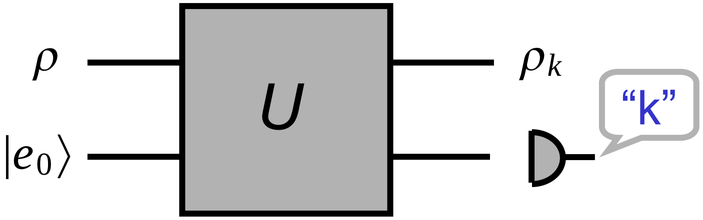
✓ Stochastic Schrödinger Equation

“Quantum Markov processes“

- Quantum operations



$$\rho \to \mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$$

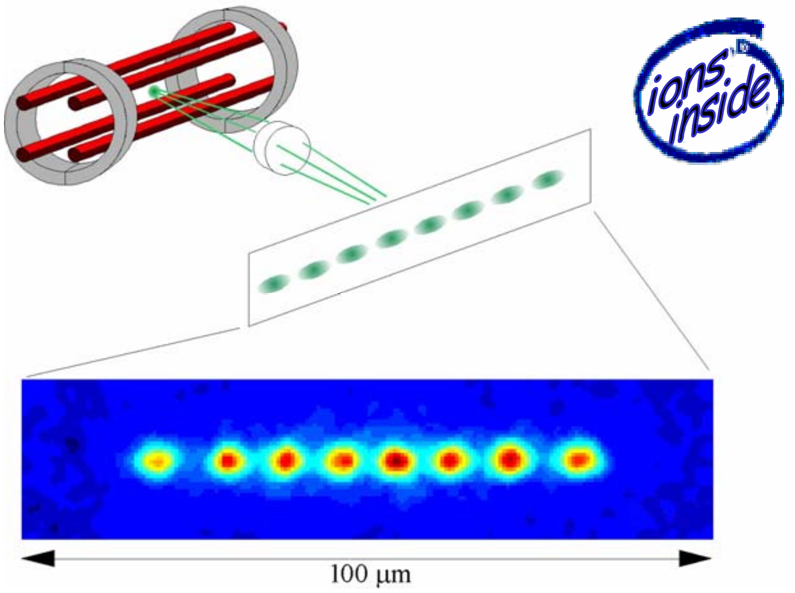


Summary: what the lectures are about …

- Theoretical modelling of quantum optical systems
  - how to describe theoretically trapped atoms and ions in various traps, CQED, atomic ensembles etc.

- Quantum state engineering / QPIC with qo systems
  - how to perform gate operations

- Preparation & Measurement in qo systems
  - state preparation and read out
  - decoherence
  - from quantum operations to stochastic Schrödinger equations, continuous measurement and all that

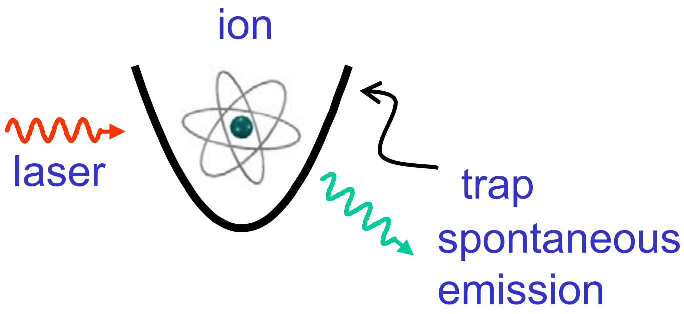


Quantum Computing with Trapped Ions

- basics: quantum optics of single ions & many ions
  - develop toolbox for quantum state engineering
- 2-qubit gates
  - from first 1995 gate proposals and realizations
  - ... geometric and „best“ coherent control gates
- spin models

1. A single trapped ion

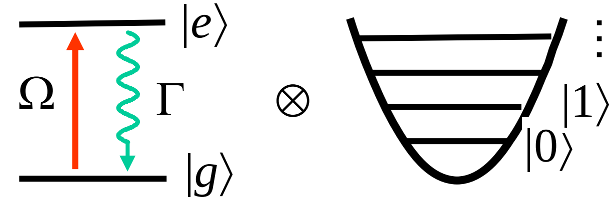
- a single laser driven trapped ion



✓ system: atom + motion in trap:
goal: quantum engineering

✓ [open quantum system]

- system: two-level atom + harmonic oscillator



two-level system

(= qubit)

phonons

$\nu \sim 10$ MHz

$$H = H_{0T} + H_{0A} + H_1$$

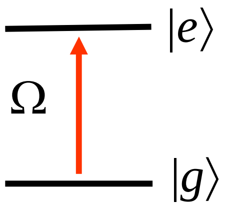
trap $$H_{0T} = \frac{\hat{P}^2}{2M} + \frac{1}{2}M\nu^2\hat{X}^2 \equiv \hbar\nu(a^\dagger a + \frac{1}{2})$$

atom $$H_{0A} = -\hbar\Delta|e\rangle\langle e|$$

laser $$H_1 = -\frac{1}{2}\hbar\Omega e^{ik_L\hat{X}}|e\rangle\langle g| + \text{h.c.}$$

$$H = \frac{\hat{p}^2}{2M} + \frac{1}{2}M\nu^2\hat{x}^2 + \hbar\omega_{eg}|e\rangle\langle e| - \hbar(\frac{1}{2}\Omega e^{ik\hat{x}-i\omega t}|e\rangle\langle g| + \text{h.c.})$$

- laser absorption & recoil



$$|g\rangle|\text{motion}\rangle \;\rightarrow\; |e\rangle e^{ik_L\hat{X}}|\text{motion}\rangle$$

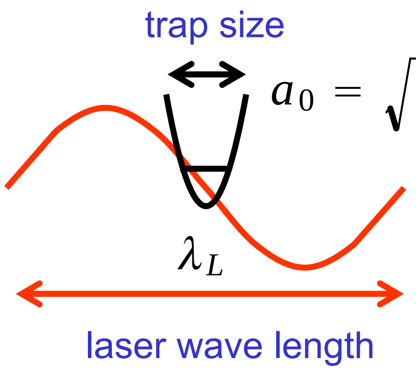
photon recoil kick

interaction $H_1 = -\frac{1}{2}\hbar\Omega e^{ik_L\hat{X}}|e\rangle\langle g|+\text{h.c.}$

$\uparrow$

laser photon recoil:
couples internal dynamics and center-of-mass

- Lamb-Dicke limit



$$a_0 = \sqrt{\frac{\hbar}{2M\nu}}$$

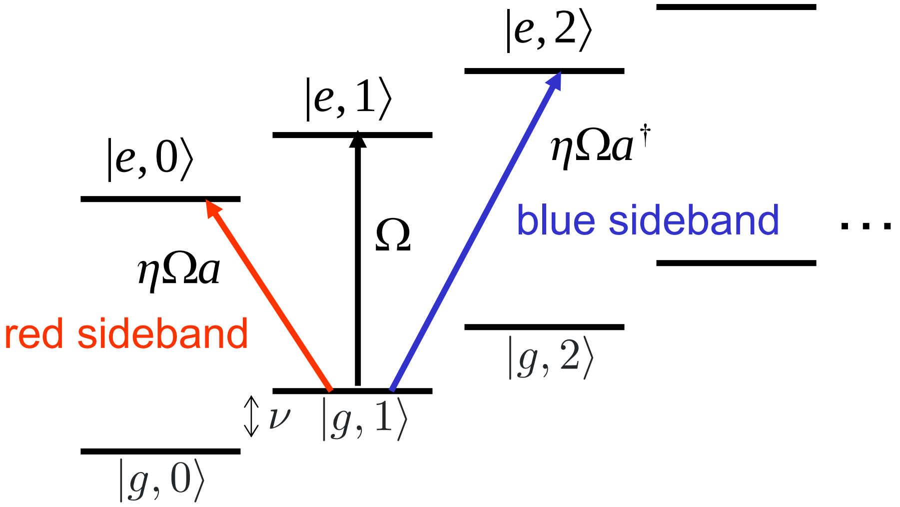
Lamb-Dicke expansion

$$\begin{aligned} e^{ik_L\hat{X}} &= e^{i\eta(a+a^\dagger)} \\ &= 1 + i\eta(a+a^\dagger) + \ldots \end{aligned}$$

$\uparrow$

$$\eta = 2\pi\frac{a_0}{\lambda_L} \equiv \sqrt{\frac{\epsilon_R}{\hbar\nu}} \quad \sim 0.1$$

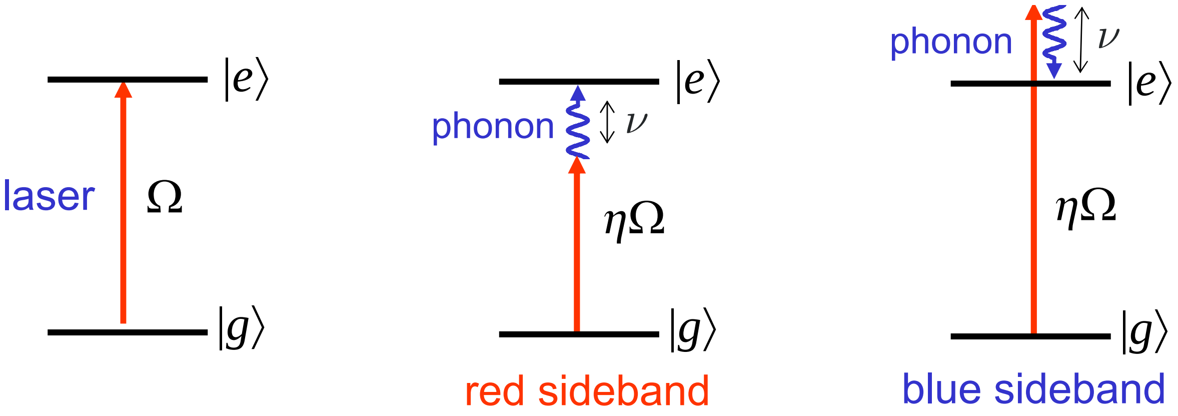
- spectroscopy: atom + trap



laser interaction

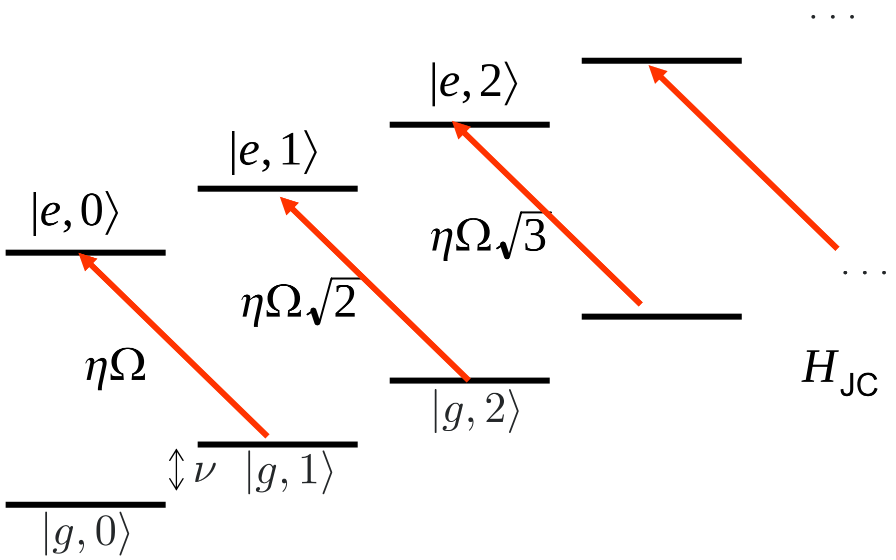
$$\frac{1}{2}\Omega e^{ik_L\hat{X}}|e\rangle\langle g| = \frac{1}{2}\Omega|e\rangle\langle g|$$
$$+\, i\frac{1}{2}\Omega\eta a\,|e\rangle\langle g|$$
$$+\, i\frac{1}{2}\Omega\eta a^\dagger|e\rangle\langle g|$$
$$+\ldots$$

- processes: "Hamiltonian toolbox for phonon-state engineering"



laser assisted phonon absorption and emission

- example: "laser tuned to red sideband"



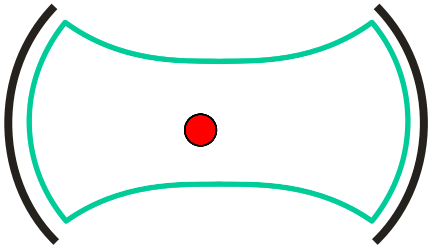
Jaynes-Cummings model

$$H_{\text{JC}} = \hbar\nu a^{\dagger}a - \hbar\Delta|e\rangle\langle e| - \tfrac{1}{2} i\hbar(\Omega\eta)|e\rangle\langle g|a + \text{h.c.}$$

trap

vacuum Rabi frequency
~ laser (switchable)

- Remark: CQED



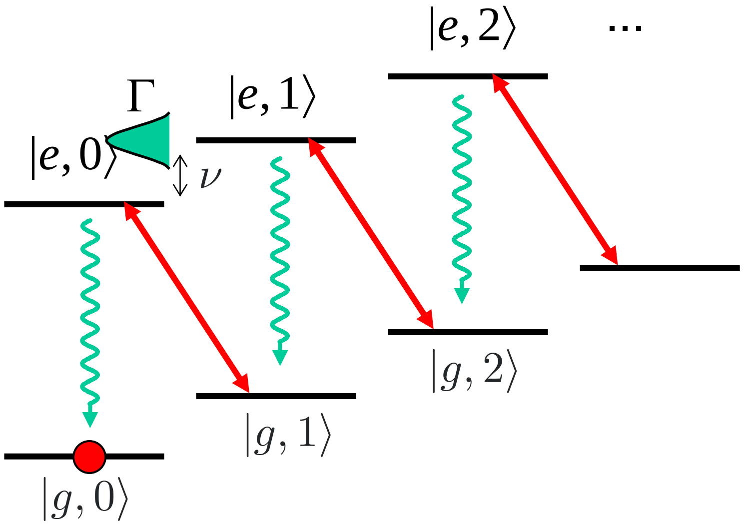
$$H_{\text{JC}} = \nu a^{\dagger}a + \omega_{eg}|e\rangle\langle e| - ig|e\rangle\langle g|a + \text{h.c.}$$

optical

vacuum Rabi frequency

[Dissipation: spontaneous emission]

- sideband cooling... as optical pumping to the ground state



preparation of pure states

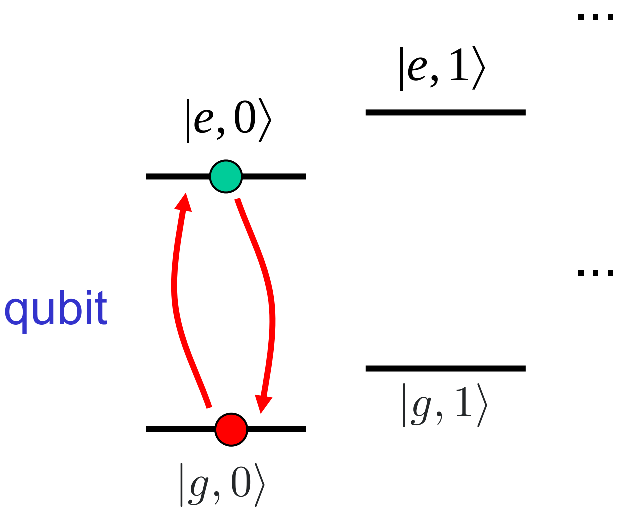
$$\rho_{\text{atom}} \otimes \rho_{\text{motion}} \longrightarrow |g\rangle\langle g| \otimes |0\rangle\langle 0|$$

- measurement of internal states: quantum jumps …

qubit read out

Excercises in quantum state engineering

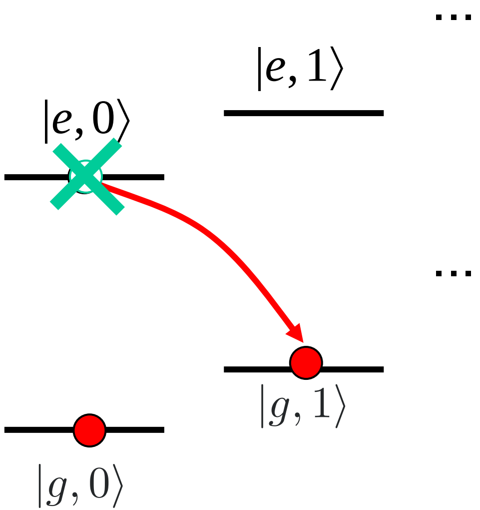
- **Example 1:** single qubit rotation



$$(\alpha|g\rangle + \beta|e\rangle) \otimes |0\rangle \overset{U_1}{\to} (\alpha'|g\rangle + \beta'|e\rangle) \otimes |0\rangle$$

(1) we can rotate the qubit without
touching the phonon state

- **Example 2:** swapping the qubit to the phonon mode



$$(\alpha|g\rangle + \beta|e\rangle) \otimes |0\rangle \longrightarrow |g\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$$

ion qubit phonon qubit

(2) Using a laser pulse we can swap
qubits stored in ions to the phonon
modes (and vice versa)

- **Example 3:** engineering arbitrary phonon superposition states

$$|g\rangle \otimes |0\rangle \xrightarrow{U} |\Psi\rangle = |g\rangle \otimes \sum_{n=0}^{N} c_n |n\rangle$$

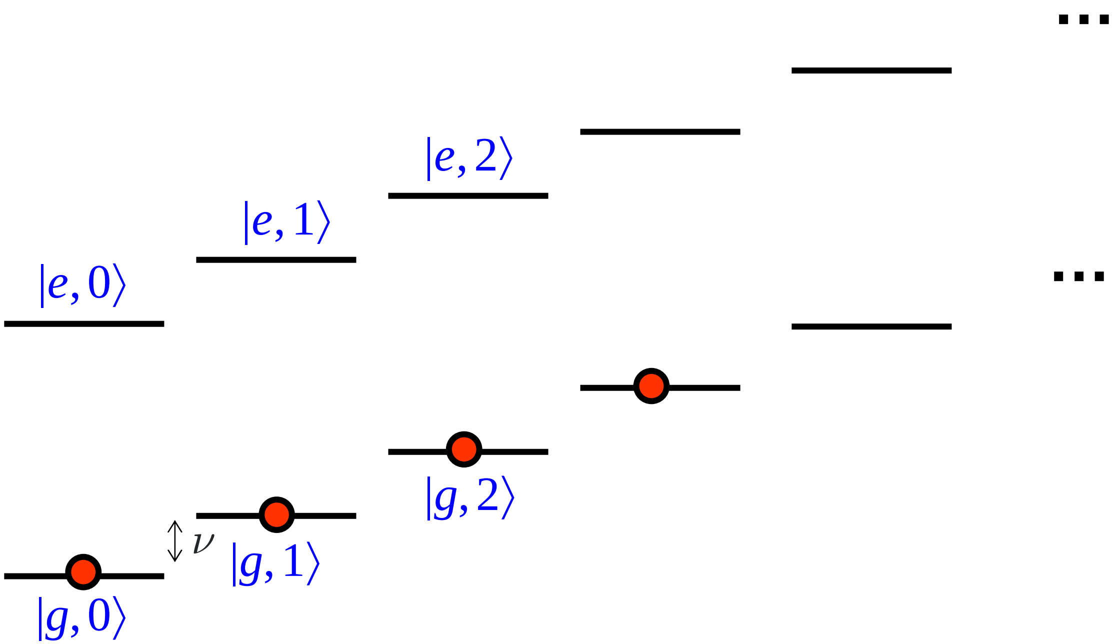
given coefficients c_n

✓ Fock states

✓ squeezed & coherent states

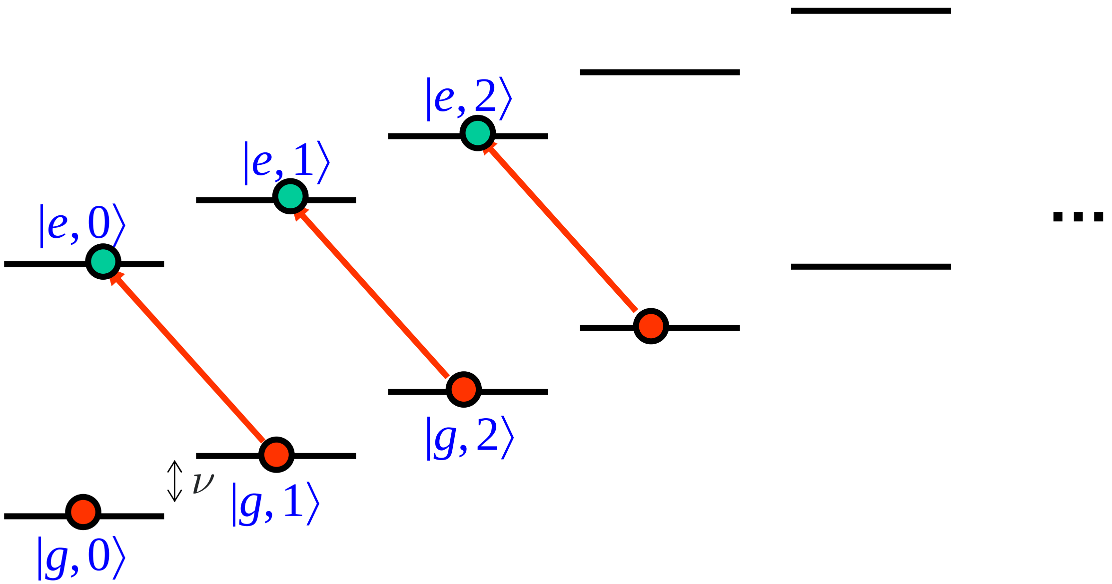
✓ Schrödinger cat states

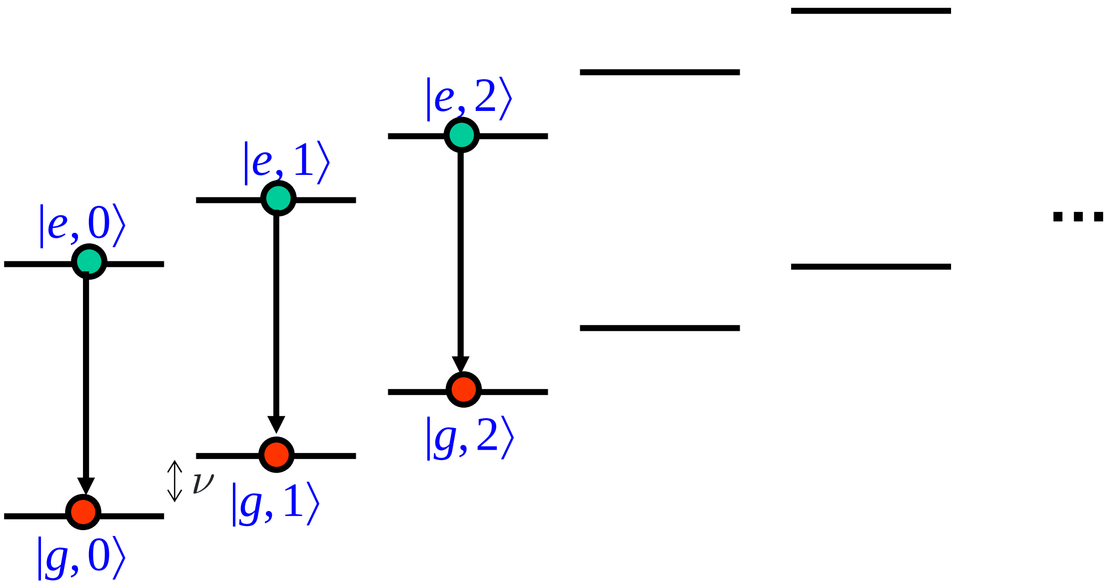
✓ ...

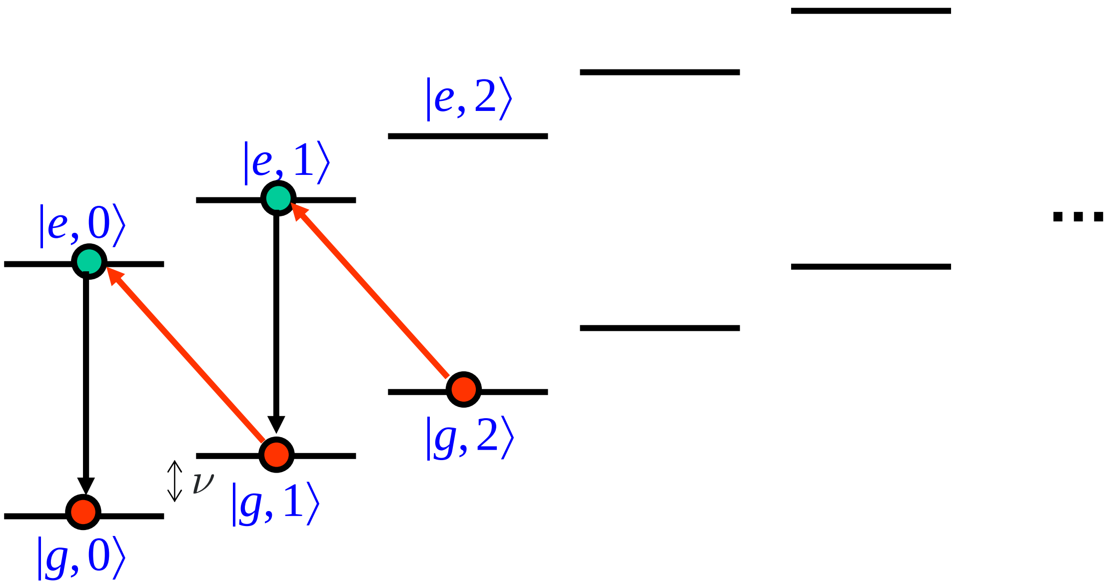


- Idea: we will look for the inverse U which transforms $|\Psi\rangle$ to $|g\rangle \otimes \sum_{n=0}^{n_{\max}} c_n |n\rangle$

Law & Eberly, Gardiner et al., Wineland et al.

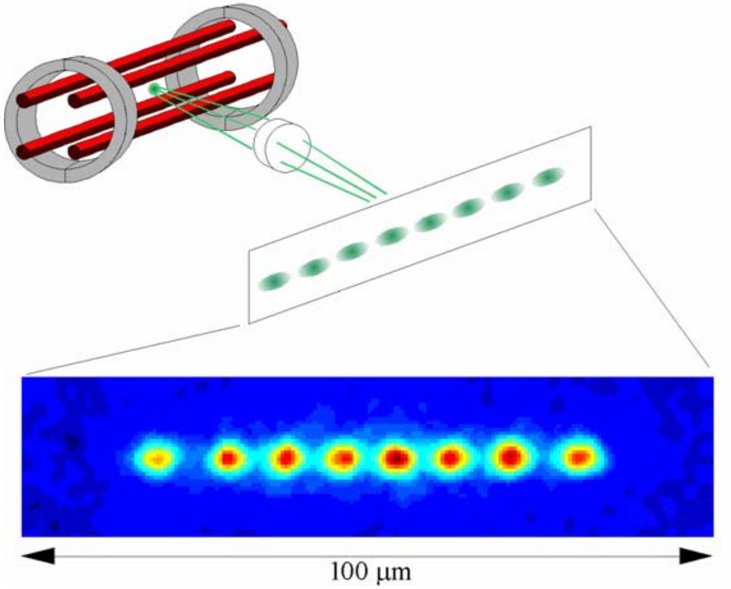

$|e,0\rangle$
$|e,1\rangle$
$|e,2\rangle$
ν
$|g,0\rangle$
$|g,1\rangle$
$|g,2\rangle$
...


$|e,0\rangle$
$|e,1\rangle$
$|e,2\rangle$
$|g,0\rangle$
$|g,1\rangle$
$|g,2\rangle$
ν
...

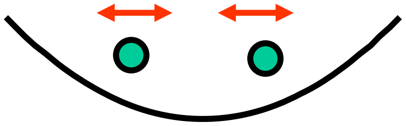

$|e,0\rangle$
$|e,1\rangle$
$|e,2\rangle$
$|g,0\rangle$
$|g,1\rangle$
$|g,2\rangle$
ν
...

HPI Zoller

2. Many Ions

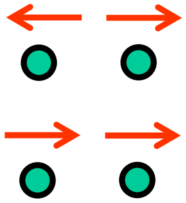


- 2 ions & collective phonon modes



stretch mode

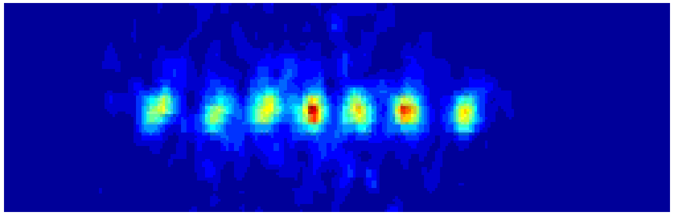
center-of-mass



$$\nu_r = \sqrt{3}\,\nu_c$$

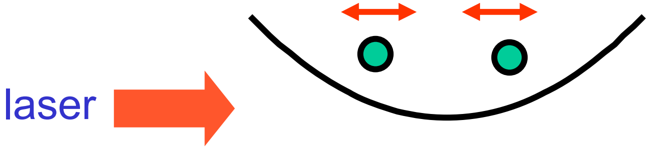
$$\nu_c = \nu$$

- example: classical ion motion

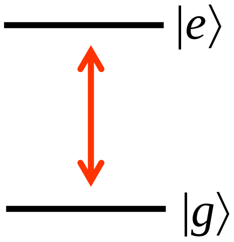


(3) We can swap a qubit to a *collective* mode via laser pulse

- **Example:** 2 ions in a 1D trap kicked by laser light



laser



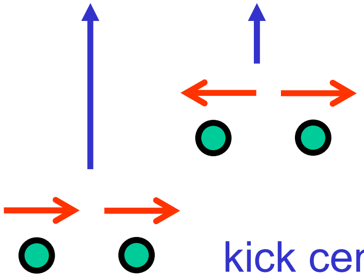
$|e\rangle$

$|g\rangle$

qubit

$$H = \nu_c a^\dagger a + \nu_r b^\dagger b$$

$$+ \frac{1}{2}\Omega(t)\sigma_1^+ e^{i\eta_c(a^\dagger+a)+\frac{1}{2}\eta_r(b^\dagger+b)} + \frac{1}{2}\Omega(t)\sigma_2^+ e^{i\eta_c(a^\dagger+a)-\frac{1}{2}\eta_r(b^\dagger+b)} + \text{h.c}$$



kick stretch mode

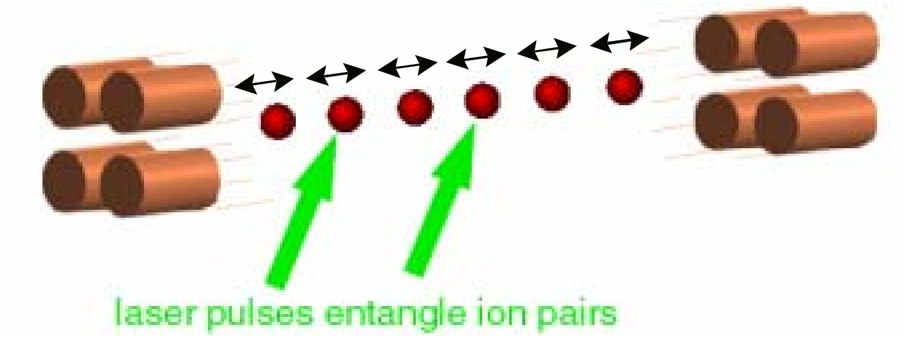
$\nu_c = \nu$

kick center-of-mass

$\nu_r = \sqrt{3}\,\nu_c$

Ion Trap Quantum Computer '95

- Cold ions in a linear trap



Qubits: internal atomic states

1-qubit gates: addressing ions with a laser

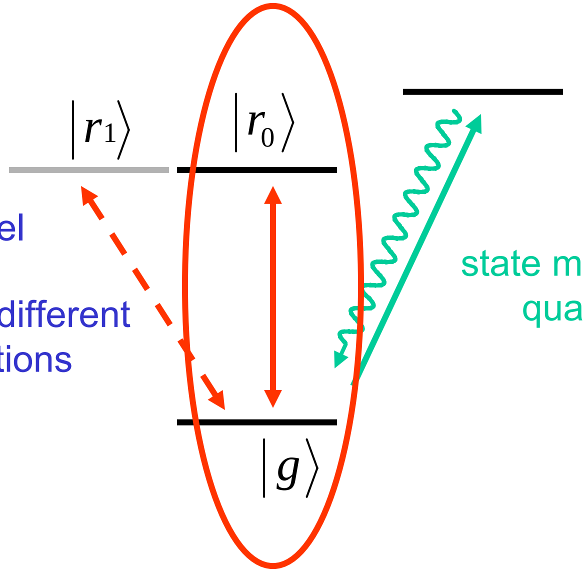
2-qubit gates: entanglement via exchange of phonons of quantized collective mode

- State vector

$$|\Psi\rangle = \underbrace{\sum c_x |x_{N-1}, \dots, x_0\rangle_{\text{atom}}}_{\text{quantum register}} \underbrace{|0\rangle_{\text{phonon}}}_{\text{databus}}$$

- QC as a time sequence of laser pulses
- Read out by quantum jumps

Level scheme



auxiliary level

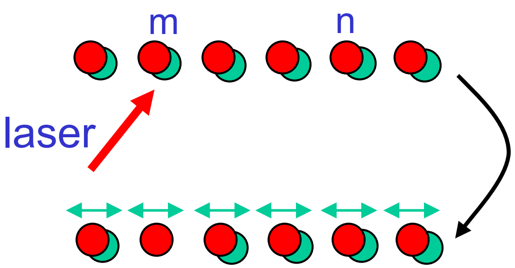
addressing with different
light polarizations

state measurement via
quantum jumps

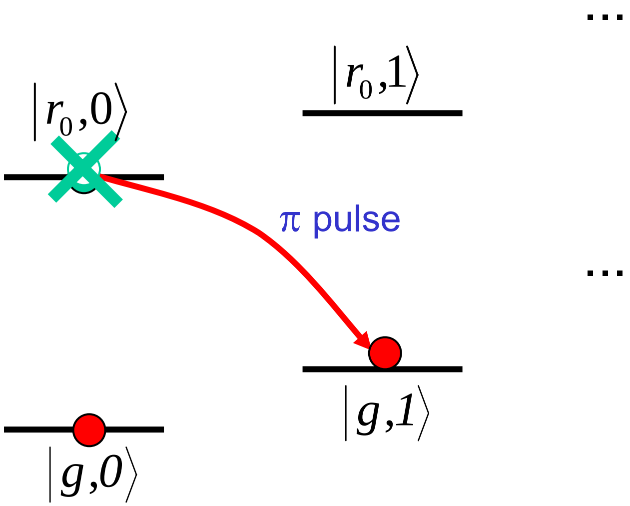
qubit

Two-qubit phase gate

- step 1: swap first qubit to phonon

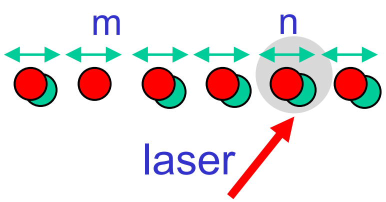


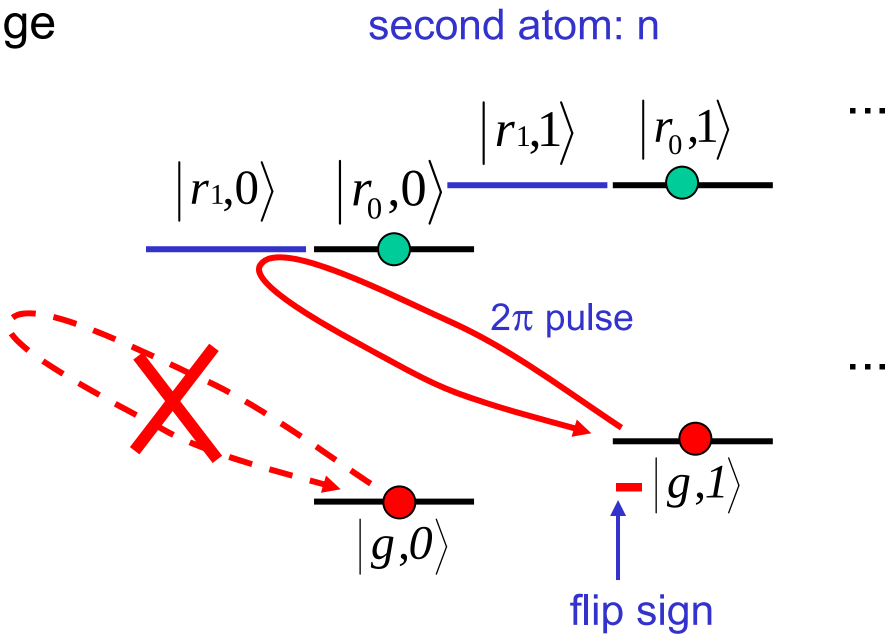
first atom: m



$$\begin{array}{ccc} |g\rangle_m|0\rangle & \xrightarrow{\hat{U}_m^{\pi,0}} & |g\rangle_m|0\rangle \\ |r\rangle_m|0\rangle & \longrightarrow & -i|g\rangle_m|1\rangle \end{array}$$

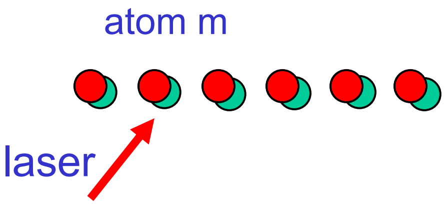
- step 2: conditional sign change





$$\begin{array}{rcr}
 & \hat{U}_n^{2\pi,1} & \\
|g\rangle_m|g\rangle_n|0\rangle & \longrightarrow & |g\rangle_m|g\rangle_n|0\rangle \\
|g\rangle_m|r\rangle_n|0\rangle & \longrightarrow & |g\rangle_m|r\rangle_n|0\rangle \\
-i|g\rangle_m|g\rangle_n|1\rangle & \longrightarrow & i|g\rangle_m|g\rangle_n|1\rangle \\
-i|g\rangle_m|r\rangle_n|1\rangle & \longrightarrow & -i|g\rangle_m|r\rangle_n|1\rangle
\end{array}$$

- step 3: swap phonon back to first qubit



$$|g\rangle_m \otimes \begin{array}{rcr} & \hat{U}_m^{\pi,0} & \\ |g\rangle_n|0\rangle & \longrightarrow & |g\rangle_m|g\rangle_n \\ |r\rangle_n|0\rangle & \longrightarrow & |g\rangle_m|r\rangle_n \\ i|g\rangle_n|1\rangle & \longrightarrow & |r\rangle_m|g\rangle_n \\ -i|r\rangle_n|1\rangle & \longrightarrow & -|r\rangle_m|r\rangle_n \end{array} \otimes |0\rangle$$

- summary: we have a phase gate between atom m and n

$$\begin{array}{lcl} |g\rangle|g\rangle\ |0\rangle & \longrightarrow & \ \ |g\rangle|g\rangle\ |0\rangle, \\ |g\rangle|r_0\rangle\ |0\rangle & \longrightarrow & \ \ |g\rangle|r_0\rangle\ |0\rangle, \\ |r_0\rangle|g\rangle\ |0\rangle & \longrightarrow & \ \ |r_0\rangle|g\rangle\ |0\rangle, \\ |r_0\rangle|r_0\rangle|0\rangle & \longrightarrow & -\ |r_0\rangle|r_0\rangle\ |0\rangle. \end{array}$$

phonon mode returned to initial state

$$|\epsilon_1\rangle|\epsilon_2\rangle \rightarrow (-1)^{\epsilon_1\,\epsilon_2}|\epsilon_1\rangle|\epsilon_2\rangle \quad (\epsilon_{1,2}=0,1)$$

Rem.: this idea translates immediately to CQED

- (addressable) 2 ion controlled-NOT + tomography

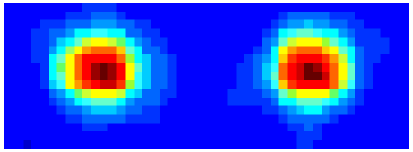
Realization of the Cirac–Zoller controlled-NOT quantum gate

Ferdinand Schmidt-Kaler, Hartmut Häffner, Mark Riebe, Stephan Gulde, Gavin P. T. Lancaster, Thomas Deuschle, Christoph Becher, Christian F. Roos, Jürgen Eschner & Rainer Blatt

Institut für Experimentalphysik, Universität Innsbruck, Technikerstraße 25, A-6020 Innsbruck, Austria

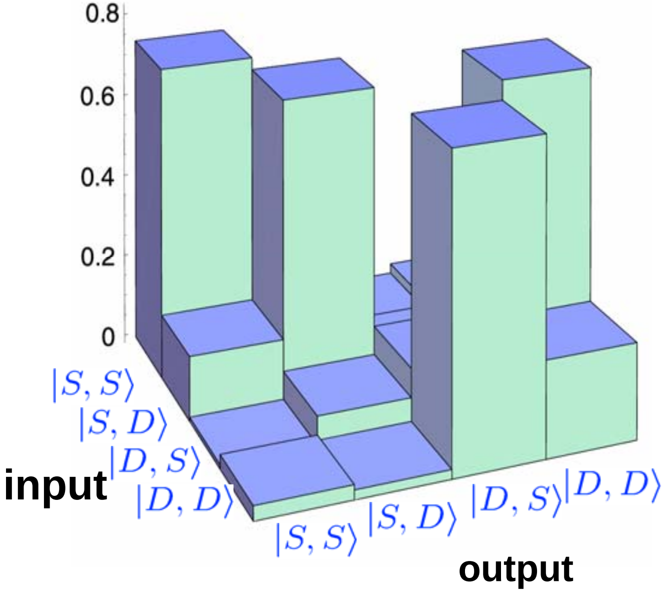
Experimental demonstration of a robust, high-fidelity geometric two ion-qubit phase gate

D. Leibfried[*,†], B. DeMarco[*], V. Meyer[*], D. Lucas[*,‡], M. Barrett[*], J. Britton[*], W. M. Itano[*], B. Jelenković[*,§], C. Langer[*], T. Rosenband[*] & D. J. Wineland[*]

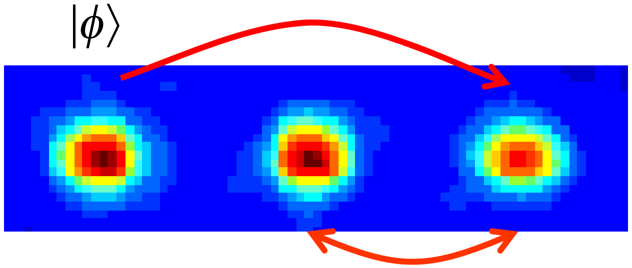


truth table CNOT

Innsbruck



- teleportation Innsbruck / Boulder



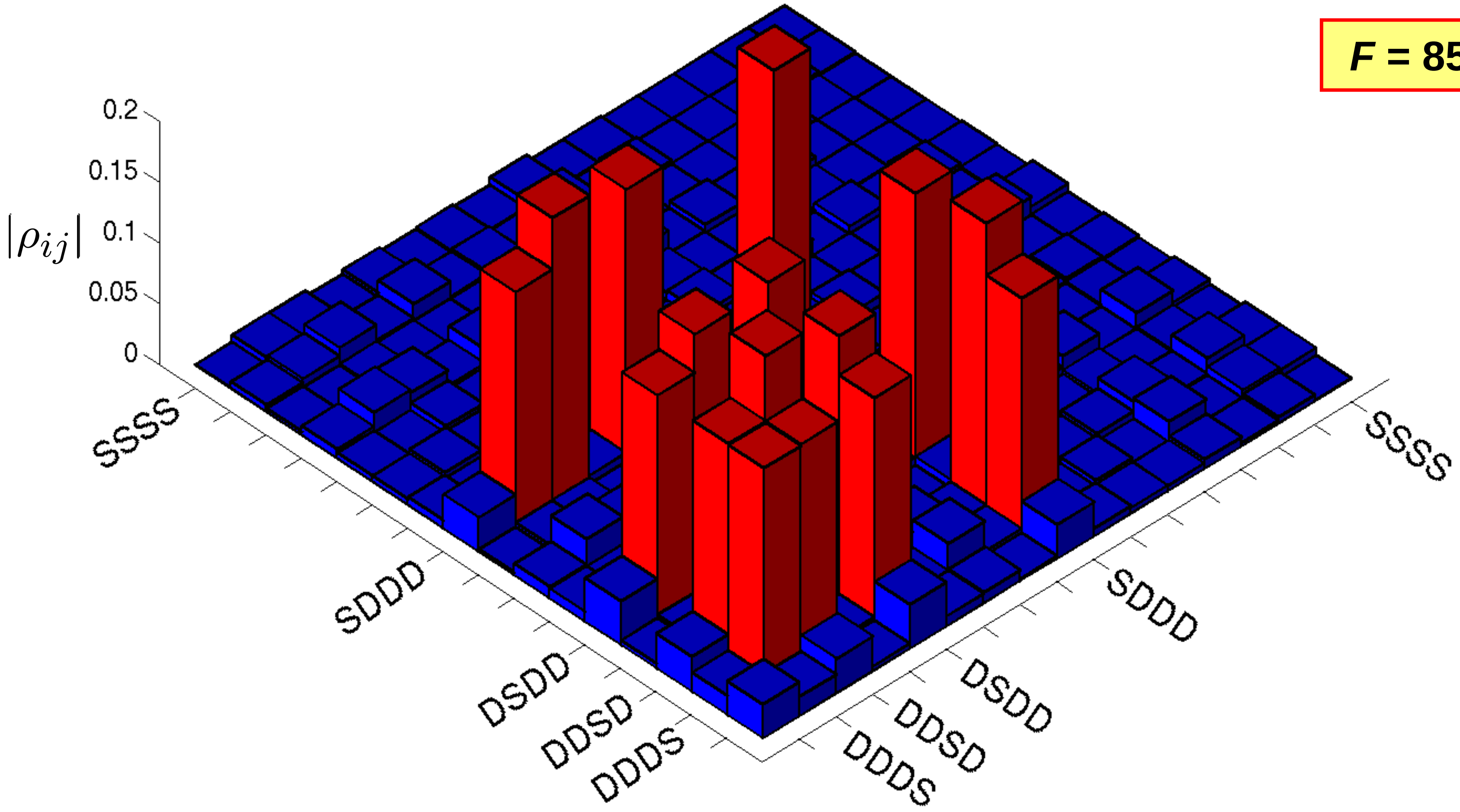
- decoherence: quantum memory DFS 20 sec

EPR pair

R. Blatt et al.
Nature 2005

Four-ion W-state

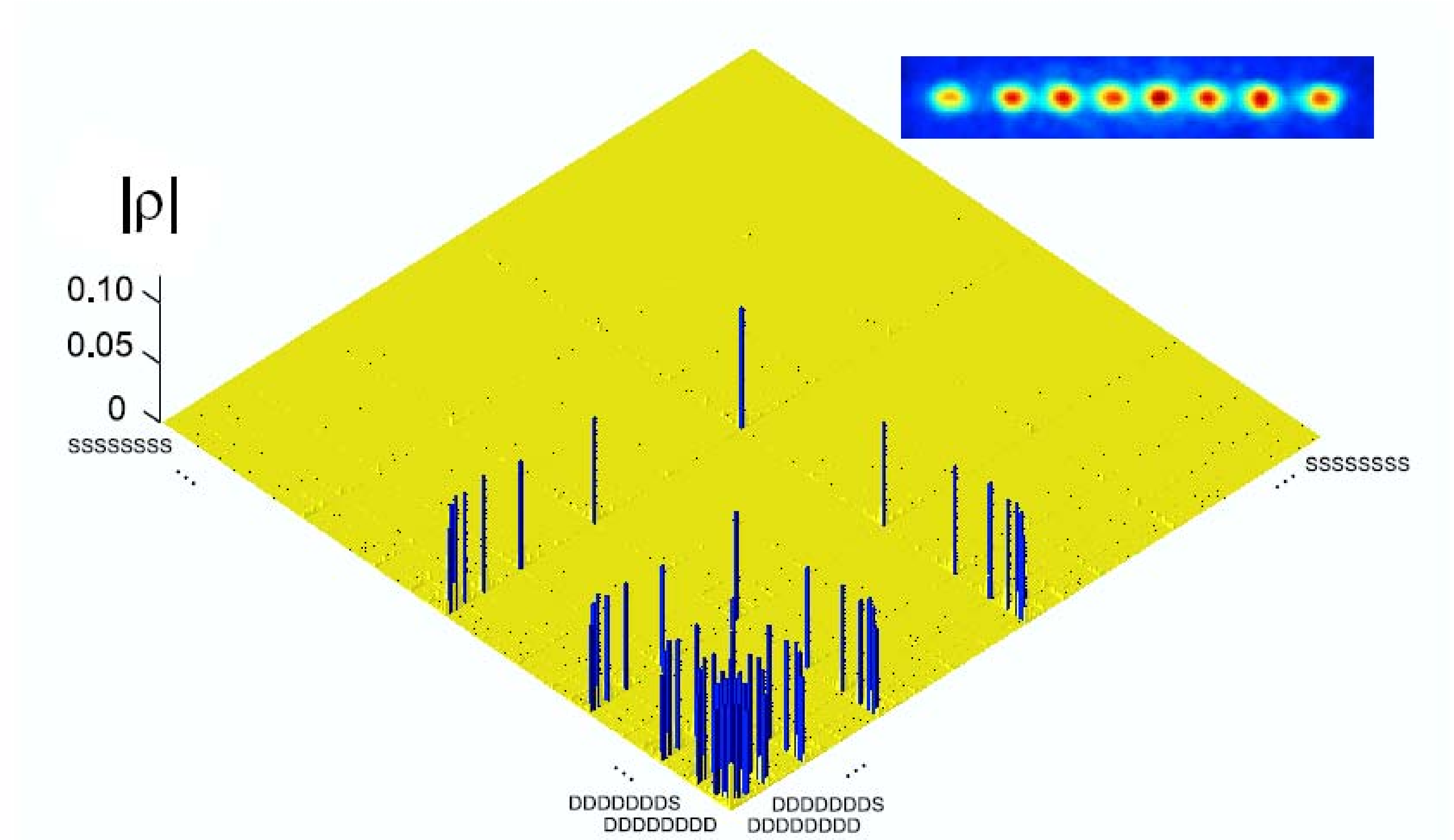
$$|\Psi_4\rangle = \frac{1}{2}(|SDDD\rangle + |DSDD\rangle + |DDSD\rangle + |DDDS\rangle)$$



***F* = 85%**

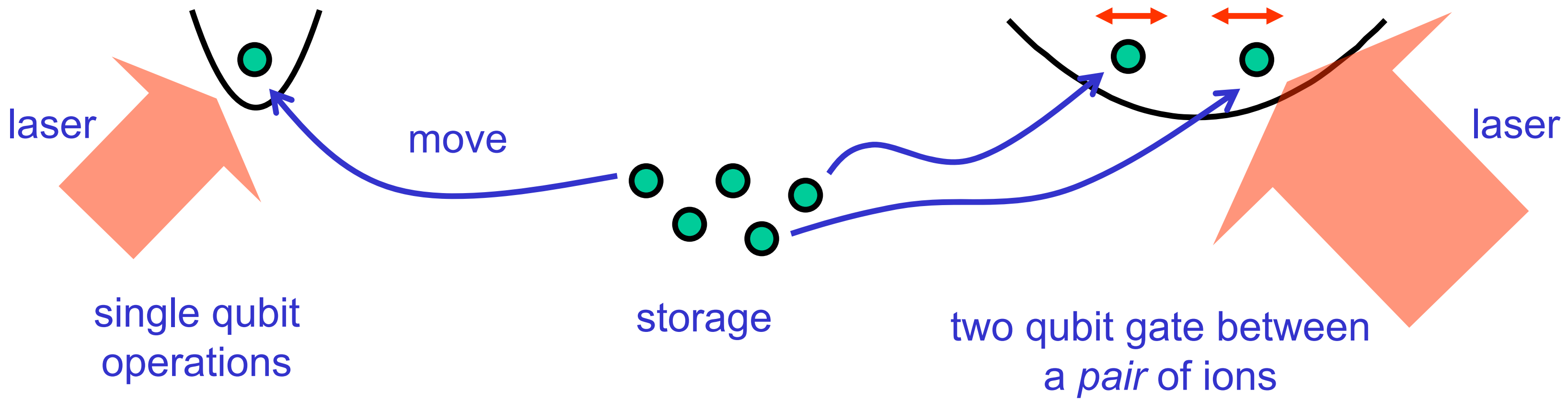
Eight ion W-state

R. Blatt et al.
Nature 2005



Scalability

- key idea: moving ions … without destroying the qubit



Two-qubit gate … the “wish list”

- fast: max # operations / decoherence [what are the limits?]

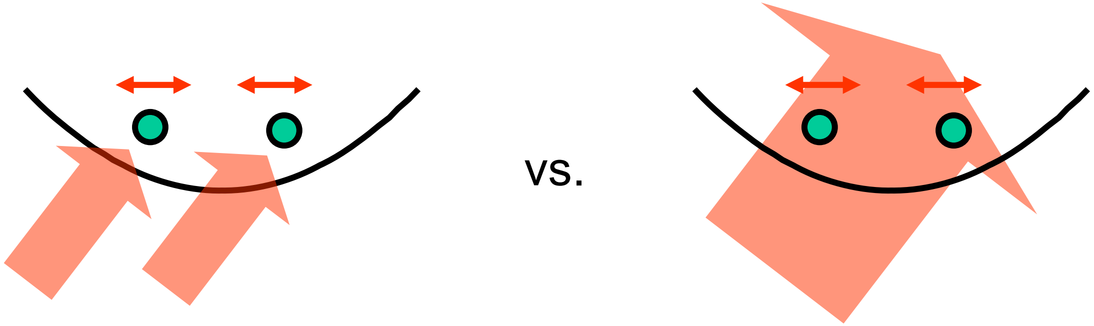
- NO temperature requirement: “hot” gate, i.e. NO ground state cooling

$$|\psi\rangle\langle\psi|\otimes\rho_{\text{motion}} \rightarrow \text{ entangle qubits via motion } \rightarrow |\psi\rangle\langle\psi|\otimes\rho'_{\text{motion}}$$

qubits motional state: e.g. thermal

motional state factors out

- NO indivdual addressing



addressing: large distance

vs.

strong coupling: small distance

Speed limits

- In all present proposals the speed limit for the gate is given by the trap frequency

$$T_{\text{gate}} \sim 1/\eta\nu$$

trap frequency

Lamb Dicke parameter $\eta = \sqrt{\frac{\epsilon_R}{\nu}}$

⇒ $$T_{\text{gate}} \sim 1/\sqrt{\nu}$$

$\nu \sim$ 10 MHz, i.e. $T_{\text{gate}} \sim \mu$ s

limits given by trap design

The rest of the lecture …

- Push gate

J.I. Cirac & PZ

- Geometric phase gates

D. Leibfried et al.
NIST

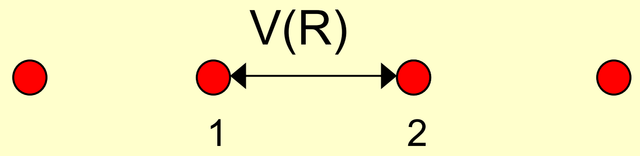
- Optimal Control Gates
  - what is the *best* gate for given resources?

J. Garcia-Ripoll
J.I. Cirac,
PZ

- [Examples]
  - fast gate with short laser pulses
  - fast gate with continuous laser pulses
  - engineering spin Hamiltonians …

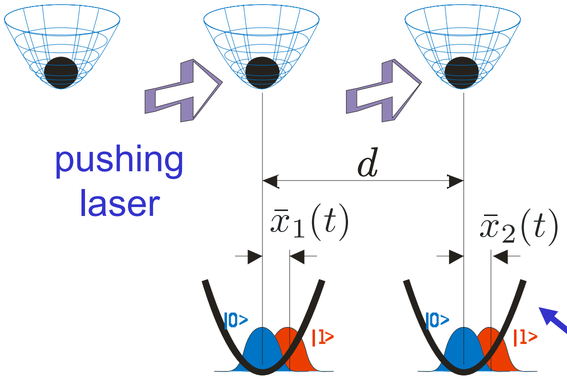
Another example for a 2-qubit gate …

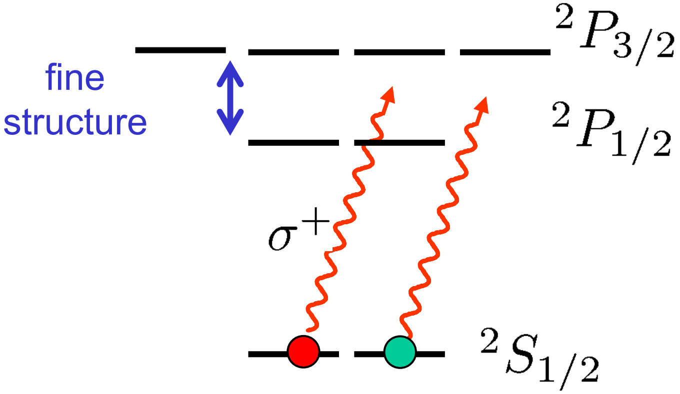
Push gate



state dependent interaction

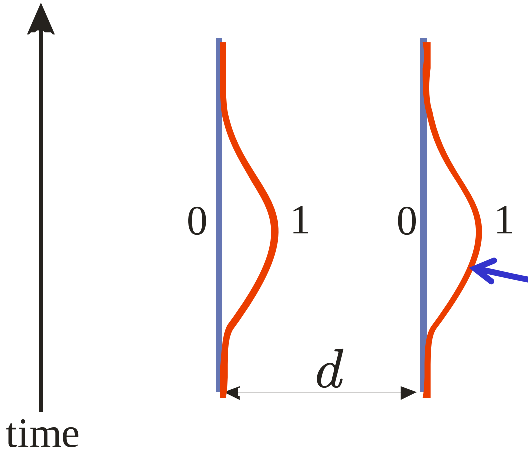
- converting "spin to charge"
- spin dependent optical potential





different AC Stark shifts

qubit dependent displacement of the ion

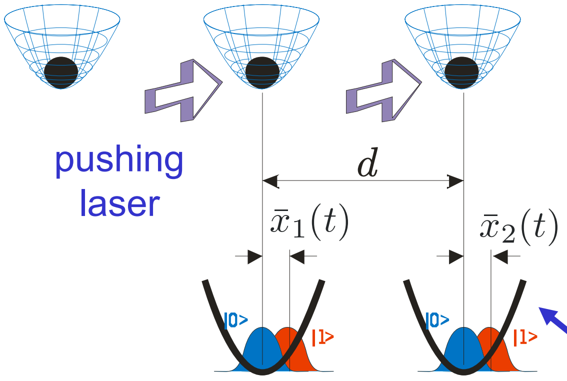


accumulate different energy shifts along different trajectories: 2-qubit gate

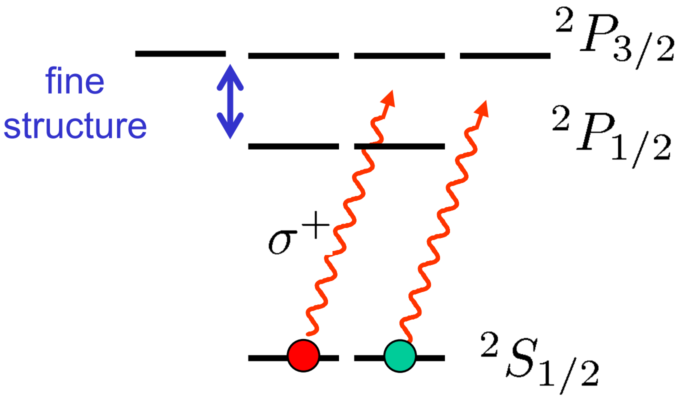
- robust: temperature insensitive ☺

Push gate

- converting "spin to charge"



- spin dependent optical potential

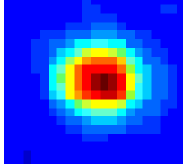


different AC Stark shifts

qubit dependent displacement of the ion

- Hamiltonian

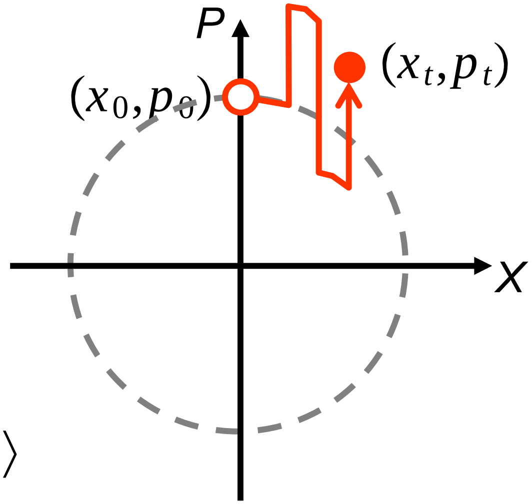
$$H = \sum_{i=1}^{N}\left[\frac{1}{2m}p_i^2 + V_{e,i}(x_i) - F_i(t)\sigma_z^i x_i\right] + \sum_{i<j}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{|x_i - x_j|}$$



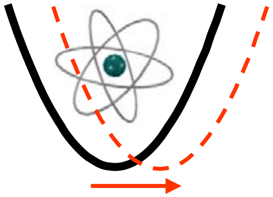
Geometric Phase [Gate]: One Ion

- Goal: geometric phase by driving a harmonic oscillator
- Hamiltonian

phase space



$$H = \tfrac{1}{2}\hbar\omega(\hat{p}^2 + \hat{x}^2) - f(t)\hat{x}$$



- Time evolution

$$|\psi_0\rangle = |z_0 \equiv x_0 + ip_0\rangle \Longrightarrow |\psi_t\rangle = e^{i\phi_t}|z_t \equiv x_t + ip_t\rangle$$

coherent state

coherent state

phase

- Solution

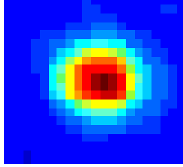
$$\frac{d}{dt}z = -i\omega z + i\frac{1}{\sqrt{2}}f(t) \Longrightarrow z_t = e^{-i\omega t}\left[z_0 + \frac{i}{\sqrt{2}}\int_0^t d\tau\, e^{i\omega\tau} f(\tau)\right]$$

$$\frac{d}{dt}\phi = \frac{1}{2\sqrt{2}}f(t)(z^* + z)$$

classical evolution

displacement

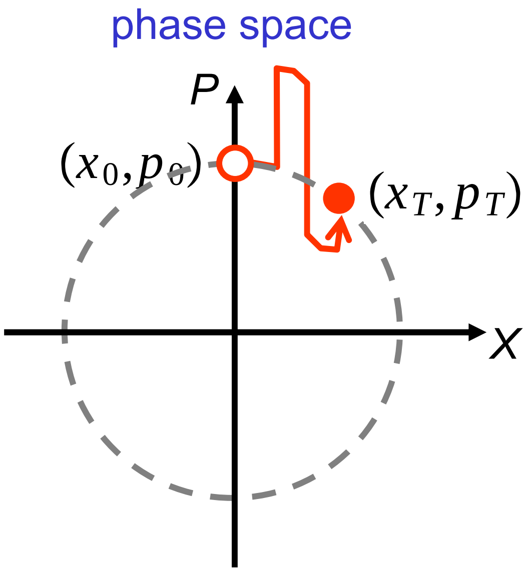
phase

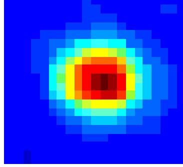


- Condition:

  After a given time *T* the coherent wavepacket is restored to the freely evolved state

$$\int_0^T d\tau\, e^{i\omega\tau} f(\tau) \overset{!}{=} 0$$

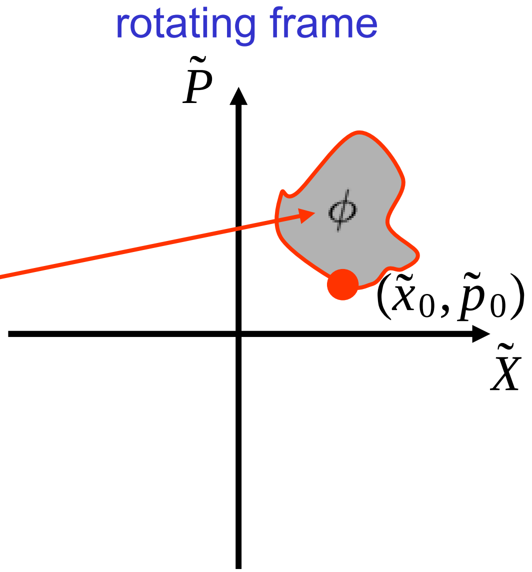




- Rotating frame $\tilde{z}_t \equiv \tilde{x}_t + i\tilde{p}_t = e^{i\omega t} z_t$

$$\frac{d\tilde{z}}{dt} = i e^{i\omega t} \frac{1}{\sqrt{2}} f(t)$$

$$\frac{d\phi}{dt} = \frac{d\tilde{p}}{dt}\tilde{x} - \frac{d\tilde{x}}{dt}\tilde{p} = 2\frac{dA}{dt}$$



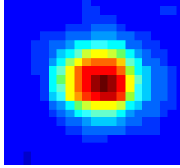
- Phase

$$\phi(T) = \mathrm{Im}\frac{i}{\sqrt{2}} \int_0^T d\tau\, e^{i\omega\tau} f(\tau)\, \tilde{z}_\tau^*$$

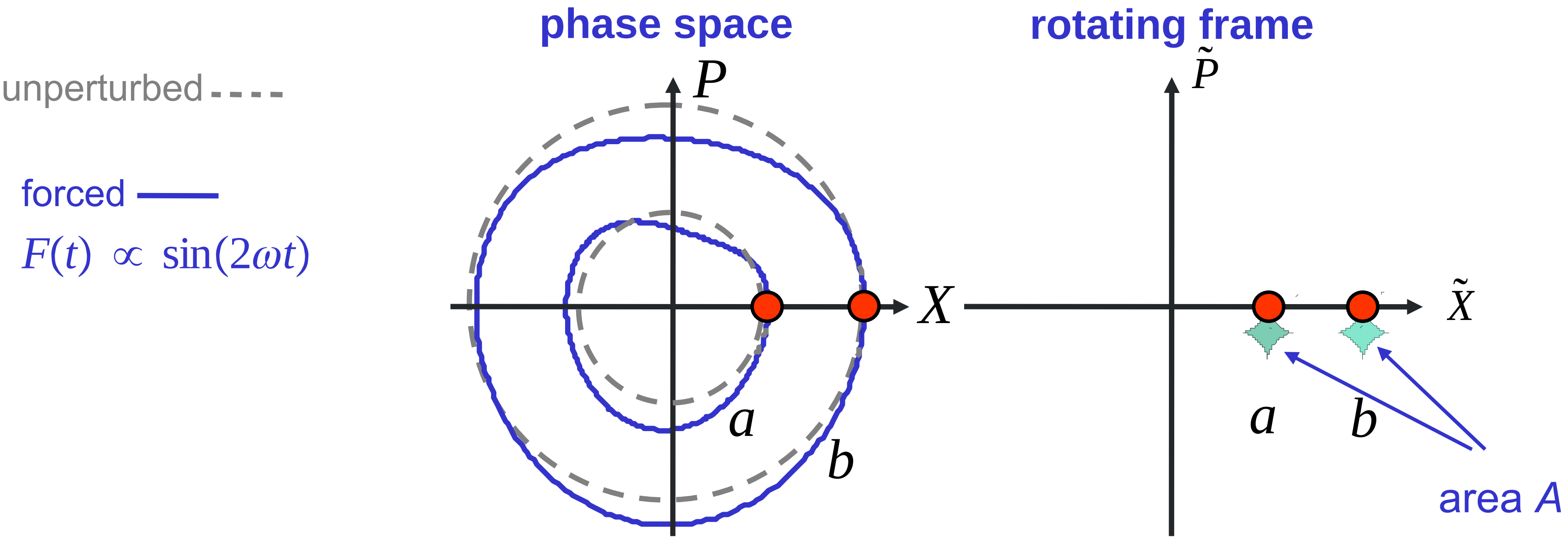
$$= \mathrm{Im}\frac{i}{\sqrt{2}} \left[\underbrace{\int_0^T d\tau\, e^{i\omega\tau} f(\tau_1)}_{=0} \right] \tilde{z}_0^* + \frac{1}{2}\underbrace{\mathrm{Im} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2\, e^{i\omega(\tau_1-\tau_2)} f(\tau_1) f(\tau_2)}$$

return condition

The phase does *not* depend on the initial state, (x_0,p_0)

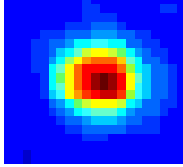


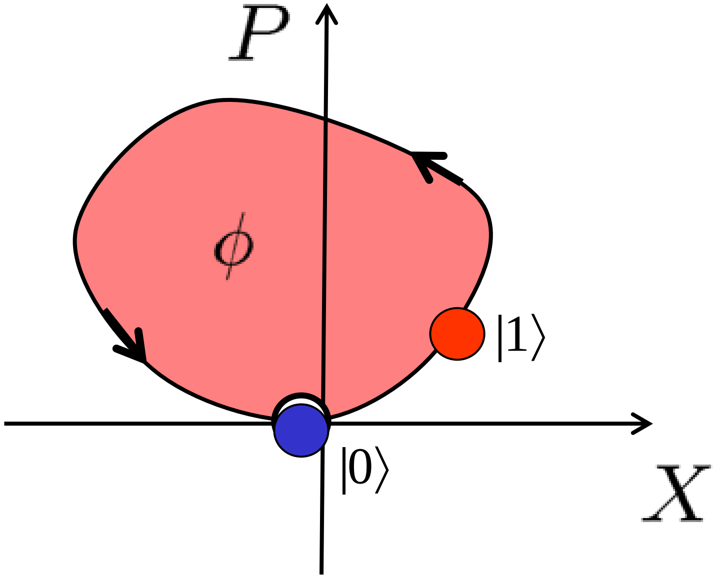
- Example



- The phase does not depend on the initial state, (x_0,p_0) ☺ (temperature independent)

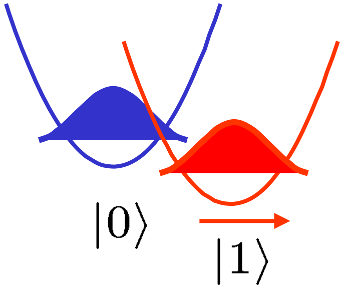
Geometric Phase Gate: Single Ion





- Hamiltonian

$$H = \tfrac{1}{2}\hbar\omega(\hat{p}^2 + \hat{x}^2) - |1\rangle\langle 1| f(t)\hat{x}$$



- Time evolution operator

$$U(T) = e^{i\phi|1\rangle\langle 1|}$$

$$(\alpha|0\rangle + \beta|1\rangle) \otimes |z_0\rangle$$

$$\xrightarrow{U(T)} (\alpha|0\rangle + \beta e^{i\phi}|1\rangle) \otimes |z_T\rangle$$

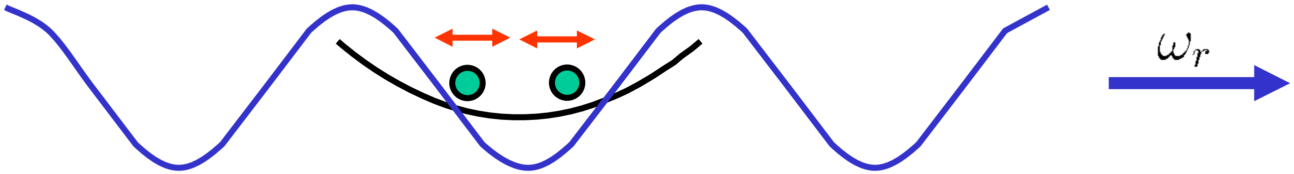
single ion phase gate

motion factors out

© **NIST** D. Leibfried et al.

NIST Gate: Leibfried *et al* Nature 2003

- 2 ions in a running standing wave tuned to ω_r



$$H = \omega_r a^\dagger a - F(t)(\sigma_z^1 + \sigma_z^2)(a_r + a_r^\dagger)$$

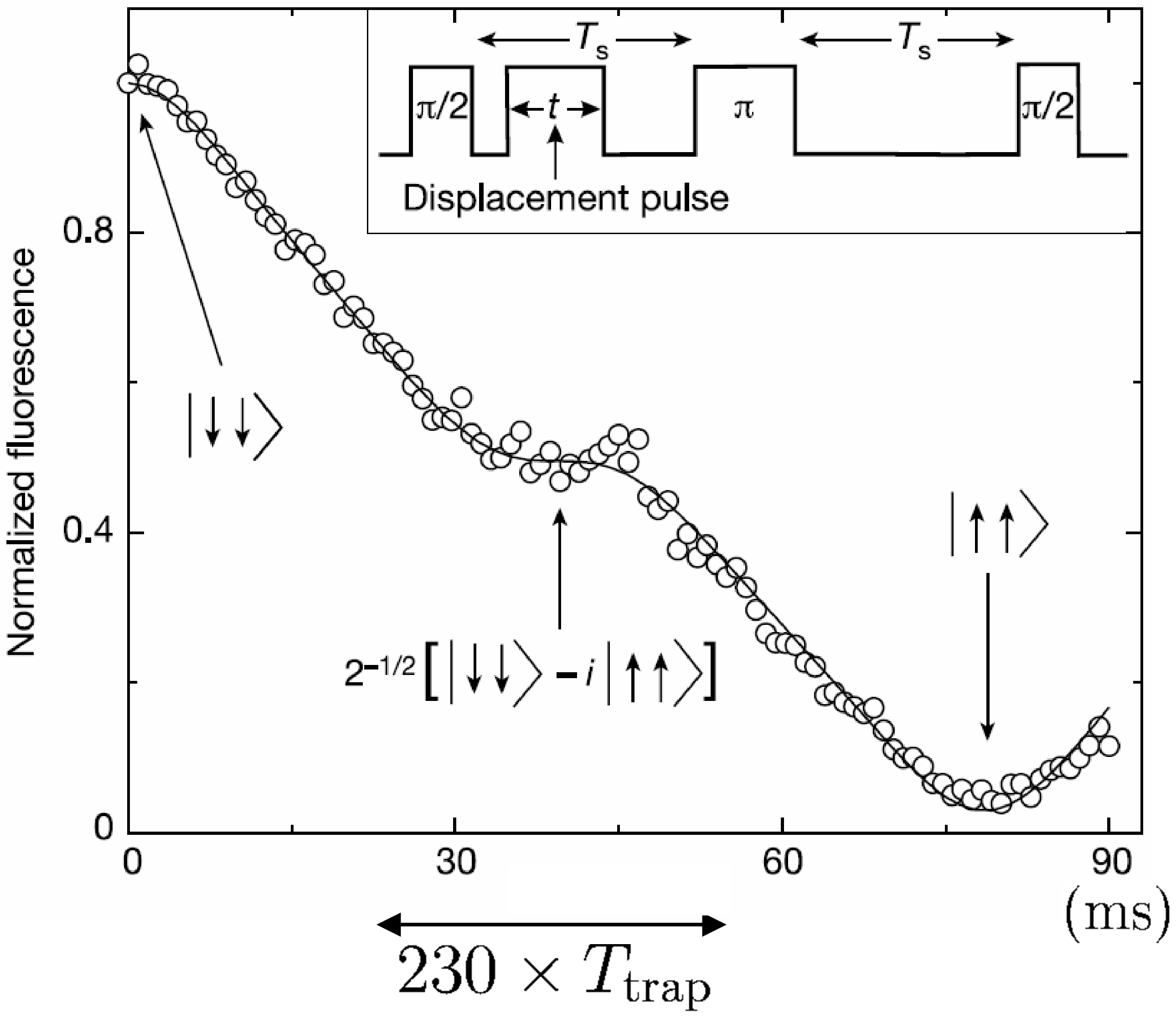
- If F(t) is periodic with a period multiple of ω_r, after some time the motional state is restored, but now the total phase is

$$\phi = A\sigma_z^1\sigma_z^2 \qquad U(T) = \exp(i\phi\sigma_1^z\sigma_2^z)$$

- To address one mode, the gate must be slow ☹

$$T \gg 2\pi/\omega_r$$

NIST Gate: Leibfried *et al.* Nature 2003



Best gate?

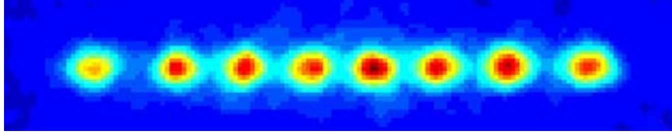
- What is the best possible gate?

  requirements: ...

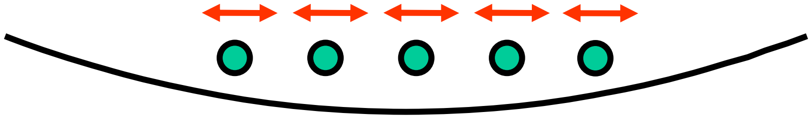
  constraints: …

- … an optimal control problem

N Ions



- We will consider N trapped ions (linear traps, microtraps…), subject to state-dependent forces:



$$H = \sum_{i=1}^{N} \left[\tfrac{1}{2m} p_i^2 + V_{e,i}(x_i) - F_i(t)\sigma_z^i x_i\right] + \sum_{i<j} \frac{e^2}{4\pi\varepsilon_0} \frac{1}{|x_i - x_j|}$$

- normal modes

$$H = \sum_i \left[\tfrac{1}{2m} P_i^2 + \tfrac{1}{2} m\nu_k^2 Q_k^2\right] - \sum_k F_i(t)\sigma_z^i M_{ik} Q_k$$

integrable

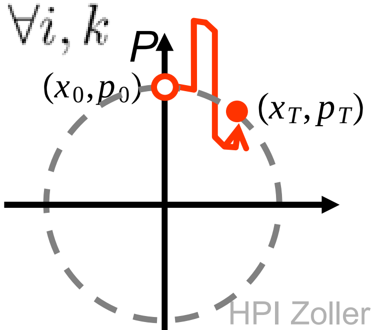
- unitary evolution operator

$$U(T) = \exp\left(i \sum_{ij} J_{ij}\sigma_z^i \sigma_z^j\right)$$

general Ising interaction

- constraints on forces

$$\int_0^T d\tau\, e^{i\omega_k \tau} F_i(\tau) = 0, \quad \forall i, k$$



Quantum Control Problem

- Target: the Ising interaction, is a function of the forces

$$J_{ij} = \frac{1}{2m\hbar}\int_0^T\int_0^T d\tau_1 d\tau_2\, F_i(\tau_1)F_j(\tau_2)\mathcal{G}_{ij}(\tau_1-\tau_2).$$

given

determine

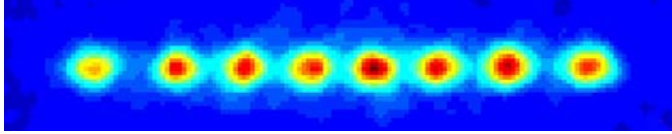
The kernel G depends only on the trapping potential.

- Constraints: displacements, z_k, depend both on the forces and on the internal states. To cancel them, we must impose

$$\int_0^T d\tau\, e^{i\omega_k\tau}F_i(\tau)=0,\quad \forall i,k$$

- Additional constraints: the total time, T; smoothness & intensity of the forces, no local addressing of ions …

fastest gate?



More results

- **Theorem:** For N ions and a given Ising interaction J_{ij} , it is always possible to find a set of forces that realize the gate

simulate spin models

$$\exp\left(-iT\sum_{ij} J_{ij}\sigma_z^i\sigma_z^j\right) ; \qquad \exp\left(-iT(\sum_{ij} J_{ij}\sigma_z^i\sigma_z^j + \sum_i h_i\sigma_z^i)\right) ,$$

although now the solution has to be found numerically.

- **Applications:** Generation of cluster states, of GHZ states, stroboscopic simulation of Hamiltonians, adiabatic quantum computing,…

The time, T, is arbitrary!

---

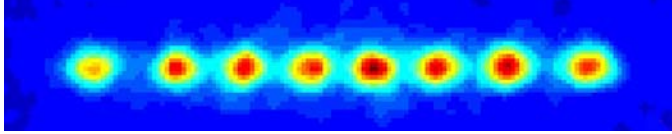
cluster state

$$|+\rangle \;=\; \frac{1}{\sqrt{2}}(|0\rangle_z + |1\rangle_z)$$

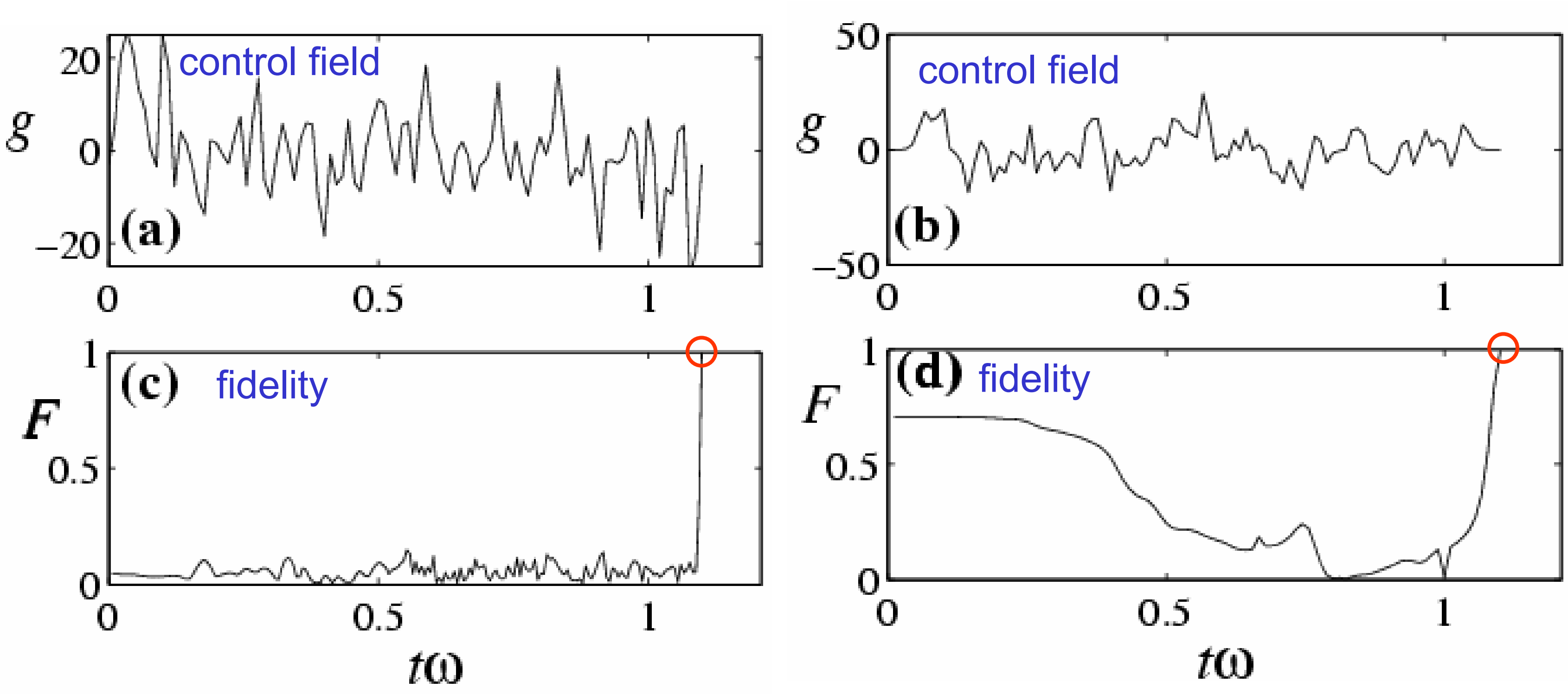
$$|\phi\rangle_c \;=\; \exp(i\int_0^t \tfrac{1}{4}\hbar g(t)dt \sum_{<a,b>} \sigma_z^{(a)} \otimes \sigma_z^{(b)} dt) \;\; (\otimes_{a\in C} |+\rangle_a)$$

GHZ state

$$|\phi\rangle_{\mathrm{GHZ}} \quad \sim \quad e^{-iJ_z^2 t}|+\rangle \equiv e^{-i(\sum_i \frac{1}{2}\sigma_z^i)^2 t} \sim |00\ldots\rangle + |11\ldots\rangle$$



Engineering cluster and GHZ states



Cluster state N=10

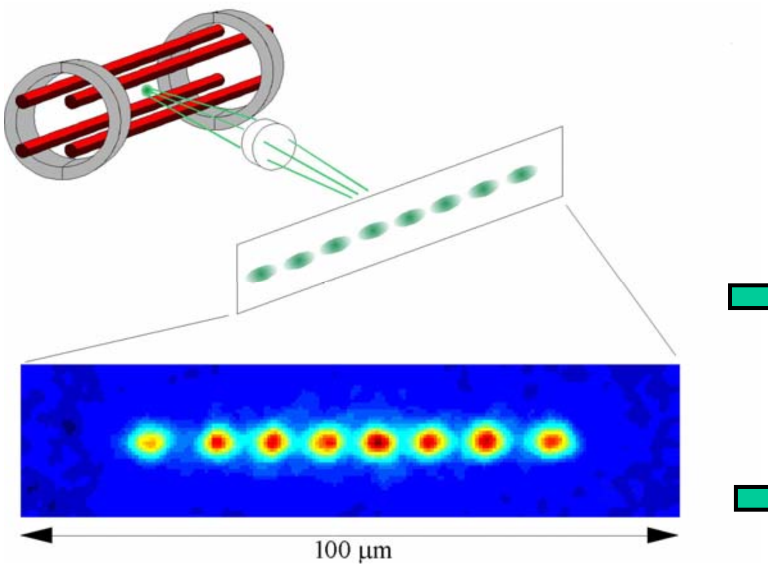
GHZ state N=20

These examples use a common force: $F_i(t) = x_i\, g(t)$

Juanjo Garcia-Ripoll has calculated this up to N=30 ions

So far … Quantum computing with trapped ions

- trapped ions



QC model:

✓ qubits: longlived atomic states

✓ single qubit gates: laser

✓ two qubit gates: via phonon bus

➡ ✓ read out: quantum jumps

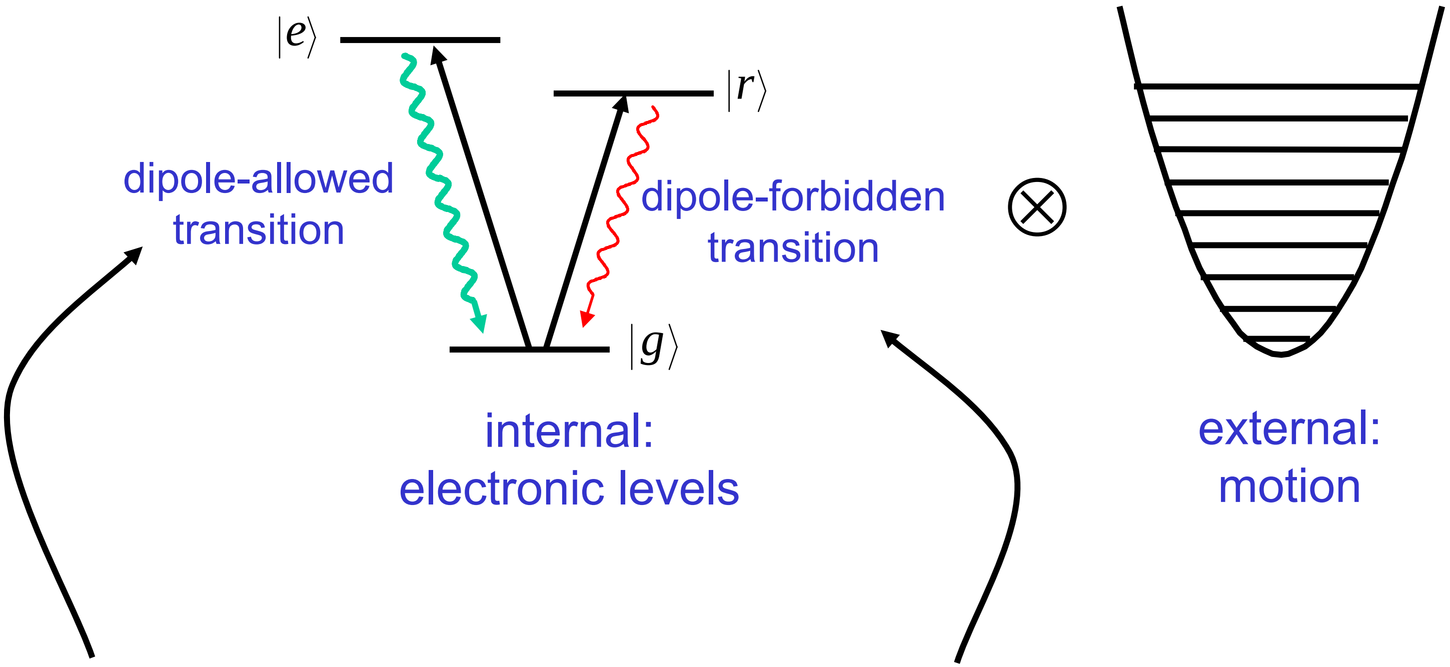
requirements:

➡ ✓ state preparation: phonon cooling

✓ [small decoherence]

Trapped ion: the system

- system = internal + external degrees of freedom



dipole-allowed transition

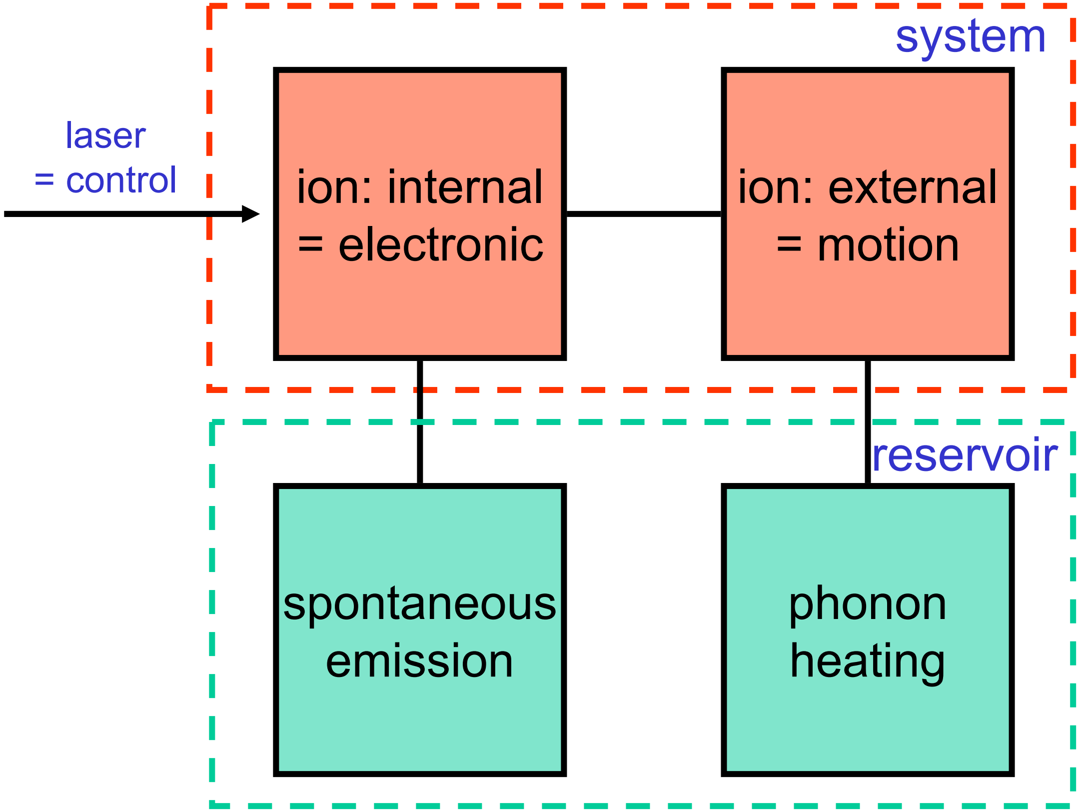
dipole-forbidden transition

internal: electronic levels

external: motion

- strong dissipation
  - ✓ laser cooling / state preparation
  - ✓ qubit / state measurement
- small dissipation
  - ✓ Hamiltonian: quantum state engineering

System + Reservoir



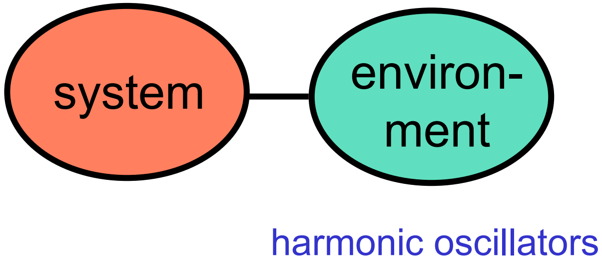
Development of the theory:

- system: Hamiltonian (control)
- reservoir: master equation + continuous measurement theory

Our approach ...

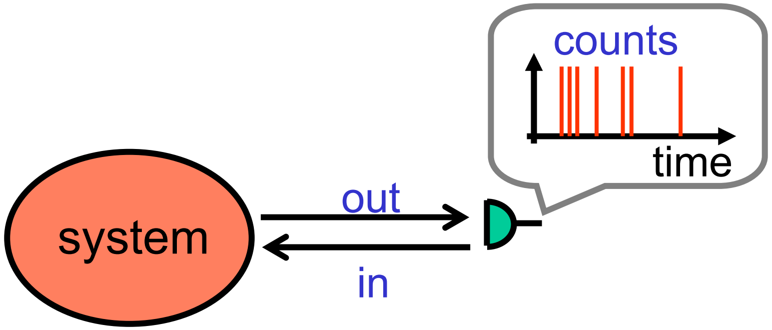
Quantum Optics ⟷ Quantum Information

- Open quantum system



✓ master equation

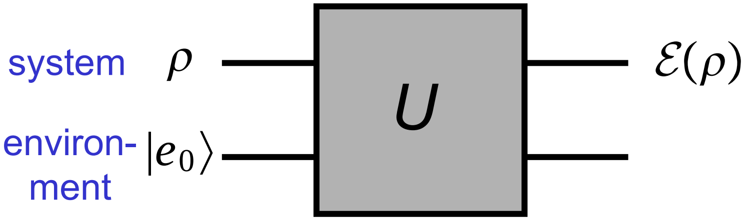
- Continuous observation



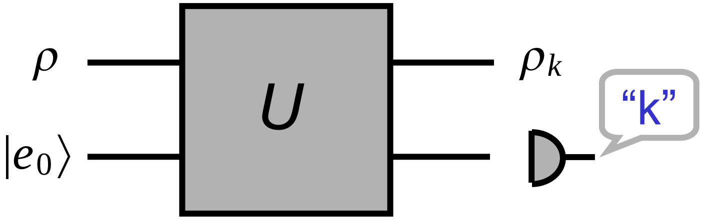
✓ Stochastic Schrödinger Equation

"Quantum Markov processes"

- Quantum operations



$$\rho \rightarrow \mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$$



Outline

- Quantum Operations
  - language of quantum information
- System + environment models in quantum optics
  - Quantum Stochastic Schrödinger Equation (QSSE)
- Solution of the QSSE
  - explicit solution: entangled state representation of system + environment
  - complete photon statistics (continuous measurement)
  - master equation

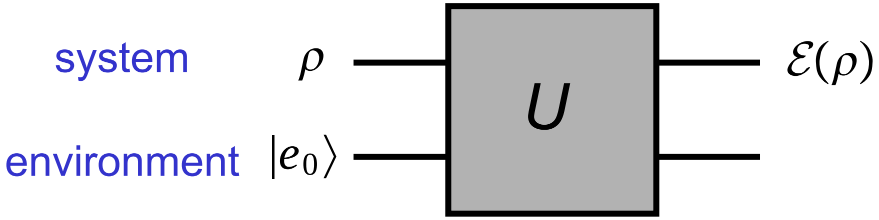
- Examples / application
  - ions: spontaneous emission, laser cooling & quantum reservoir engineering, qubit readout
- Cascaded quantum systems (advanced topic)
  - from QSSE to master equations etc.
  - application: qubit transmission in a quantum network
- [Extra topics]
  - homodyne, quantum feedback

1. Quantum Operations

Ref.: Nielsen & Chuang, Quantum Information and Quantum Computation

Quantum operations

Evolution of a quantum system coupled to an environment:
open quantum system



$$\rho \longrightarrow \mathcal{E}(\rho) = \mathrm{tr}_{\mathrm{env}}[U(\rho \otimes \rho_B)U^\dagger]$$

quantum operation

Operator sum representation:

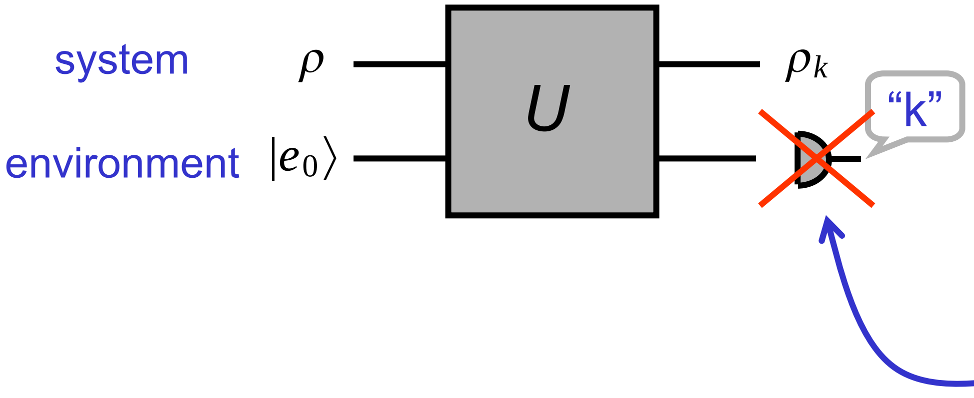
$$\begin{aligned}\rho \longrightarrow \mathcal{E}(\rho) &= \mathrm{tr}_{\mathrm{env}}[U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger] \\ &= \sum_k \langle e_k|U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger|e_k\rangle \\ &= \sum_k E_k \rho E_k^\dagger \quad \text{with } E_k = \langle e_k|U|e_0\rangle\end{aligned}$$

operation elements

Properties: $\sum_k E_k^\dagger E_k = 1$

Quantum operations

Measurement of the environment: $P_k \equiv |e_k\rangle\langle e_k|$



state

$$\rho_k \sim \mathrm{tr_{env}}(|e_k\rangle\langle e_k|U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger)$$
$$= E_k \rho E_k^\dagger$$
$$\rho_k = E_k \rho E_k^\dagger \,/\, \mathrm{tr_{sys}}(E_k \rho E_k^\dagger) \qquad \text{(normalized)}$$

probability

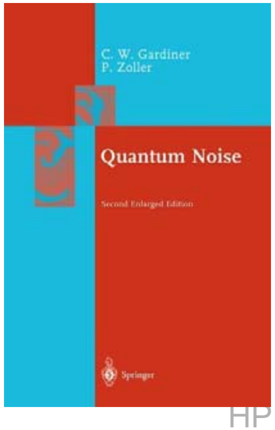
$$p_k = \mathrm{tr_{sys+env}}(|e_k\rangle\langle e_k|U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger)$$
$$= \mathrm{tr_{sys}}(E_k \rho E_k^\dagger)$$

Remark: if we do not read out the measurement

$$\rho \rightarrow \mathcal{E}(\rho) = \sum p_k \rho_k$$
$$= \sum_k E_k \rho E_k^\dagger$$

2. System + environment models in Quantum Optics

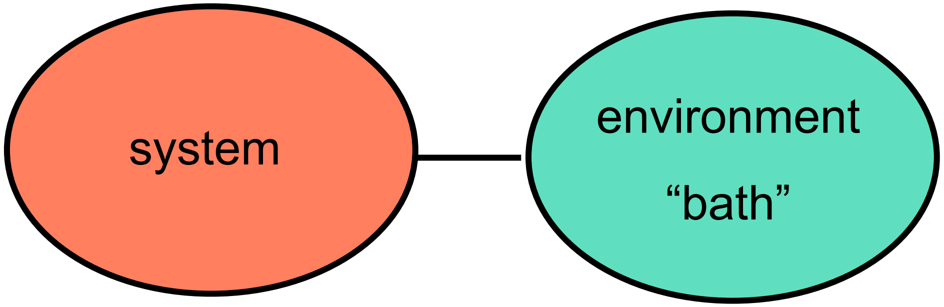
- formulation
  - ➡ – operator / c-number stochastic Schrödinger equation
  - – [(operator) Langevin equation]



System + environment model

Hamiltonian:

$$H_{\text{tot}} = H_{\text{sys}} + H_B + H_{\text{int}}$$



H_{sys} unspecified

$$H_B = \int_{\omega_0-\vartheta}^{\omega_0+\vartheta} d\omega\, \omega\, b^\dagger(\omega) b(\omega) \quad \text{with } [b(\omega), b^\dagger(\omega')] = \delta(\omega-\omega')$$

harmonic oscillators

$$H_{\text{int}} = i\int_{\omega_0-\vartheta}^{\omega_0+\vartheta} d\omega\, \kappa(\omega)[c b^\dagger(\omega) - c^\dagger b(\omega)]$$

system operator

Assumptions:

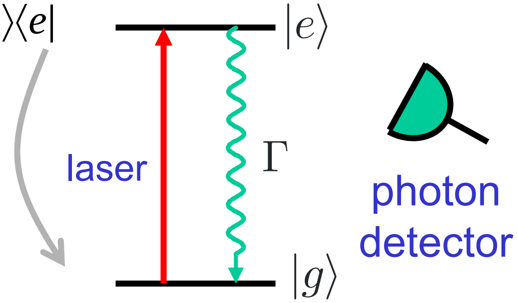
- rotating wave approximation

Simplest possible ...

Example: spontaneous emission

- driven two-level system undergoing spontaneous emission

$c \rightarrow \sigma_- = |g\rangle\langle e|$

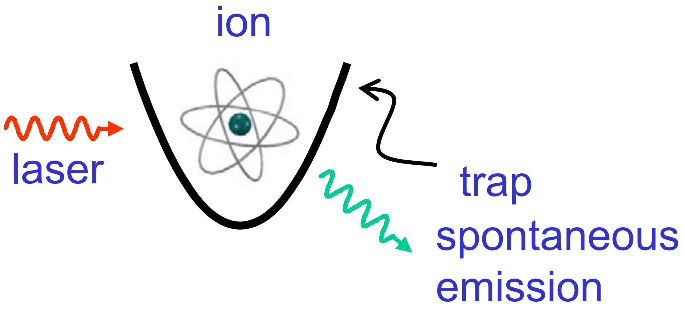


$$H_{\text{sys}} = \omega_{eg}|e\rangle\langle e| - \left(\tfrac{1}{2}\Omega e^{-i\omega_L t}\sigma_+ + \text{h.c.}\right)$$

$$H_{\text{int}} = -\vec{\mu}_{eg} \bullet \vec{E}^{(+)}(0)\sigma_+ + \text{h.c.}$$

$$\rightarrow i\int_{\omega_{eg}-\vartheta}^{\omega_{eg}+\vartheta} d\omega\,\kappa(\omega)b^\dagger(\omega)\sigma_+ + \text{h.c.}$$

- ... including the recoil from spontaneous emission



$$H_{\text{int}} = -\vec{\mu}_{eg} \bullet \vec{E}^{(+)}(\vec{x})\sigma_+ + \text{h.c.}$$

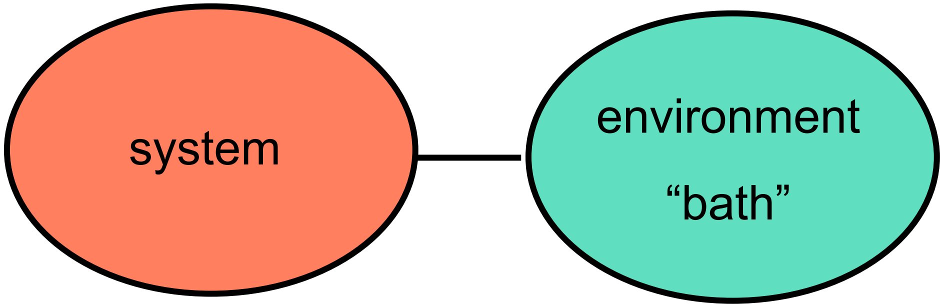
$$\rightarrow \sum_\lambda \int d^3k \ldots b_{\lambda\vec{k}} e^{i\vec{k}\cdot\vec{x}}\sigma_+ + \text{h.c.}$$

recoil

System + environment model

Hamiltonian:

$$H_{\text{tot}} = H_{\text{sys}} + H_B + H_{\text{int}}$$



system

environment

“bath”

H_{sys} unspecified

$$H_B = \int_{\omega_0-\vartheta}^{\omega_0+\vartheta} d\omega\, \omega\, b^\dagger(\omega) b(\omega) \quad \text{with } [b(\omega), b^\dagger(\omega')] = \delta(\omega - \omega')$$

harmonic oscillators

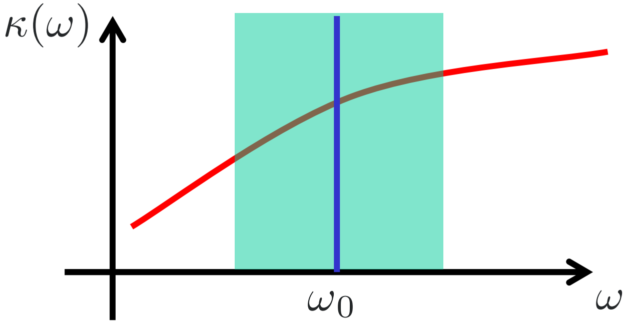
$$H_{\text{int}} = i \int_{\omega_0-\vartheta}^{\omega_0+\vartheta} d\omega\, \kappa(\omega) [c b^\dagger(\omega) - c^\dagger b(\omega)]$$

system operator

Assumptions:

- rotating wave approximation
- flat spectrum: $\kappa(\omega) \to \sqrt{\gamma/2\pi}$

flat over bandwidth

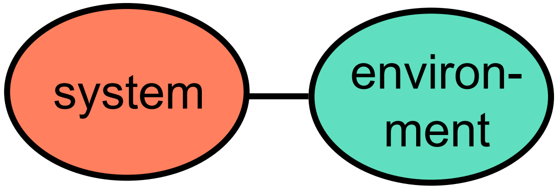


system frequency

$\omega_0 - \vartheta$ $\omega_0 + \vartheta$

reservoir bandwidth B

Schrödinger Equation



- Schrödinger equation

$$\frac{d}{dt}|\Psi_t\rangle = -i\left[H_{\text{sys}} + H_B + H_{\text{int}}\right]|\Psi_t\rangle$$

$|\psi\rangle \otimes |\text{vac}\rangle$

initial condition

- convenient to transform …

$$\Psi_t\rangle \longrightarrow e^{-iH_Bt}|\Psi_t\rangle$$

interaction picture
with respect to bath

$$b(\omega) \longrightarrow b(\omega)e^{-i\omega t}$$

$$H_{\text{sys}} \longrightarrow \tilde{H}_{\text{sys}}$$

"rotating frame“
(transform optical
frequencies away)

$$c \longrightarrow ce^{-i\omega_0 t}$$

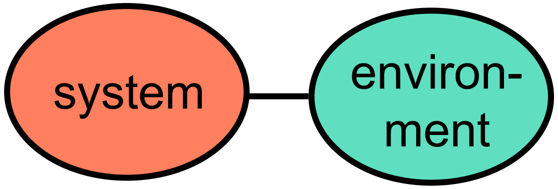
$$\frac{d}{dt}|\tilde\Psi_t\rangle = \left[-i\tilde{H}_{\text{sys}} + \left(\underline{\int_{\omega_0-\vartheta}^{\omega_0+\vartheta} d\omega}\,\boxed{\kappa(\omega)}\,\underline{b(\omega)^{\dagger}e^{\imath(\omega-\omega_0)t}}\right)c - \text{h.c.}\right]|\tilde\Psi_t\rangle$$

$$\kappa(\omega) \rightarrow \sqrt{\gamma/2\pi}$$

flat over bandwidth

$$b(t) := \frac{1}{\sqrt{2\pi}}\int_{\omega_0-\vartheta}^{\omega_0+\vartheta} d\omega\, b(\omega)e^{-\imath(\omega-\omega_0)t}$$

“noise operators”



- **Schrödinger Equation**

$$\frac{d}{dt}|\Psi_t\rangle = \left[-iH_{\text{sys}} + \sqrt{\gamma}\, b(t)^\dagger c - \sqrt{\gamma}\, c^\dagger b(t)\right]|\Psi_t\rangle$$

$$b(t) := \frac{1}{\sqrt{2\pi}} \int_{\omega_0-\vartheta}^{\omega_0+\vartheta} d\omega\, b(\omega) e^{-\imath(\omega-\omega_0)t}$$

"noise operators"

White noise limit $\vartheta \to \infty$

$$[b(t), b^\dagger(s)] = \delta(t-s)$$

$$\langle b(t) b^\dagger(s)\rangle = \delta(t-s)$$

vacuum

$$\tau_{\text{sys}} \gg 1/\vartheta \gg \cancel{\tau_{\text{opt}}}$$

white noise limit $\vartheta \to \infty$

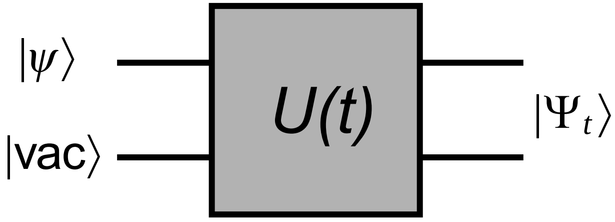
transformed away after RWA

Remarks:

- [We can give precise meaning as a "Quantum Stochastic Schrödinger Equation" within a stochastic Stratonovich calculus]
- We can integrate this equation exactly
  - counting statistics
  - master equation

⟺ quantum operations

3. Integrating the “Quantum Stochastic Schrödinger Equation“

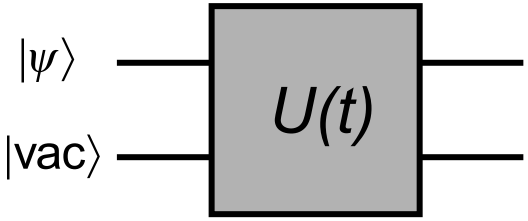


$$|\Psi_0\rangle \rightarrow |\Psi_t\rangle = e^{-iH_{\text{tot}}t}|\Psi_0\rangle$$

Schrödinger equation:
system + environment

What we want to calculate ...

- **We do not observe the environment:** reduced density operator



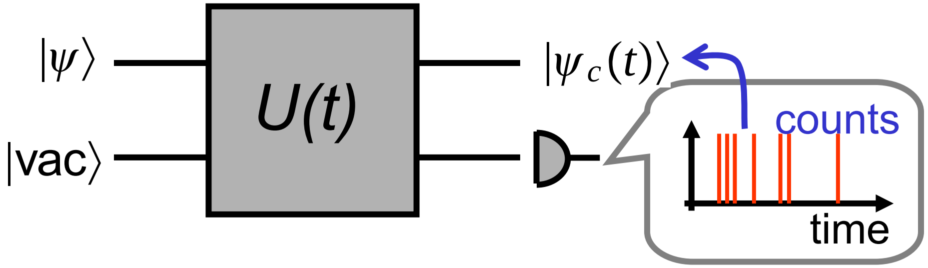
$$\rho_t = \text{tr}_B|\Psi_t\rangle\langle\Psi_t|$$

master equation:

✓ decoherence

✓ preparation of the system (e.g. laser cooling to ground state)

- **We measure the environment:** continuous measurement



conditional wave function:

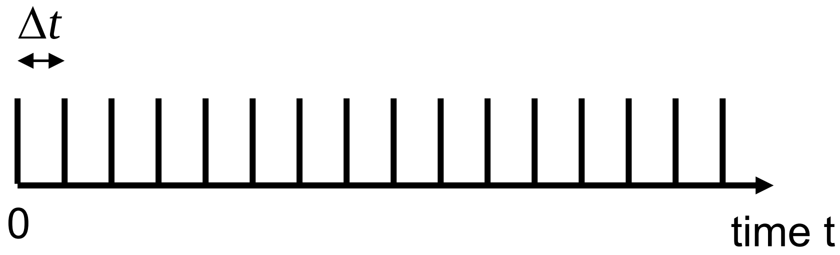
✓ counting statistics

✓ effect of observation on system evolution (e.g. preparation of the (single quantum) system)

Remark: simple man‘s version of conversion from Stratonovich to Ito

Integration in small timesteps

- We integrate the Schrödinger equation in small time steps



$$|\Psi(t=t_f)\rangle = U(\Delta t_f)\ldots U(\Delta t_1)U(\Delta t_0)|\Psi(0)\rangle$$

- Remark: choice of time step

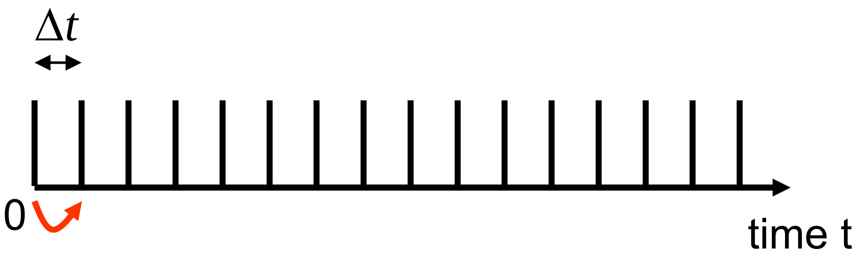
$$\tau_{\rm sys} \gg \Delta t \gtrsim 1/\vartheta \gg \cancel{\tau_{\rm opt}}$$

allows us to use perturbation theory in Δt

white noise limit $\vartheta \to \infty$

transformed away after RWA

1st time step



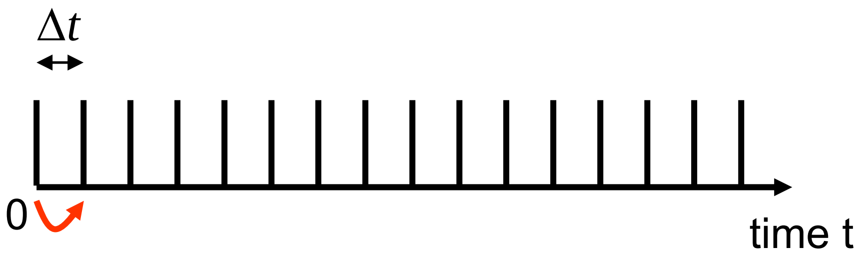
- **First time step:** first order in *Δt*

$$U(\Delta t)|\Psi(0)\rangle = \Big\{\hat{1} - iH_{\mathrm{sys}}\Delta t + \sqrt{\gamma}\, c\int_0^{\Delta t} b^\dagger(t)\,dt - \sqrt{\gamma}\, c^\dagger\int_0^{\Delta t} \cancel{b(t)}\,dt$$

$$\ldots\Big\}|\Psi(0)\rangle$$

$|\psi\rangle \otimes |\mathrm{vac}\rangle$

1st time step



- **First time step:** first order in *Δt*

$$U(\Delta t)|\Psi(0)\rangle = \Big\{\hat{1} - iH_{\mathrm{sys}}\Delta t + \sqrt{\gamma}\, c \int_0^{\Delta t} b^\dagger(t)\, dt - \sqrt{\gamma}\, c^\dagger \int_0^{\Delta t} \cancel{b(t)}\, dt$$

$$+(-i)^2 \gamma c^\dagger c \int_0^{\Delta t} dt \int_0^{t_2} dt'\, b(t) b^\dagger(t') + \ldots \Big\}|\Psi(0)\rangle$$

$|\psi\rangle \otimes |\mathrm{vac}\rangle$

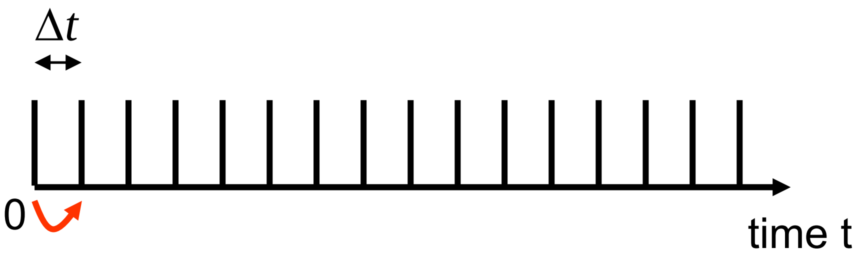
$$\int_0^t dt_2 \int_0^{t_2} dt_1\, b(t_2)\, b^\dagger(t_1)|\mathrm{vac}\rangle = \int_0^t dt_2 \int_0^{t_2} dt_1\, [b(t_2), b^\dagger(t_1)]|\mathrm{vac}\rangle$$

$$= \int_0^t dt_2 \int_0^{t_2} dt_1\, \delta(t_2 - t_1)|\mathrm{vac}\rangle$$

$$= \frac{1}{2}\Delta t|\mathrm{vac}\rangle$$

first order in Δt

1st time step



- **First time step:** to first order in *Δt*

$$|\Psi(\Delta t)\rangle = \hat{U}(\Delta t)|\Psi(0)\rangle$$
$$= \left\{\hat{1} - iH_{\text{eff}}\,\Delta t + \sqrt{\gamma}\; c\,\Delta B(0)^{\dagger}\right\}|\Psi(0)\rangle$$

We define:

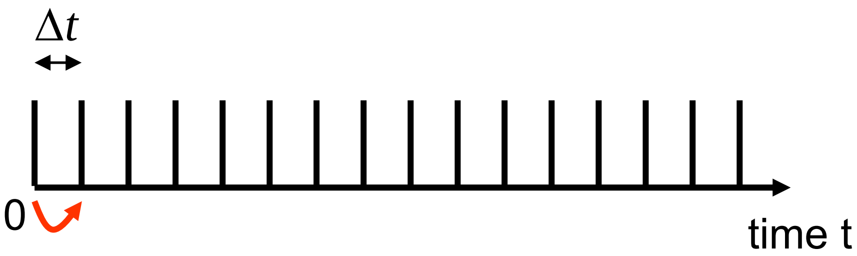
- effective (non-hermitian) system Hamiltonian

$$H_{\text{eff}} := H_{\text{sys}} - \frac{i}{2}\gamma c^{\dagger} c$$

- annihilation / creation operator for a photon in the time slot Δt :

$$\Delta B(t) := \int_{t}^{t+\Delta t} b(s)\, ds$$

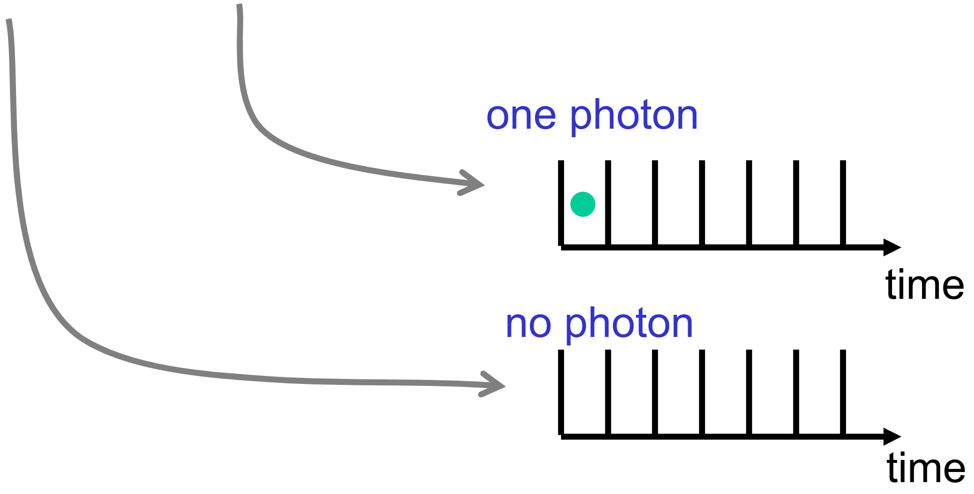
1st time step



- **First time step:** to first order in *Δt*

$$
\begin{aligned}
|\Psi(\Delta t)\rangle &= \hat{U}(\Delta t)|\Psi(0)\rangle \\
&= \left\{\hat{1} - iH_{\text{eff}}\,\Delta t + \sqrt{\gamma}\; c\Delta B(0)^{\dagger}\right\}|\Psi(0)\rangle
\end{aligned}
$$

interpretation: superposition of vacuum and one-photon state

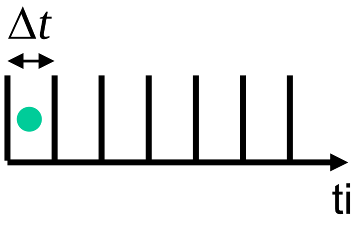


1st time step

Discussion:

annihilation / creation operator for a photon in the time slot Δt :

$$\Delta B(t) := \int_t^{t+\Delta t} b(s)\, ds$$



Remarks and properties:

- commutation relations:

$$[\Delta B(t), \Delta B^\dagger(t')] = \begin{cases} \Delta t & t = t' \quad \text{overlapping intervals} \\ 0 & t \neq t' \quad \text{nonoverlapping intervals} \end{cases}$$

- one-photon wave packet in time slot Δt

$$\frac{\Delta B^\dagger(t)}{\sqrt{\Delta t}} |\text{vac}\rangle \equiv |1\rangle_t \quad \text{(normalized)}$$

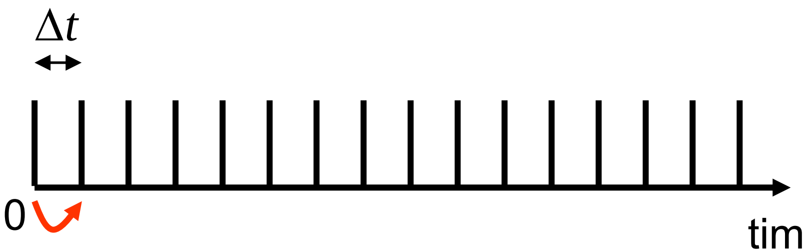
- number operator of photon in time slot t:

$$N(t) = \frac{\Delta B^\dagger(t)}{\sqrt{\Delta t}} \frac{\Delta B(t)}{\sqrt{\Delta t}}$$

- $N(t)$ as set up commuting operators, $[N(t), N(t')] = 0$,
which can be measured "simultaneously"

1st time step:

quantum operations



- **Summary of first time step:** to first order in *Δt*

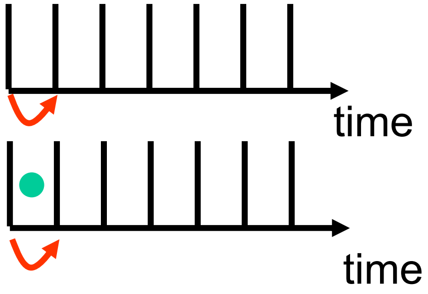
$$
\begin{aligned}
|\Psi(\Delta t)\rangle &= \left[1 - iH_{\text{eff}}\Delta t + \sqrt{\gamma}\, c\, \Delta B^{\dagger}(0)\right]|\Psi(0)\rangle \\
&= |\text{vac}\rangle \otimes \left(1 - iH_{\text{eff}}\Delta t\right)|\psi(0)\rangle + |1\rangle_t \otimes \left(\sqrt{\gamma\Delta t}\, c|\psi(0)\rangle\right) \\
&\equiv |\text{vac}\rangle \otimes E_0|\psi(0)\rangle + |1\rangle_t \otimes E_1|\psi(0)\rangle
\end{aligned}
$$

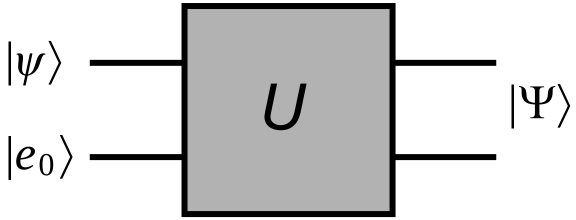
operation elements

where we read off the operation elements

$$E_0 = 1 - iH_{\text{eff}}\Delta t \quad \text{(no photon)}$$

$$E_1 = \sqrt{\gamma\Delta t}\, c \quad \text{(1 photon)}$$



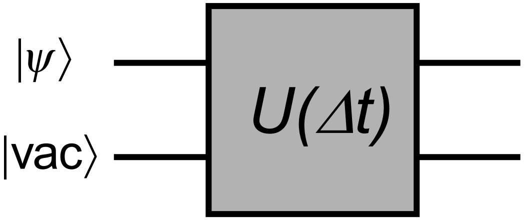


$$
\begin{aligned}
|\psi\rangle|e_0\rangle \rightarrow |\Psi\rangle &= U|\psi\rangle|e_0\rangle \\
&= \sum_k |e_k\rangle\langle e_k|U|e_0\rangle|\psi\rangle \equiv \boxed{\sum |e_k\rangle E_k|\psi\rangle}
\end{aligned}
$$

1st time step

Discussion 1:

- **We do not read the detector:** reduced density operator

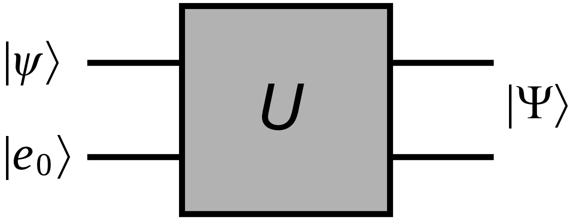


$$
\begin{aligned}
\rho(\Delta t) &= \mathrm{tr}_B |\Psi(\Delta t)\rangle\langle\Psi(\Delta t)| \\
&= E_0 \rho(0) E_0^\dagger + E_1 \rho(0) E_1^\dagger \\
&= \left(1 - iH_{\text{eff}}\Delta t\right)\rho(0)\left(1 - iH_{\text{eff}}\Delta t\right)^\dagger + \gamma c \rho(0) c^\dagger \Delta t
\end{aligned}
$$

no photon one photon

master equation:

$$
\begin{aligned}
\rho(\Delta t) - \rho(0) &= -i\left(H_{\text{eff}}\rho(0) - \rho(0)H_{\text{eff}}^\dagger\right)\Delta t + \gamma c \rho(0) c^\dagger \Delta t \\
&\equiv -i\left[H_{\text{sys}}, \rho(0)\right]\Delta t + \frac{1}{2}\gamma(2c\rho(0)c^\dagger - c^\dagger c \rho(0) - \rho(0)c^\dagger c)\Delta t
\end{aligned}
$$

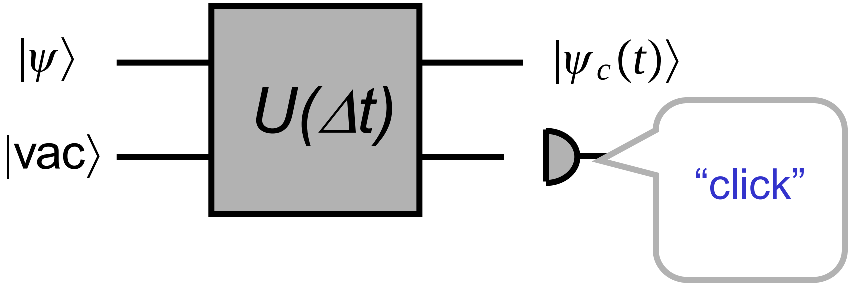


$$
\begin{aligned}
\rho \longrightarrow \mathcal{E}(\rho) &= \mathrm{tr}_{\text{env}}[U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger] \\
&= \textstyle\sum_k E_k \rho E_k^\dagger
\end{aligned}
$$

1st time step

Discussion 2:

- **We read the detector:**



- **Click:** resulting state

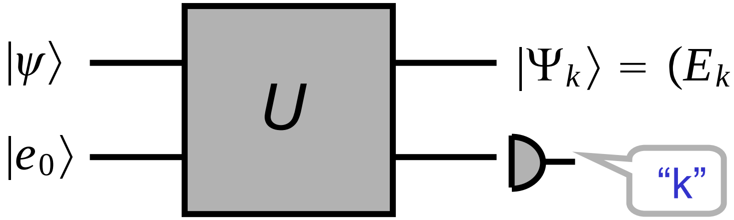
quantum jump operator

$$E_1|\psi(0)\rangle \equiv |\psi^{\text{click}}(\Delta t)\rangle = \sqrt{\gamma\Delta t}\, c|\psi(0)\rangle \quad \text{(quantum jump)}$$

with probability

$$p^{\text{click}} = \text{tr}_{\text{sys}}(E_1\rho(0)E_1) = \gamma\Delta t\, \|c\psi(0)\|^2$$

Rem.: density matrix $\rho_1(0) = E_1\rho(0)E_1/\text{tr}(\dots)$



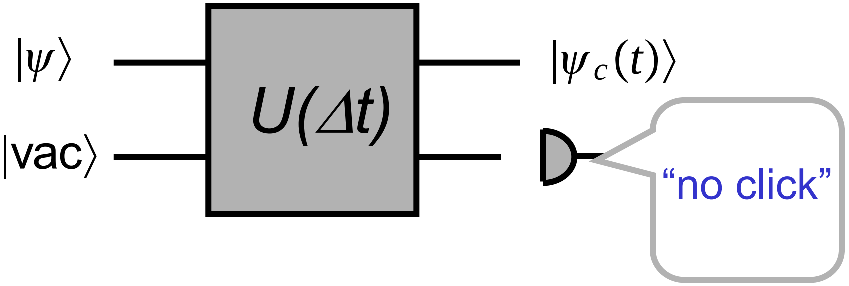
$$|\Psi_k\rangle = (E_k|\psi\rangle)|e_k\rangle/\|\dots\|$$

$$p_k = \|E_k\psi\|^2$$

1st time step

Discussion 2:

- **We read the detector:**



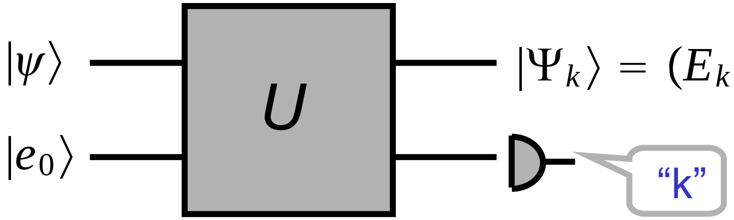
- **No click:** resulting state

decaying norm

$$E_0|\psi(0)\rangle \equiv |\psi^{\text{no click}}(\Delta t)\rangle = \left(1 - iH_{\text{eff}}\Delta t\right)|\psi(0)\rangle \approx e^{-iH_{\text{eff}}\Delta t}|\psi(0)\rangle$$

with probability

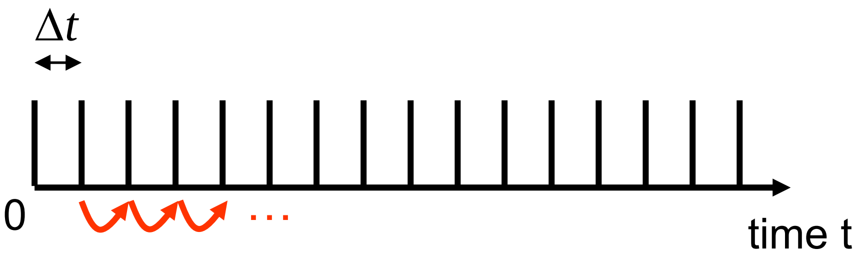
$$p^{\text{no click}} = \text{tr}_{\text{sys}}(E_0\rho(0)E_0) = \left\| e^{-iH_{\text{eff}}\Delta t}\psi(0) \right\|^2$$



$$|\Psi_k\rangle = (E_k|\psi\rangle)|e_k\rangle/\|\ldots\|$$

$$p_k = \|E_k\psi\|^2$$

2nd time step etc.



- **Second and more time steps:**

$$|\Psi(n\Delta t)\rangle = \left[1 - iH_{\text{eff}}\Delta t + \sqrt{\gamma}\, c\, \Delta B^\dagger((n-1)\Delta t)\right]|\Psi((n-1)\Delta t\rangle$$

stroboscopic integration

$$\equiv \left[1 - iH_{\text{eff}}\Delta t + \sqrt{\gamma}\, c\, \Delta B^\dagger((n-1)\Delta t)\right] \times$$
$$\ldots \times \left[1 - iH_{\text{eff}}\Delta t + \sqrt{\gamma}\, c\, \Delta B^\dagger(0)\right]|\Psi(0)\rangle$$

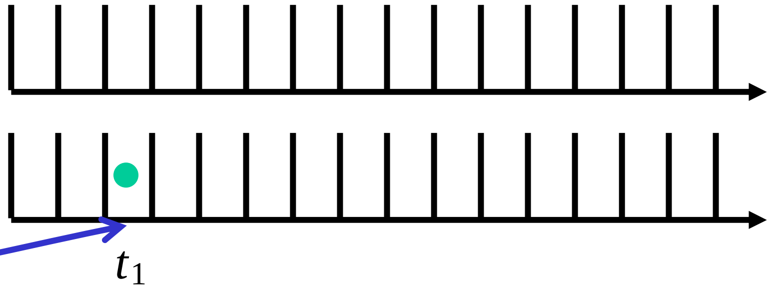
✓ Note: remember … commute in different time slots

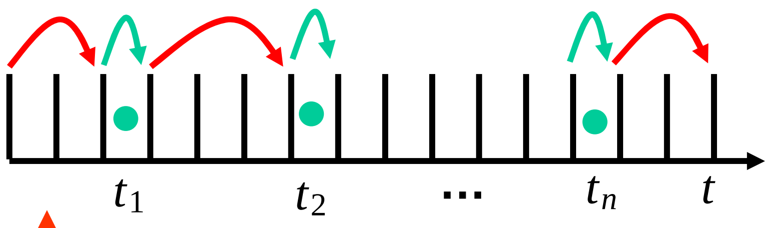
$$[\Delta B(t), \Delta B^\dagger(t')] = \begin{cases} \Delta t & t = t' \quad \text{overlapping intervals} \\ 0 & t \neq t' \quad \text{nonoverlapping intervals} \end{cases}$$

Final result for solution of SSE

- **Wave function of the system + environment: entangled state**

$$
\begin{aligned}
|\Psi(t)\rangle \;=\; & |\mathrm{vac}\rangle \otimes e^{-iH_{\mathrm{eff}}t}|\psi(0)\rangle \\
& + (\gamma\Delta t)^{1/2} \sum_{t_1} |1_{t_1}\rangle \otimes e^{-iH_{\mathrm{eff}}(t-t_1)} c\, e^{-iH_{\mathrm{eff}}t_1}|\psi(0)\rangle \\
& + \ldots \\
& + (\gamma\Delta t)^{n/2} \sum_{t_n>\ldots>t_1} |1_{t_1}1_{t_2}\ldots 1_{t_n}\rangle \otimes \underbrace{e^{-iH_{\mathrm{eff}}(t-t_n)} c \ldots c\, e^{-iH_{\mathrm{eff}}t_1}|\psi(0)\rangle} \\
& \ldots
\end{aligned}
$$





1. system time evolution $|\psi(t|t_1 t_2 \ldots t_n)\rangle$ for a specfic count sequence

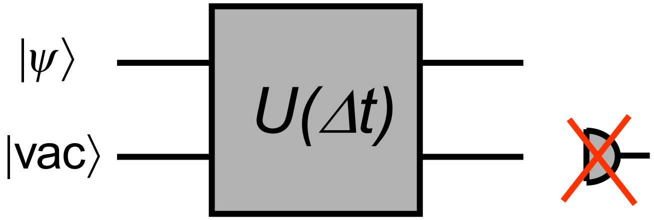
2. photon count statistics: probability densities

$$p_{(0,t]}(t_1, t_2, \ldots, t_n) = \| \psi(t|t_1 t_2 \ldots t_n) \|^2$$

click: $|\psi_t\rangle \to \sqrt{\gamma}\, c|\psi_{t+\Delta t}\rangle$

no click: $|\psi_0\rangle \to |\psi_t\rangle = e^{-iH_{\mathrm{eff}}t}|\psi_0\rangle$

- **Tracing over the environment we obtain the master equation**



$$\frac{d}{dt}\rho(t) = -i\left[H_{\text{sys}}, \rho(t)\right] + \tfrac{1}{2}\gamma\left(2c\rho(t)c^\dagger - c^\dagger c\rho(t) - \rho(t)c^\dagger c\right)$$

master equation

✓ Lindblad form

✓ coarse grained time derivative

For theorists …

Ito-Quantum Stochastic Schrödinger Equation

- taking the limit …

$$\Delta t \longrightarrow dt$$

$$\Delta B(t) \longrightarrow dB(t)$$

$$\Delta B^\dagger(t) \longrightarrow dB(t)^\dagger$$

Ito operator noise increments

- Quantum Stochastic Schrödinger Equation

$$d|\Psi(t)\rangle = \left[-\frac{i}{\hbar}H_{\text{sys}}\,dt + \sqrt{\gamma}\,c\,dB^\dagger(t) - \sqrt{\gamma}\,c^\dagger dB(t)\right]|\Psi(t)\rangle$$

- Properties of Ito increments:
  - point to the future:

$$dB(t)|\Psi(t)\rangle = 0$$

  - Ito rules:

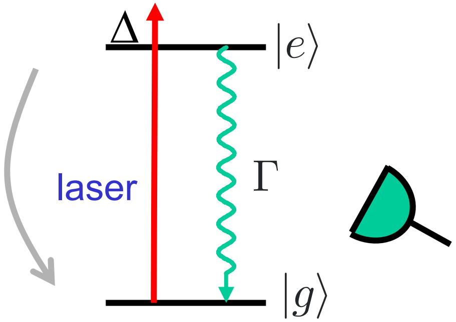
$$[dB(t)]^2 = [dB^\dagger(t)]^2 = 0,$$

$$dB(t)\,dB^\dagger(t) = dt,$$

$$dB^\dagger(t)\,dB(t) = 0.$$

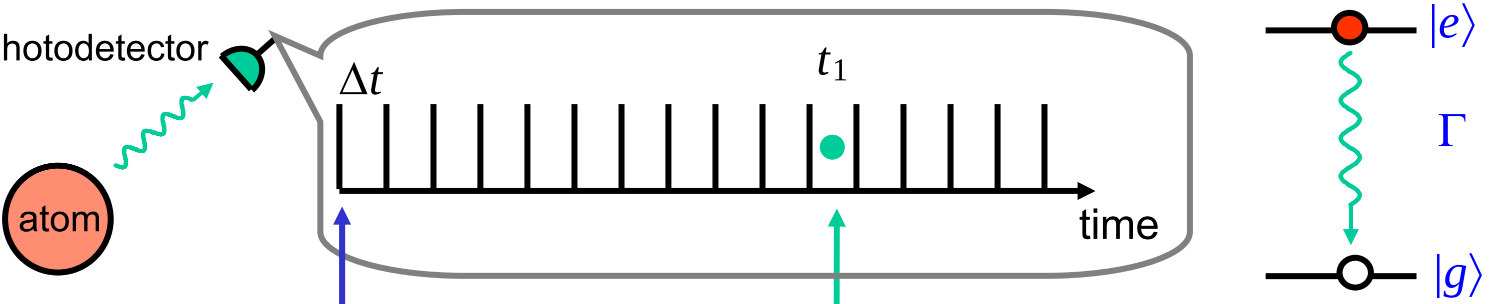
4. Examples:

- Two-level atom undergoing spontaneous emission
- Driven two-level atom: Optical Bloch Equations



- laser cooling and reservoir engineering of single trapped ion
  - ground state cooling
  - squeezed state generation by reservoir engineering

Example 1: two-level atom undergoing spontaneous decay



initial state $|\psi_c(0)\rangle = c_g|g\rangle + c_e|e\rangle$

while no photon is detected

$$|\psi_c(t)\rangle = \frac{e^{-iH_{\mathrm{eff}}t/\hbar}|\psi_c(0)\rangle}{\|\ldots\|} = \frac{c_g|g\rangle + c_e e^{-\Gamma t/2}|e\rangle}{\|\ldots\|}$$

our knowledge increases that the atom is not in the excited state

a photon is detected $|\psi_c(t+\Delta t)\rangle = \frac{\sqrt{\Gamma}\sigma_-|\tilde{\psi}_c(t)\rangle}{\|\ldots\|} = |g\rangle$

probability that a photon is detected in (t,t+Δt] $\mathcal{P}_1^{(t,t+\Delta t]} = \Gamma|c_e|^2 e^{-\Gamma t}\Delta t$

Example 2: driven two-level atom + spontaneous emission

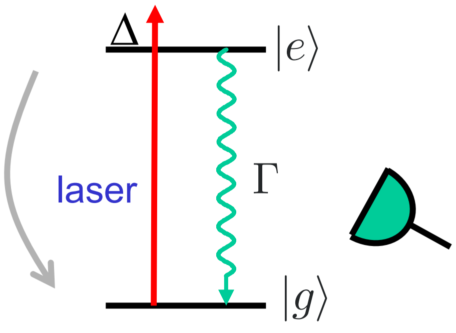
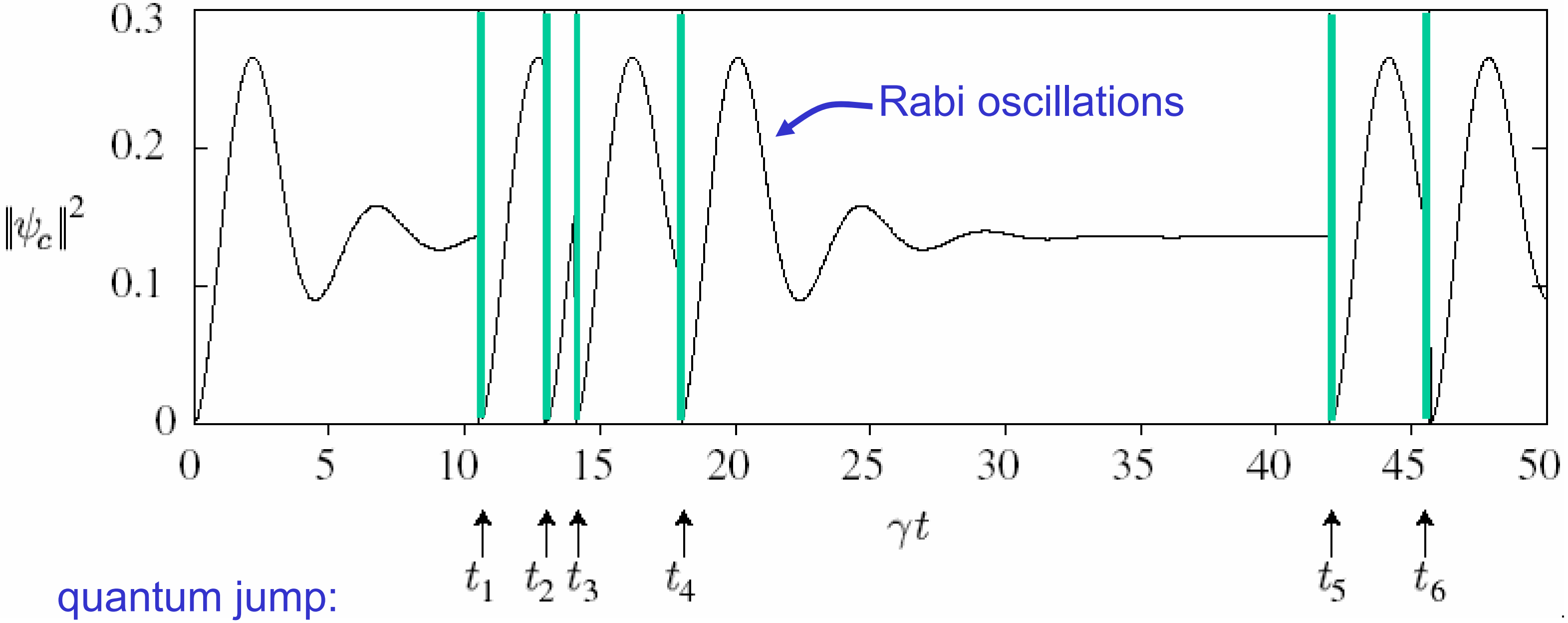


Fig.: typical quantum trajectory: upper state population

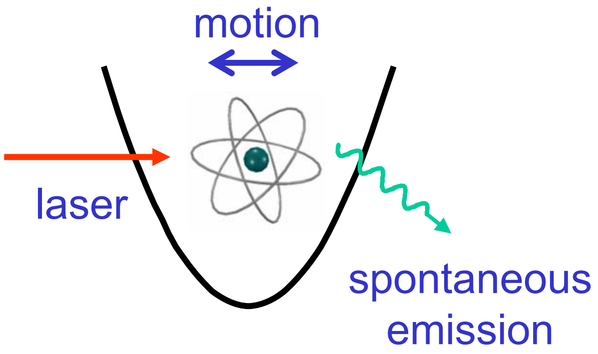


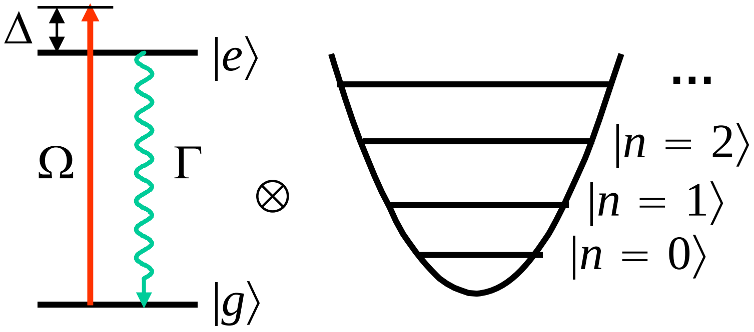
Rabi oscillations

quantum jump:
electron returns to the
ground state

(prepares the system)

Example 3: laser cooling of a trapped ion



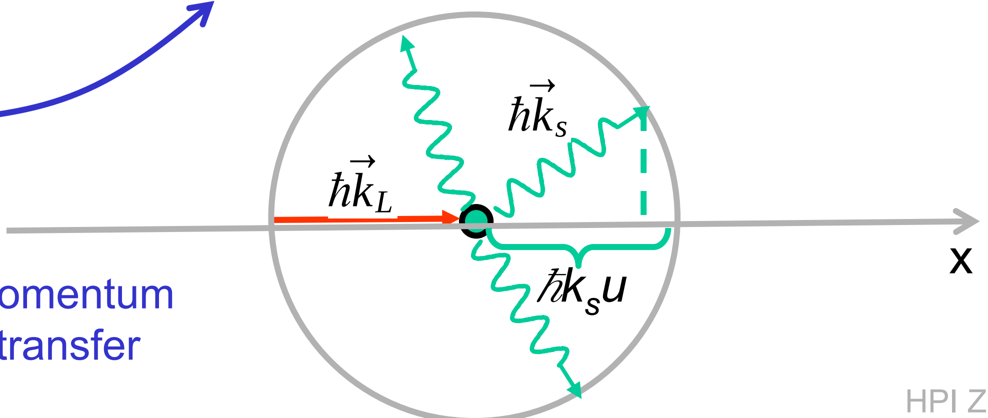


$$H_{\text{sys}} = \left(\frac{\hat{P}^2}{2m} + \tfrac{1}{2} m \nu^2 \hat{X}^2\right) - \Delta |e\rangle\langle e| - \left(\tfrac{1}{2}\Omega e^{ik\hat{X}}\sigma_- + \text{h.c.}\right)$$

- Master equation (1D):

$$\frac{d}{dt}\rho = -i\left[H_{\text{sys}}, \rho\right] + \frac{1}{2}\Gamma\left(2\int_{-1}^{+1} du\, N(u)\left(e^{ik\hat{X}u}\sigma_-\right)\rho\left(\sigma_+ e^{ik\hat{X}u}\right) - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-\right)$$

quantum jump operator:
recoil from spontaneous emission



momentum
transfer

- Lamb-Dicke limit: adiabatic elimination of internal dynamics

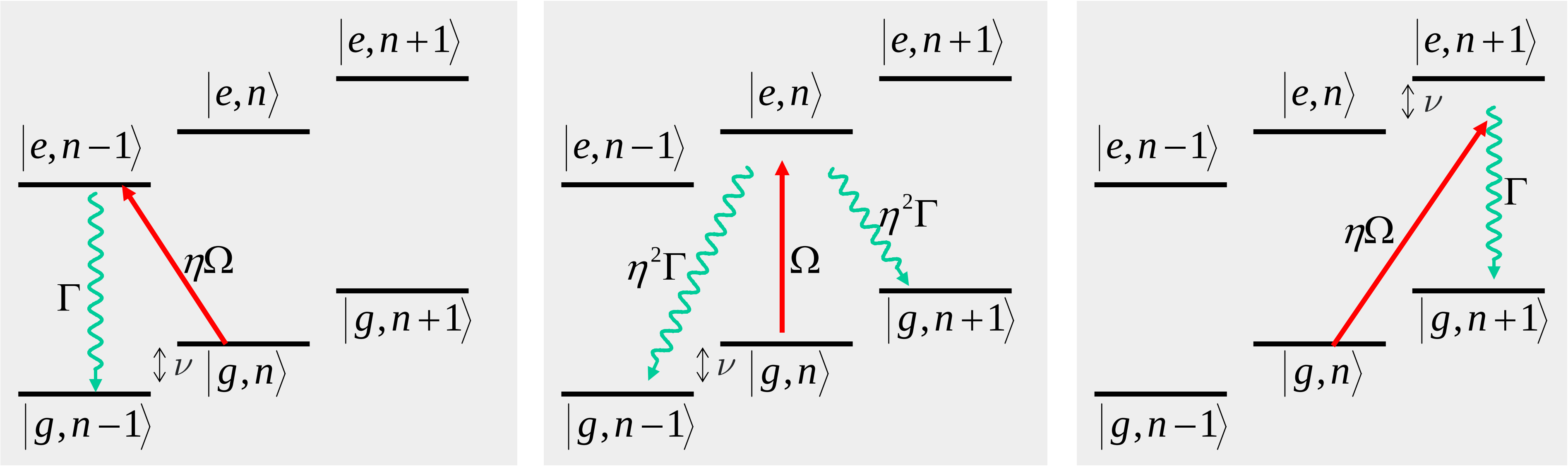
$$\dot{\rho} = A_+\left(a\rho a^\dagger - \frac{1}{2}a^\dagger a\rho - \rho\frac{1}{2}a^\dagger a\right)$$

cooling term

$$+ A_-\left(a^\dagger\rho a - \frac{1}{2}aa^\dagger\rho - \rho\frac{1}{2}aa^\dagger\right)$$

heating term

- processes contributing at low intensity

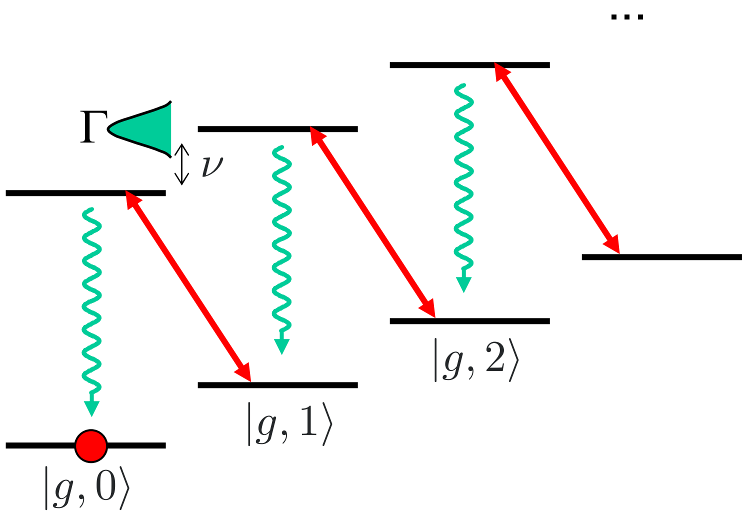


cooling $2\,\mathrm{Re}\,S(-\nu)$ diffusion D heating $2\,\mathrm{Re}\,S(+\nu)$

$$A_\pm = 2\mathrm{Re}[S(\mp\nu) + D]$$

sideband cooling

- ... as optical pumping to the ground state



- master equation

$$\dot{\rho} = A_+\left(a\rho a^\dagger - \tfrac{1}{2}a^\dagger a\rho - \rho\tfrac{1}{2}a^\dagger a\right) \quad (A_+ \gg A_-)$$

- final state

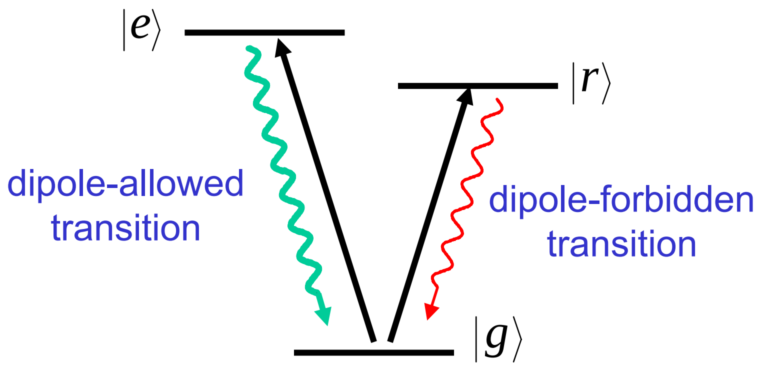
$$\rho_{\text{osc}} \to |0\rangle\langle 0| \qquad (\Gamma \ll \nu, \text{ sideband cooling})$$

"dark state" of the jump operator *a*:

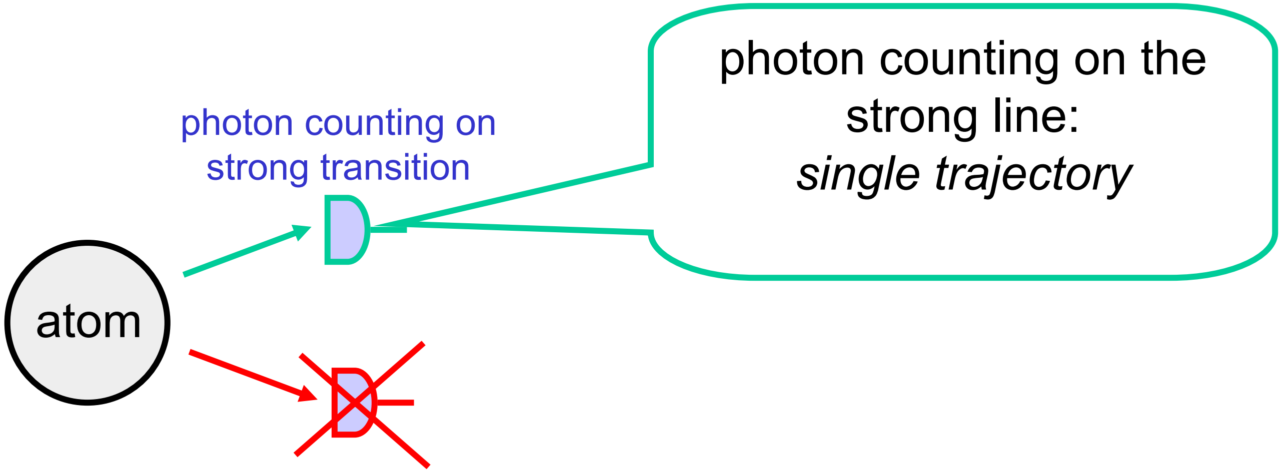
$$a|0\rangle = 0$$

Example 4: State measurement & quantum jumps in 3-level systems

- three level atom

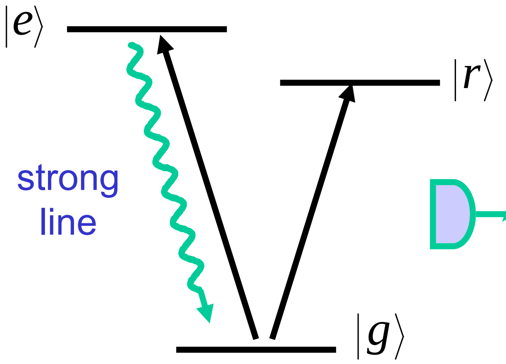


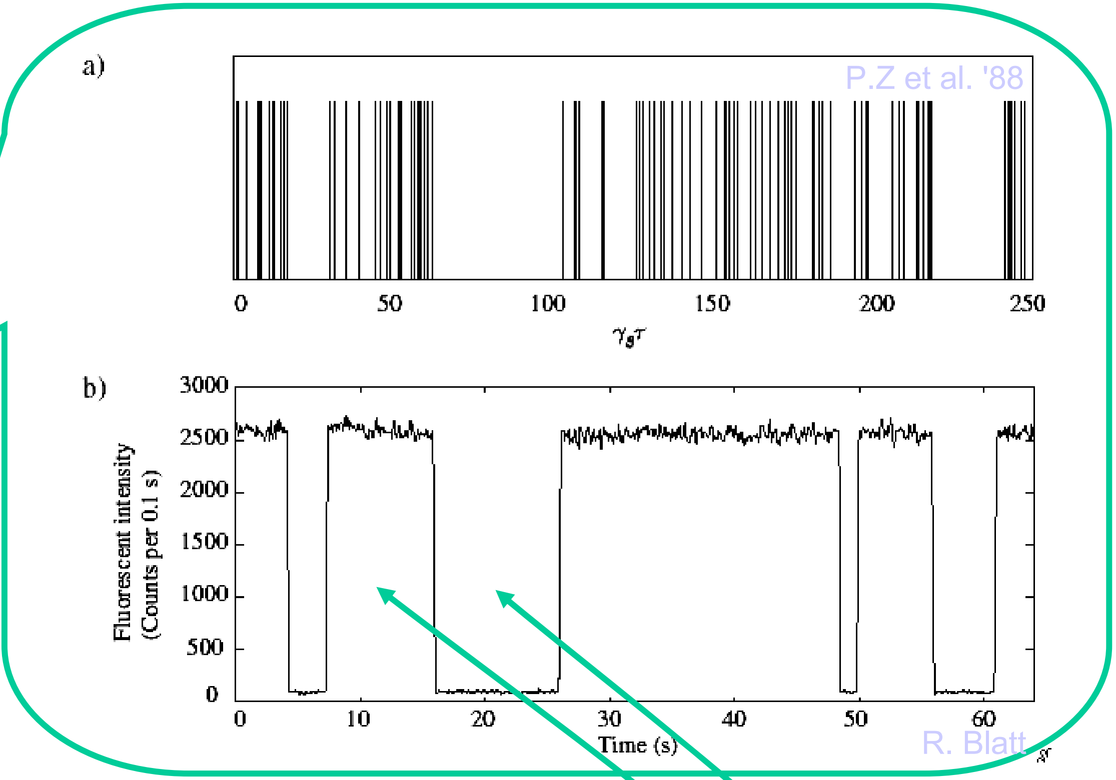
- single atom photon counting



photon counting on the strong line:
single trajectory

photon counting on strong transition





here: with a weak driving field g - r

- ✓ atomic density matrix conditional to observing an emission window

$$\rho_c(t) \longrightarrow |r\rangle\langle r|$$

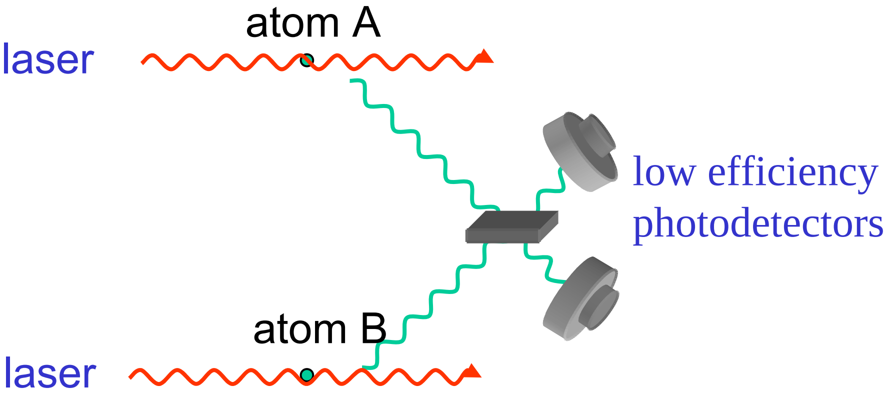
preparation in metastable state

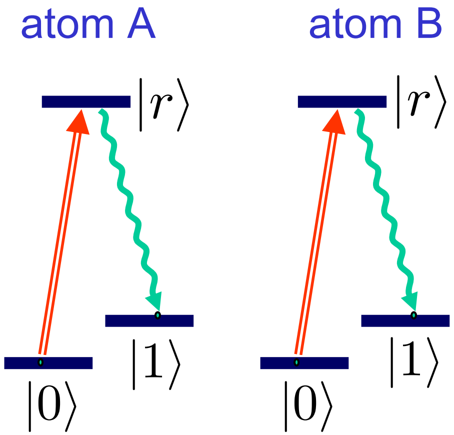
- ✓ state measurement with 100% efficiency

$$\psi = \alpha|g\rangle + \beta|r\rangle \qquad \begin{array}{ll} |\alpha|^2 & \ldots \text{ probability } \textbf{NO} \text{ window} \\ |\beta|^2 & \ldots \text{ probability window} \end{array}$$

Example 5: Preparation of 2 atoms in a Bell state via measurement

- **System:** two atoms with ground states $|0\rangle$, $|1\rangle$ and excited state $|r\rangle$





- Weak (short) laser pulse, so that the excitation probability is small.

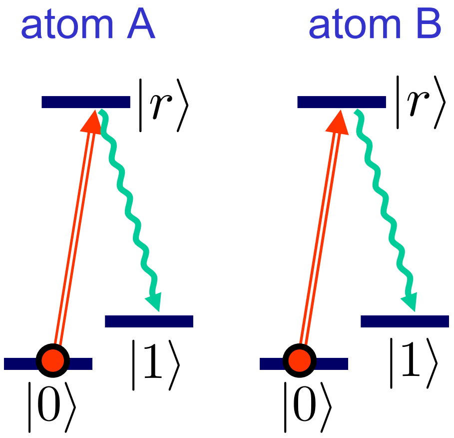
- If no detection, pump back and start again.

- If detection, an entangled state is created.

$$\sim |0,1\rangle + |1,0\rangle$$

for a first exp step:
Monroe et al, Nature 2004

Process:



- preparation (by optical pumping)

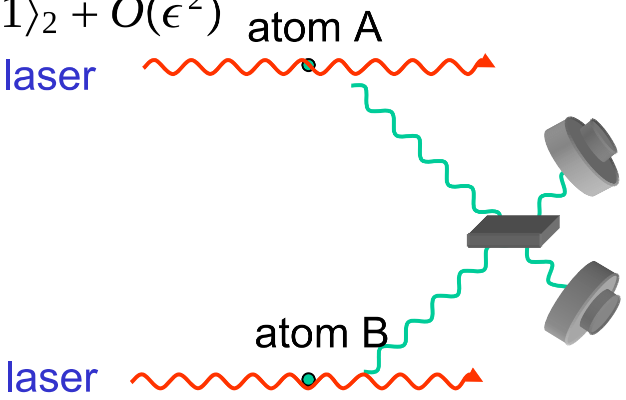
$$|\Psi(t=0)\rangle = |\mathrm{vac}\rangle|0\rangle_1|0\rangle_2$$

- excitation by a weak short laser pulse

$$\begin{aligned}|\Psi(t=0^+)\rangle &= |\mathrm{vac}\rangle\,(|0\rangle_2+\epsilon|r\rangle_2)(|0\rangle_2+\epsilon|r\rangle_2)\\ &= |\mathrm{vac}\rangle\,[|0\rangle_1|0\rangle_2+\epsilon(|r\rangle_1|0\rangle_2+|0\rangle_1|r\rangle_2)+O(\epsilon^2)]\end{aligned}$$

- spontaneous emission

$$\begin{aligned}|\Psi(t>0^+)\rangle = {} & [|0\rangle_1|0\rangle_2+\epsilon e^{-\gamma t/2}(|r\rangle_1|0\rangle_2+|0\rangle_1|r\rangle_2)]\otimes|\mathrm{vac}\rangle\\ & +\sum_{t_1}\Delta B_1^\dagger(t_1)|\mathrm{vac}\rangle\otimes\epsilon\sqrt{\gamma}\,e^{-\gamma t_1/2}|1\rangle_1|0\rangle_2\\ & +\Delta B_2^\dagger(t_1)|\mathrm{vac}\rangle\otimes\epsilon\sqrt{\gamma}\,e^{-\gamma t_1/2}|0\rangle_1|1\rangle_2+O(\epsilon^2)\end{aligned}$$



- We observe the fluorescence through a beam splitter

$$\Delta B_{1,2}^\dagger \rightarrow \frac{1}{\sqrt{2}}(\Delta B_1^\dagger \pm \Delta B_2^\dagger)$$

- Observation of a click prepares Bell state

$$|1\rangle_1|0\rangle_2+|0\rangle_1|1\rangle_2$$

5. Cascaded Quantum Systems

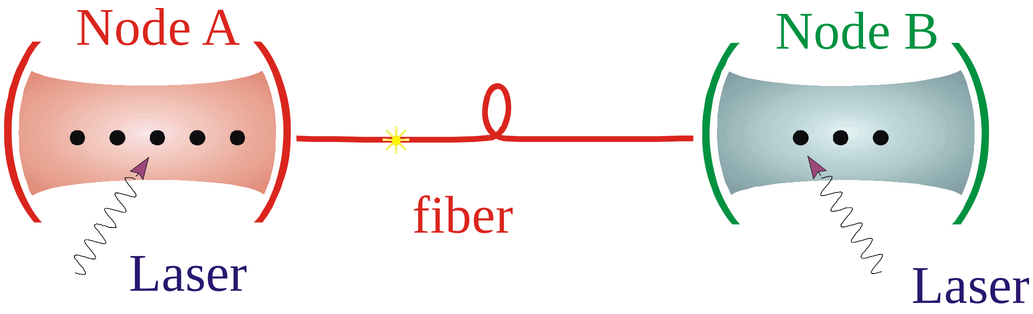
- formal theory
- example
  - optical interconnects

Motivation: Theory of Optical Interconnects

J.I. Cirac, P.Z. H.J. Kimble and H. Mabuchi PRL '97

- A cavity QED implementation

Optical cavities connected by a quantum channel

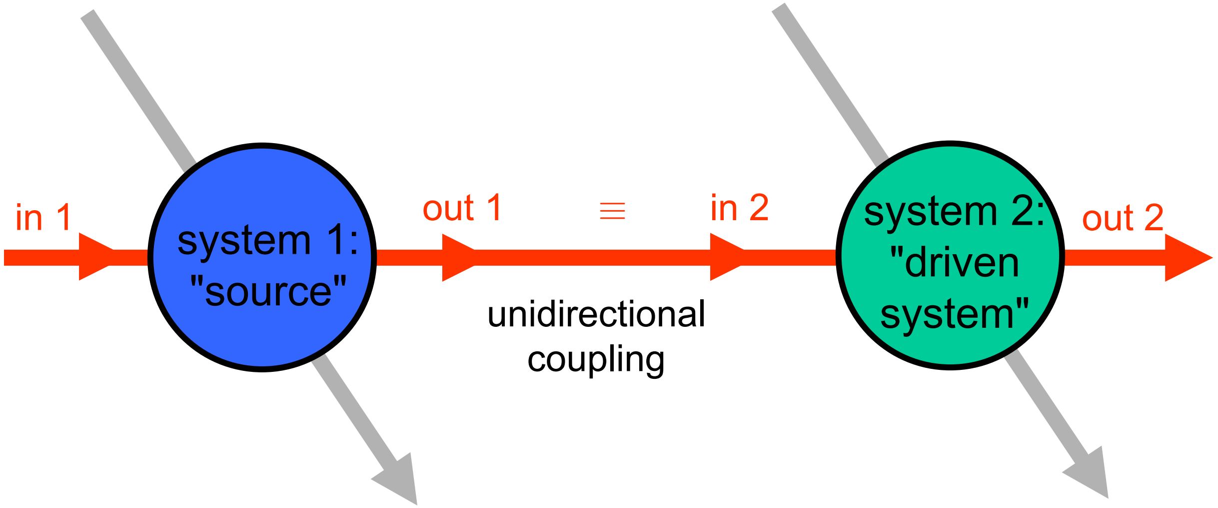


- memory: atoms →
- databus: photons →
- memory: atoms

- We call this protocol *photonic channel*

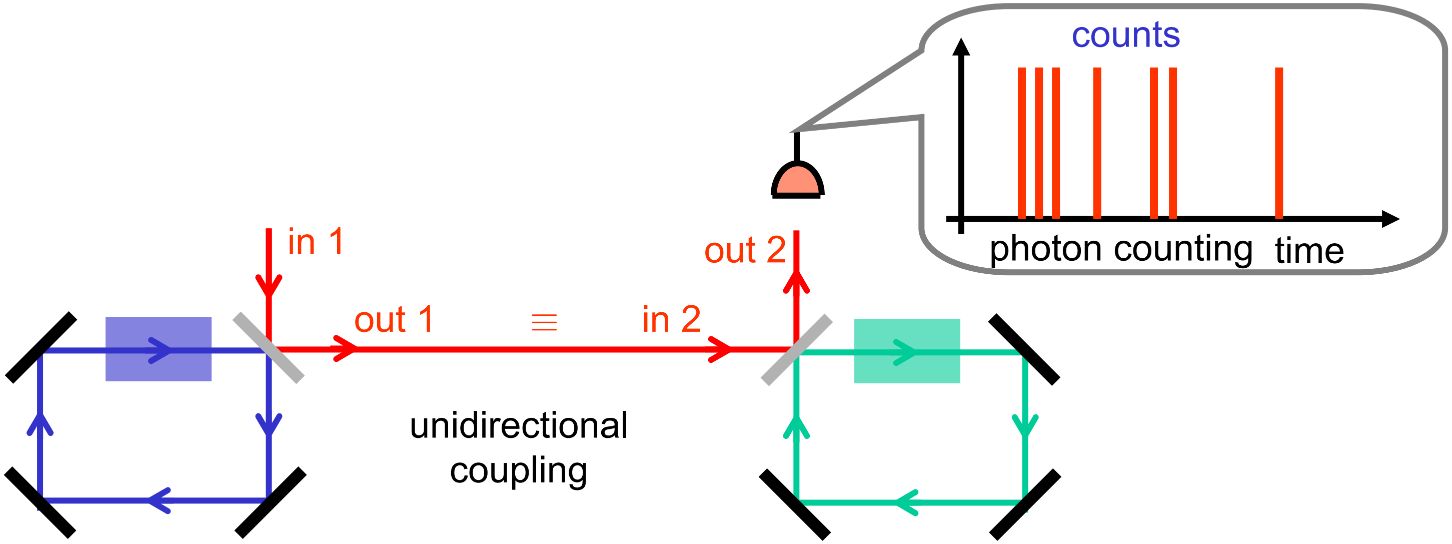
Cascaded Quantum Systems

- cascaded quantum system = first quantum system drives a second quantum system: *unidirectional* coupling



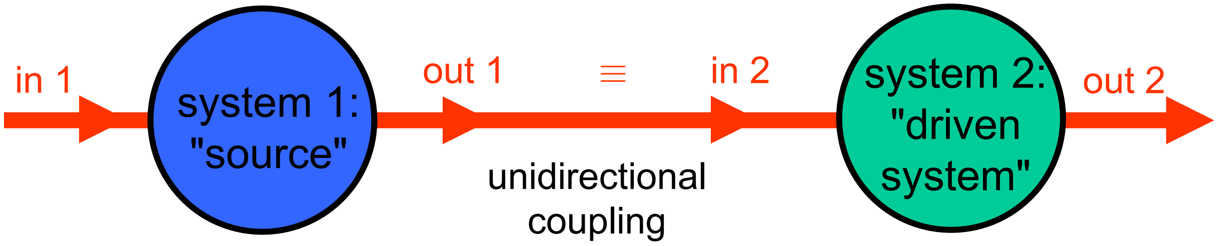
Cascaded Quantum Systems

- example of a cascaded quantum system



system 1:
"source"

system 2:
"driven system"



in 1

system 1: "source"

out 1 $\equiv$ in 2

unidirectional coupling

system 2: "driven system"

out 2

Hamiltonian

$$H = H_{\mathrm{sys}}(1) + H_{\mathrm{sys}}(2) + H_{\mathrm{B}} + H_{\mathrm{int}}$$

$$H_B = \int_{\omega_0-\vartheta}^{\omega_0+\vartheta} d\omega\, \hbar\omega\, b^\dagger(\omega) b(\omega)$$

with $b(\omega)$ the annihilation operator

$$[b(\omega), b^\dagger(\omega')] = \delta(\omega-\omega')$$

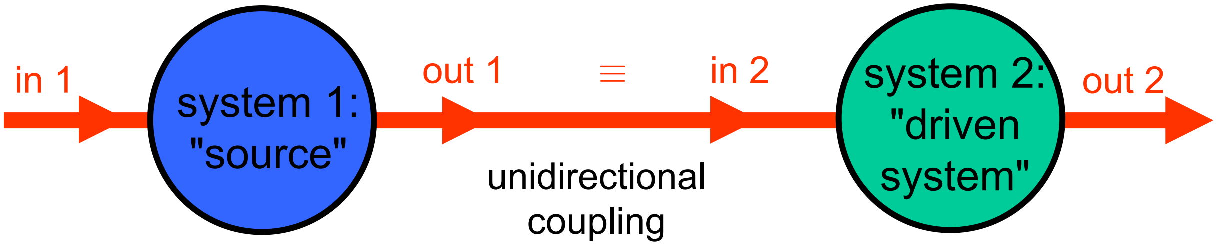
interaction part

position of first system

position of second system

$$H_{\mathrm{int}}^{(\vartheta)}(t) = i\hbar\int d\omega\kappa_1(\omega)[b^\dagger(\omega)e^{-i\omega/cx_1}c_1 - c_1^\dagger b(\omega)e^{+i\omega/cx_1}]$$

$$+i\hbar\int d\omega\kappa_2(\omega)[b^\dagger(\omega)e^{-i\omega/cx_2}c_2 - c_2^\dagger b(\omega)e^{+i\omega/cx_2}] \qquad (x_2 > x_1)$$



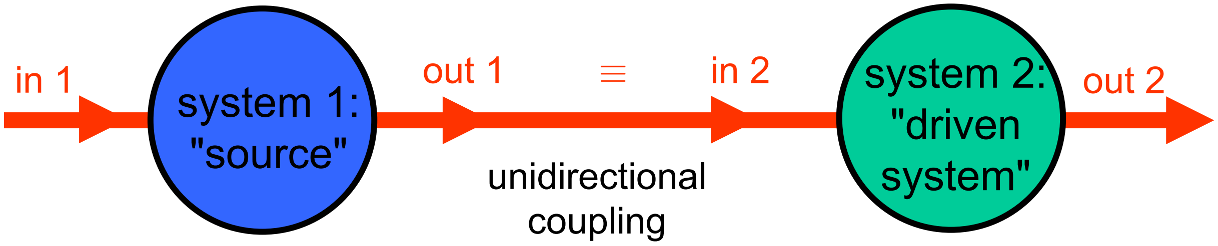
interaction picture

time delay

$$H_{\mathrm{int}}(t) = i\hbar\sqrt{\gamma_1}\,[b^\dagger(t)c_1 - b(t)c_1^\dagger] + i\hbar\sqrt{\gamma_2}\,[b^\dagger(t^-)c_2 - b(t^-)c_2^\dagger]$$

with $t^- = t - \tau$ where $\tau \to 0^+$

$$b(t) \equiv b_{\mathrm{in}}(t) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega b(\omega)e^{-i(\omega-\omega_0)t}$$



in 1

system 1: "source"

out 1 ≡ in 2

unidirectional coupling

system 2: "driven system"

out 2

Stratonovich SSE

time delay

$$\frac{d}{dt}\Psi(t) = \left\{ -\frac{i}{\hbar}\left(H_{\mathrm{sys}}(1) + H_{\mathrm{sys}}(2)\right) + \right.$$

$$\left. + \sqrt{\gamma_1}\,[b^\dagger(t)c_1 - b(t)c_1^\dagger] + \sqrt{\gamma_2}\,[b^\dagger(t^-)c_2 - b(t^-)c_2^\dagger] \right\} \Psi(t)$$

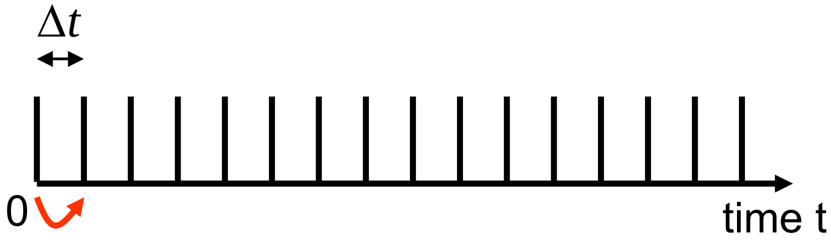
Initial condition:

$$|\Psi\rangle = |\psi\rangle \otimes |\mathrm{vac}\rangle$$

Notation:

$$\sqrt{\gamma_1}\,c_1 \to c_1, \quad \sqrt{\gamma_2}\,c_2 \to c_2, \quad \hbar = 1$$

First time step



$$U(\Delta t)|\Psi(0)\rangle \;=\; \Big\{\hat{1} - i[H(1)+H(2)]\Delta t + (\,c_2 + c_1\,)\int_0^{\Delta t} dt\, b^\dagger(t)$$

$$(-i)^2\int_0^{\Delta t} dt_1 \int_0^{t_2} dt_2 \underbrace{(-b(t_1)c_1^\dagger - b(t_{\bar{1}})c_2^\dagger)}_{\text{destruction}}\underbrace{(b^\dagger(t_2)c_1 + b^\dagger(t_{\bar{2}})c_2)}_{\text{creation}} + \ldots\Big\}|\Psi(0)\rangle$$

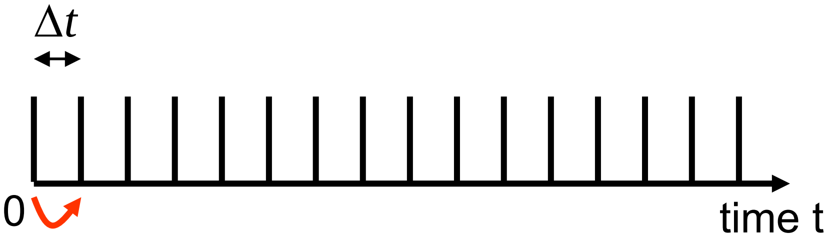
$$= \int_0^{\Delta t} dt_1 \int_0^{t_2} dt_2 (-\delta(t_1-t_2)c_1^\dagger c_1 + \delta(t_1-t_2+\tau)c_1^\dagger c_2$$

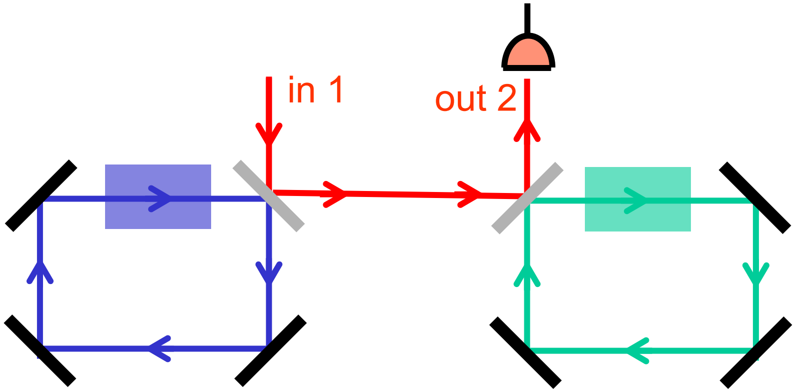
time delay! →

$$-\,\delta(t_1-\tau-t_2)c_2^\dagger c_1 - \delta(t_1-t_2)c_2^\dagger c_2)|\mathrm{vac}\rangle$$

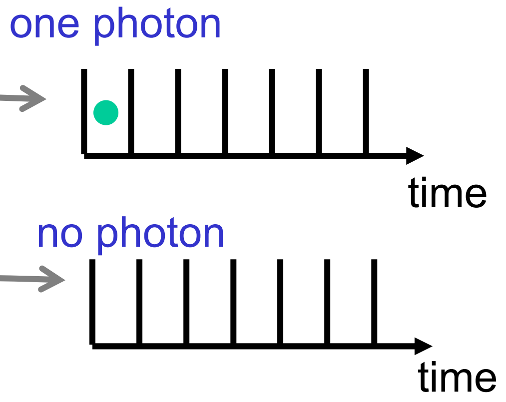
$$= \Big(-\frac{1}{2}c_1^\dagger c_1 + 0 \underbrace{- c_2^\dagger c_1}_{\text{reabsorption}} - \frac{1}{2}c_2^\dagger c_2\Big)|\mathrm{vac}\rangle\Delta t$$

First time step





$$U(\Delta t)|\Psi(0)\rangle \;=\; \Big\{\hat{1} - iH_{\text{eff}}\Delta t + (\,c_2 + c_1\,)\Delta B^\dagger(0)\Big\}|\Psi(0)\rangle$$



- effective Hamiltonian

$$H_{\text{eff}} = H(1) + H(2) - i\frac{1}{2}c_1^\dagger c_1 - i\frac{1}{2}c_2^\dagger c_2 \underline{- ic_2^\dagger c_1}\ \text{reabsorption}$$

$$= \Big\{H(1) + H(2) + \underbrace{i\frac{1}{2}(c_1^\dagger c_2 - c_2^\dagger c_1)}_{\text{hermitian}}\Big\} \underbrace{- i\frac{1}{2}c^\dagger c}_{\text{decay}} \quad (\text{with } c = c_2 + c_1)$$

- we identify $(\,c_2 + c_1\,)$ with the "jump operator"

Summary of results:

- Ito-type stochastic Schrödinger equation:

$$
\begin{aligned}
d|\Psi(t)\rangle &= |\Psi(t+dt)\rangle - |\Psi(t)\rangle \\
&= \left\{\hat{1} - i\underset{\uparrow}{H_{\text{eff}}}\,dt + (c_1 + c_2)\,dB^\dagger(t)\right\}|\Psi(0)\rangle
\end{aligned}
$$

$$
H_{\text{eff}} = H_{\text{sys}} + i\tfrac{1}{2}(c_1^\dagger c_2 - c_2^\dagger c_1) - i\tfrac{1}{2}c^\dagger c
$$

- master equation for source + system:

Version 1:

$$
\tfrac{d}{dt}\rho = -i(H_{\text{eff}}\rho - \rho H_{\text{eff}}^\dagger) + \tfrac{1}{2}(2c\rho c^\dagger - c^\dagger c\rho - \rho c^\dagger c)
$$

Lindblad form

Version 2:

$$
\begin{aligned}
\frac{d}{dt}\rho &= -i[H_{\text{sys}},\rho] \\
&\quad + \frac{1}{2}\{2c_1\rho c_1^\dagger - \rho c_1^\dagger c_1 - c_1^\dagger c_1\rho\} + \frac{1}{2}\{2c_2\rho c_2^\dagger - \rho c_2^\dagger c_2 - c_2^\dagger c_2\rho\} \\
&\quad - \{\underline{[c_2^\dagger, c_1\rho] + [\rho c_1^\dagger, c_2]}\}.
\end{aligned}
$$

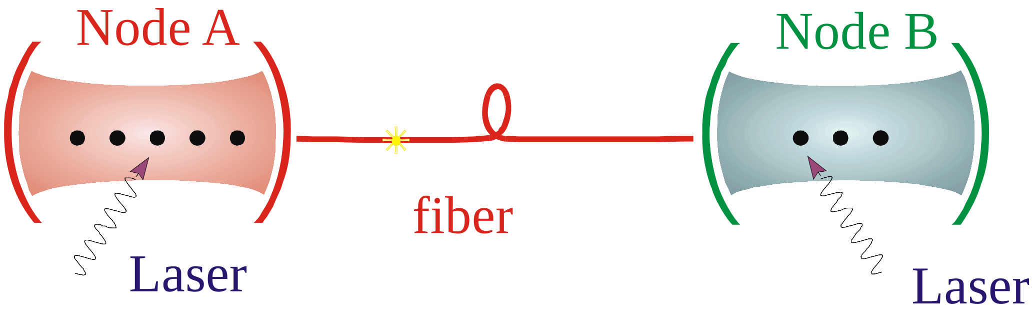
unidirectional coupling of source to system

Example: Optical Interconnects

J.I. Cirac, P.Z. H.J. Kimble and H. Mabuchi PRL '97

- A cavity QED implementation

Optical cavities connected by a quantum channel

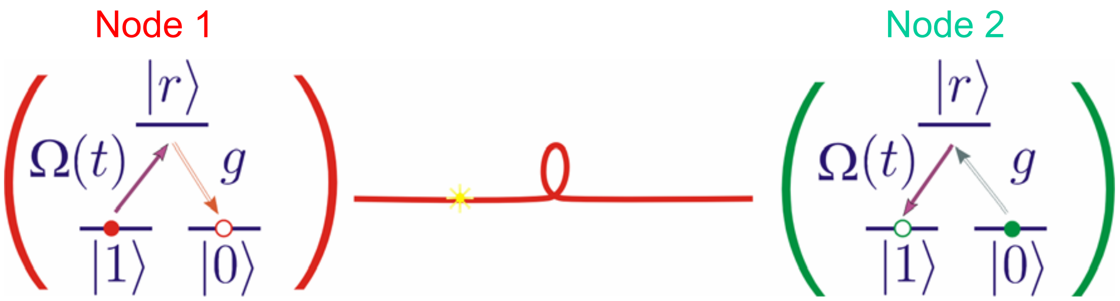


- memory: atoms → - databus: photons → - memory: atoms

- We call this protocol *photonic channel*

System

- System



- Hamiltonian: eliminate the excited state adiabatically

Hamiltonian $H = H_1 + H_2$

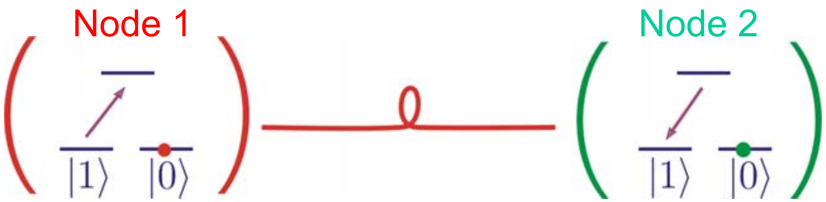
node i $\hat{H}_i = -\delta \hat{a}_i^\dagger \hat{a}_i - i g_i(t) \left[|1\rangle_i \langle 0| a - \text{h.c.} \right] \quad (i = 1, 2)$

Raman detuning $\delta = \omega_L - \omega_c$

Rabi frequency $g_i(t) = \frac{g\Omega_i(t)}{2\Delta}$

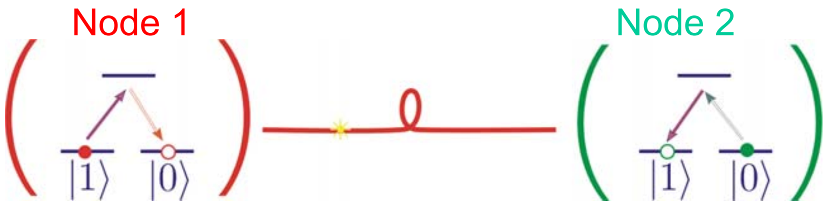
Ideal transmission

- sending the qubit in state 0



$$|0\rangle|0\rangle \quad \rightarrow \quad |0\rangle\ |0\rangle$$

- sending the qubit in state 1



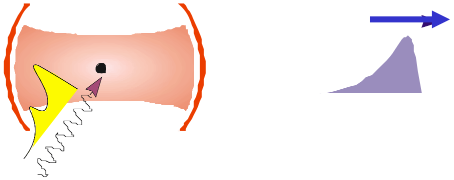
$$|1\rangle|0\rangle \quad \rightarrow \quad |0\rangle\ |1\rangle$$

- superpositions

$$[\alpha\ |0\rangle + \beta|1\rangle]\ \ |0\rangle \quad \rightarrow \quad |0\rangle\,[\alpha\ |0\rangle + \beta\ |1\rangle]$$

Physical picture as guideline for solution

- Ideal transmission = no reflection from the second cavity
- Physical picture as guideline for solution: "time reversing cavity decay"
  - consider one cavity alone



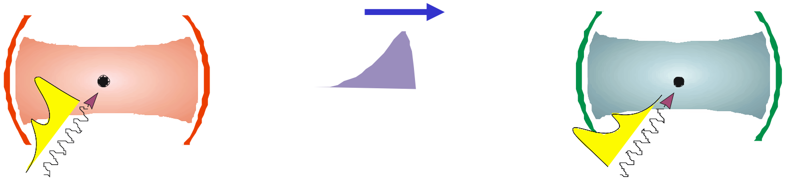
decay

  - run the movie backwards

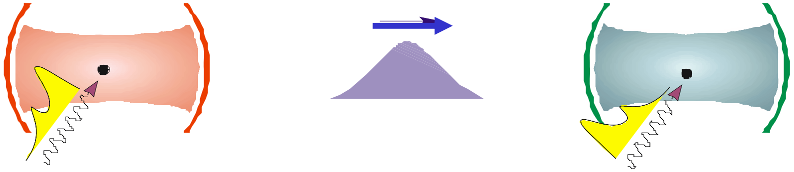
restore

inverse laser pulse

- two cavities



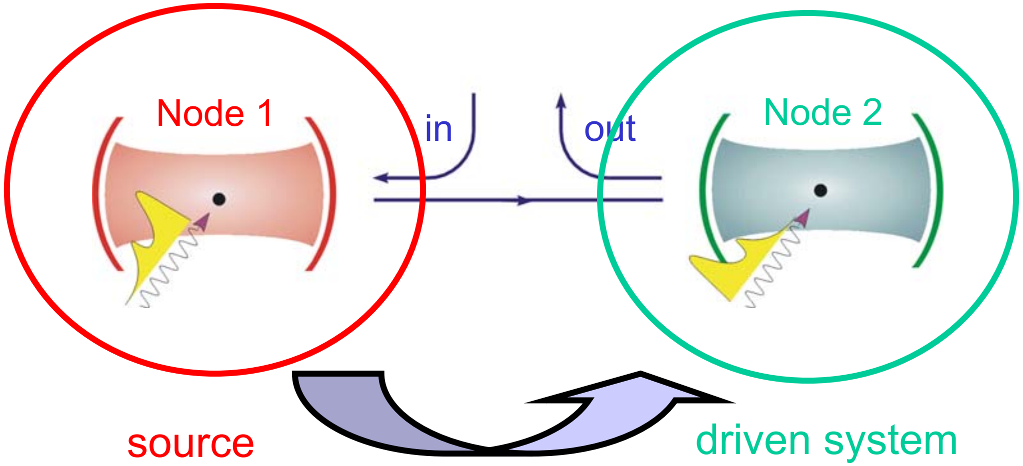
- design laser pulses to make the outgoing wavepacket symmetric



- we try a solution where the laser pulses are the time reverse of each other

Description ... as a cascaded quantum systems

- cascaded quantum system

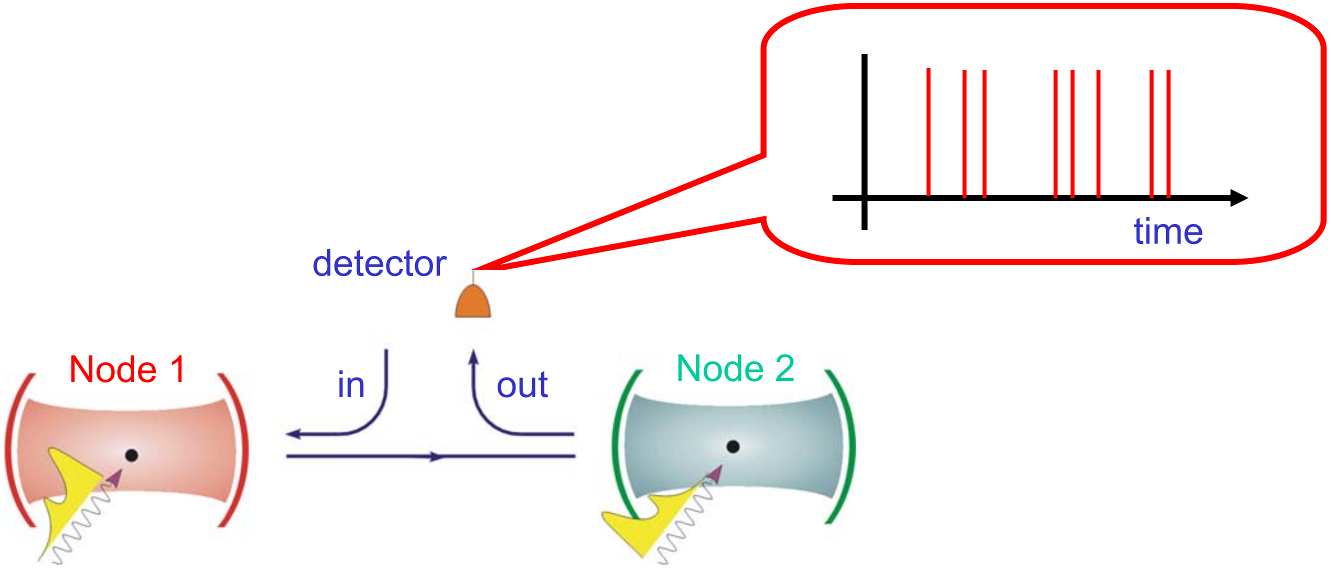


unidirectional coupling

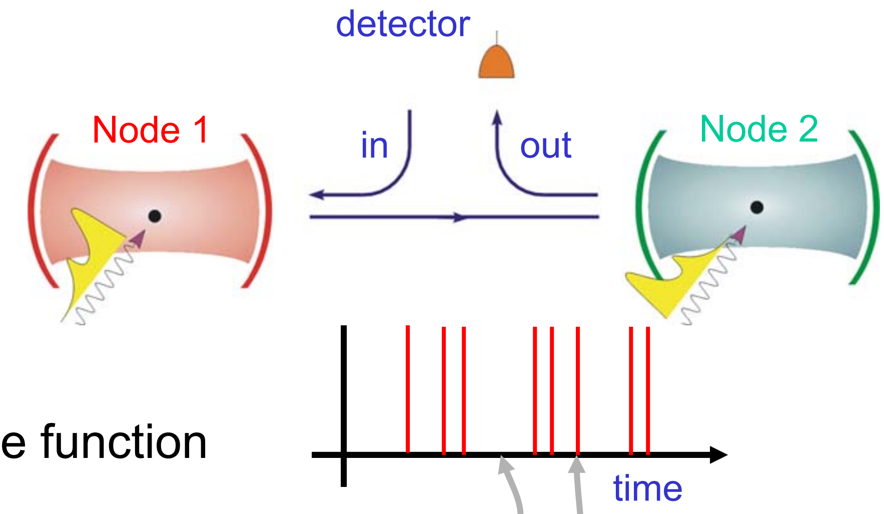
- a theory of cascaded quantum systems H. Carmichael and C. Gardiner, PRL '94

... quantum trajectories

- Quantum trajectory picture: *evolution conditional to detector clicks*



- We want *no reflection*: this is equivalent to requiring that the detector never clicks (= dark state of the cascaded quantum system)



- system wave function $|\Psi_c(t)\rangle$

- between the quantum jumps the wave function evolves with

$$\hat{H}_{\rm eff}(t) = \hat{H}_1(t) + \hat{H}_2(t) - i\kappa\left(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 2\ \hat{a}_2^\dagger\ \hat{a}_1 \right)$$

- quantum jump

$$|\psi_c(t+dt)\rangle \propto \hat{c}|\psi_c(t)\rangle \qquad (\text{with } \hat{c} = \hat{a}_1 + \hat{a}_2)$$

- probability for a jump $\propto \langle\psi_c(t)|\hat{c}^\dagger\hat{c}|\psi_c(t)\rangle$

- condition that no jump occurs

$$\langle\psi_c(t)|\hat{c}^\dagger\hat{c}|\psi_c(t)\rangle \overset{!}{=} 0 \quad \Longrightarrow \quad \hat{c}|\psi_c(t)\rangle = 0 \quad \forall t$$

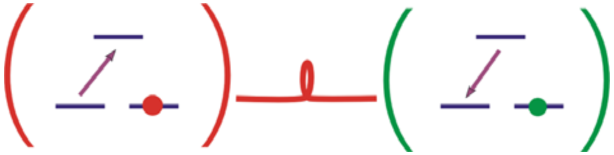
no reflection

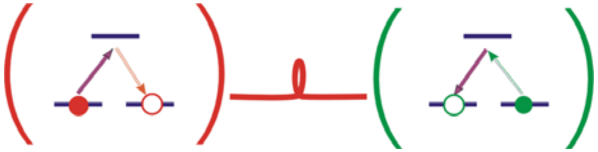
Equations

- Wave function for quantum trajectories: ansatz

atoms cavity modes

$$
\begin{aligned}
|\Psi_c(t)\rangle \; &= \; |00\rangle|00\rangle \\
&+ \; \Big[\alpha_1(t)|10\rangle|00\rangle + \alpha_2(t)|01\rangle|00\rangle \\
&\qquad + \beta_1(t)|00\rangle|10\rangle + \beta_2(t)|00\rangle|01\rangle\Big].
\end{aligned}
$$



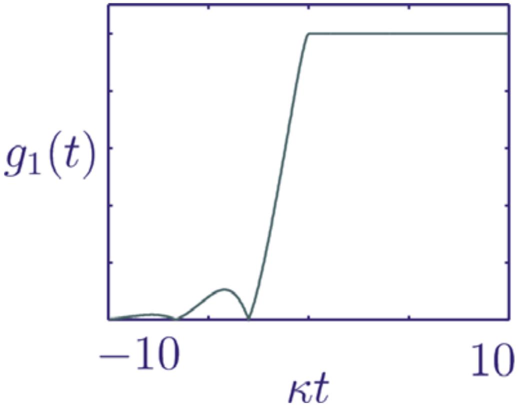


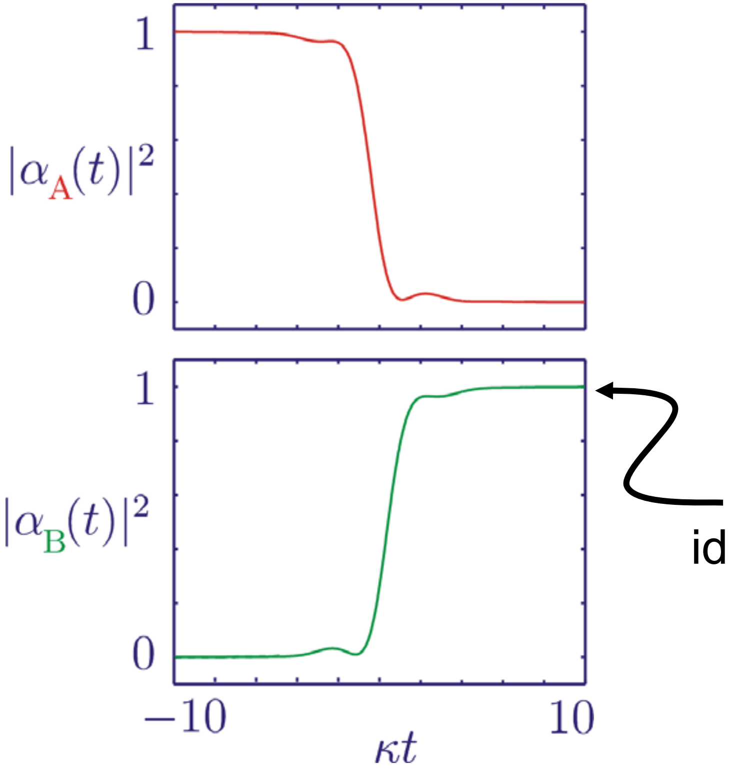
ONE excitation in system

- we derive equations of motion ... and impose the dark state conditions
- we find exact analytical solutions for pulse shapes leading to "no reflection" ...

Results

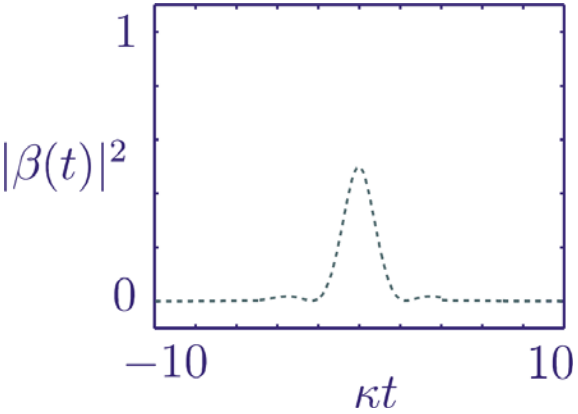
pulse shape





ideal transmission

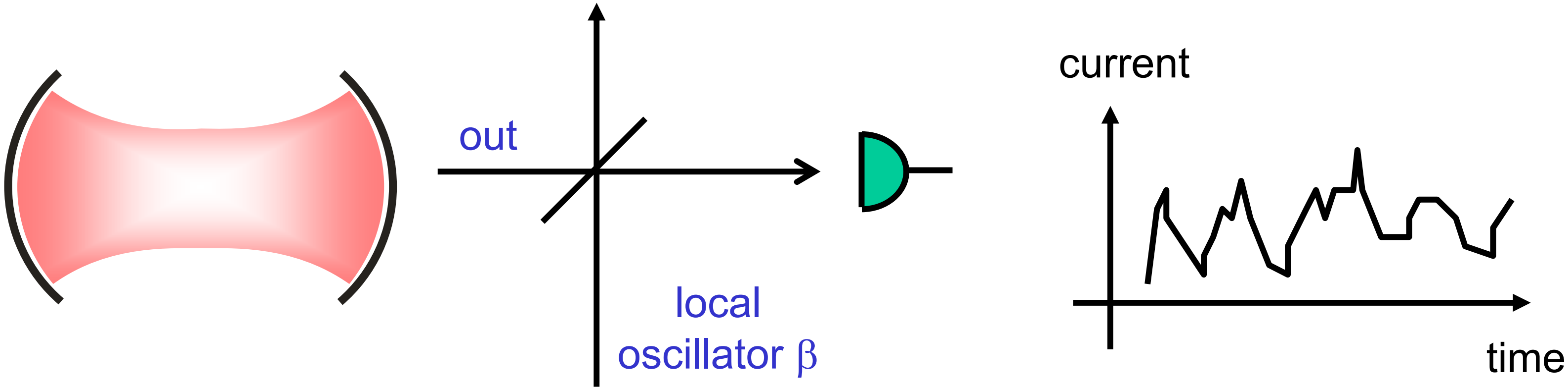
wave packet



similar theory developed for …

6. Homodyne Detection

- homodyne detection



- conditional system wave function

$$d|\psi_X(t)\rangle = \left[(-iH - \tfrac{1}{2}\gamma c^{\dagger}c)dt + \sqrt{\gamma}\, c\, dX(t)\right]|\psi_X(t)\rangle$$

with $dX(t) = \sqrt{\gamma}\,\langle x(t)\rangle_c dt + dW(t)$ and $dW(t)$ a Wiener increment

homodyne current

$c+c^{+}$

shot noise

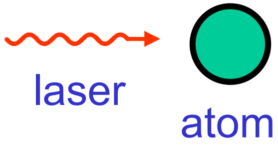
A few slides on …

“How to write effective Hamiltonians for atom-light interactions“

mini-tutorial

Elementary atomic QO Hamiltonians (without dissipation)

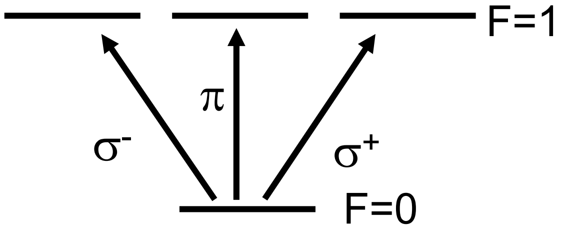
- atom interacting with classical laser light



$$H = H_{0A} - \vec{\mu}\cdot\vec{E}_{\rm cl}(\vec{x}=0,t)$$

dipole interaction

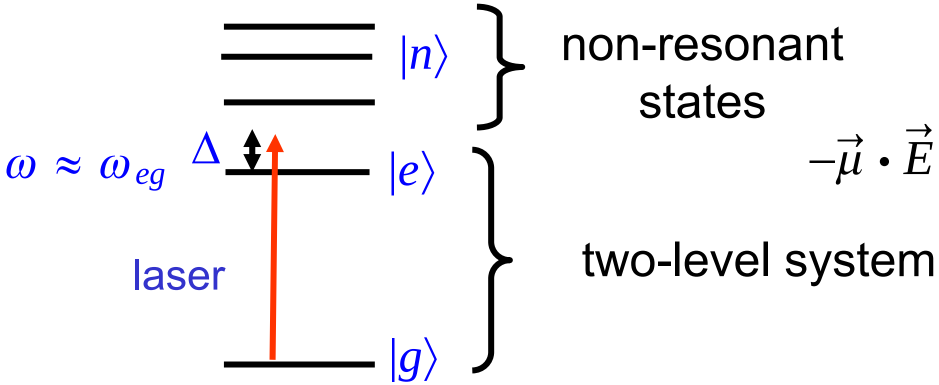
- laser: electric field



selection rules / light polarization

$$\vec{E}_{\rm cl}(\vec{x}=0,t) = \mathcal{E}(t)\vec{\epsilon}e^{-i\omega t} + \text{c.c.}$$

- two-level system + rotating wave approximation

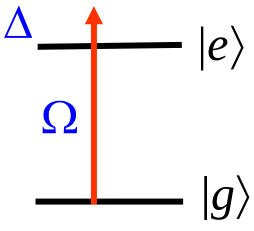


step 1: resonant couplings (nonperturbative)

$$-\vec{\mu}\cdot\vec{E}_{\rm cl} \longrightarrow -\vec{\mu}_{eg}\vec{\epsilon}\mathcal{E}(t)e^{-i\omega t}|e\rangle\langle g| - \vec{\mu}_{ge}\vec{\epsilon}\mathcal{E}^*(t)e^{i\omega t}|g\rangle\langle e|$$

step 2: rest in perturbation theory

- “Two-level atom + rotating wave approximation“ as effective Hamiltonian



$$H = \hbar\omega_{eg}|e\rangle\langle e| - \underbrace{\vec{\mu}_{eg}\vec{\epsilon}\mathcal{E}(t)}e^{-i\omega t}|e\rangle\langle g| - \vec{\mu}_{ge}\vec{\epsilon}\mathcal{E}^*(t)e^{i\omega t}|g\rangle\langle e|$$

$$H_{\mathrm{TLS+RWA}} = -\frac{1}{2}\hbar\Delta\sigma_z + \frac{1}{2}\hbar\Omega e^{i\varphi}\sigma_- + \frac{1}{2}\hbar\Omega e^{-i\varphi}\sigma_+$$

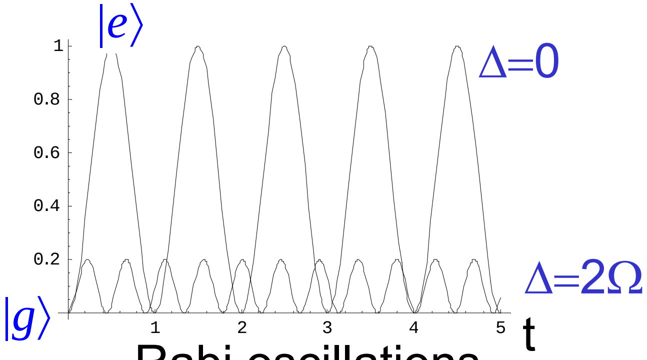
detuning

Rabi frequency

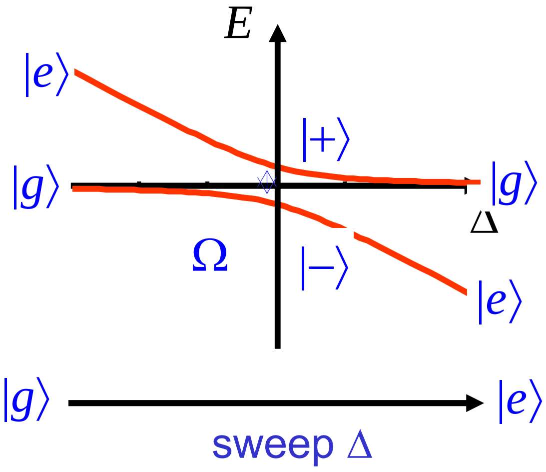
Remarks:

- optical frequencies transformed away
- validity $|\vec{\mu}_{ng,e}\vec{\epsilon}\mathcal{E}(t)| \ll$ |detunings off-resonant states|

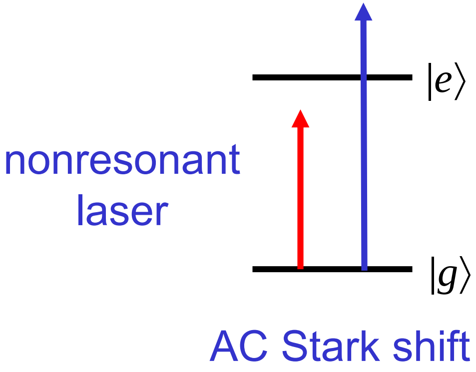
- Dynamics: Rabi oscillations vs. adiabatic sweep

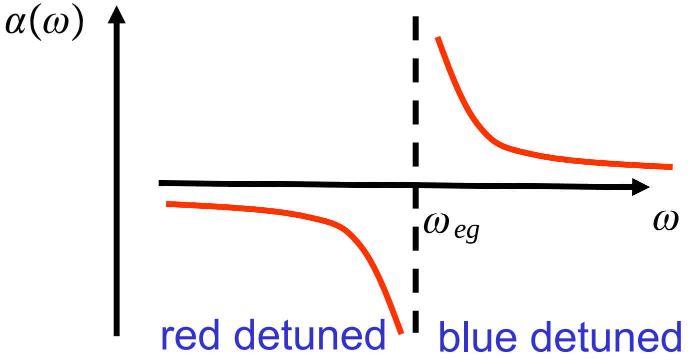


Rabi oscillations



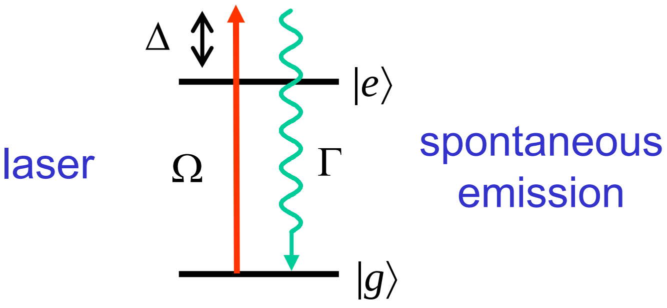
- perturbation theory for the non-resonant states:
  example: AC Starkshift





$$H = [\hbar\omega_g + \hbar\delta\omega_g(t)]|g\rangle\langle g| + \ldots \qquad \hbar\delta\omega_g(t) = \sum_n \frac{|\vec{\mu}_{ng}\vec{\epsilon}\mathcal{E}|^2}{\hbar(\omega_{gn}+\omega)} + \frac{|\vec{\mu}_{ng}\vec{\epsilon}\mathcal{E}|^2}{\hbar(\omega_{gn}-\omega)} \equiv \alpha(\omega)|\mathcal{E}|^2$$

- decoherence: spontaneous emission



$$\Delta E_g = \frac{1}{4}\frac{\Omega^2}{\Delta - \frac{1}{2}i\Gamma} = \delta E_g - i\frac{1}{2}\gamma_g$$

$$\frac{\text{good}}{\text{bad}} = \frac{\delta E_g}{\gamma_g} \sim \frac{|\Delta|}{\Gamma} \gg 1$$

typical off-resonant lattice : $\gamma \sim$ sec^{-1}

In a blue detuned lattice this can be strongly suppressed

mini-tutorial

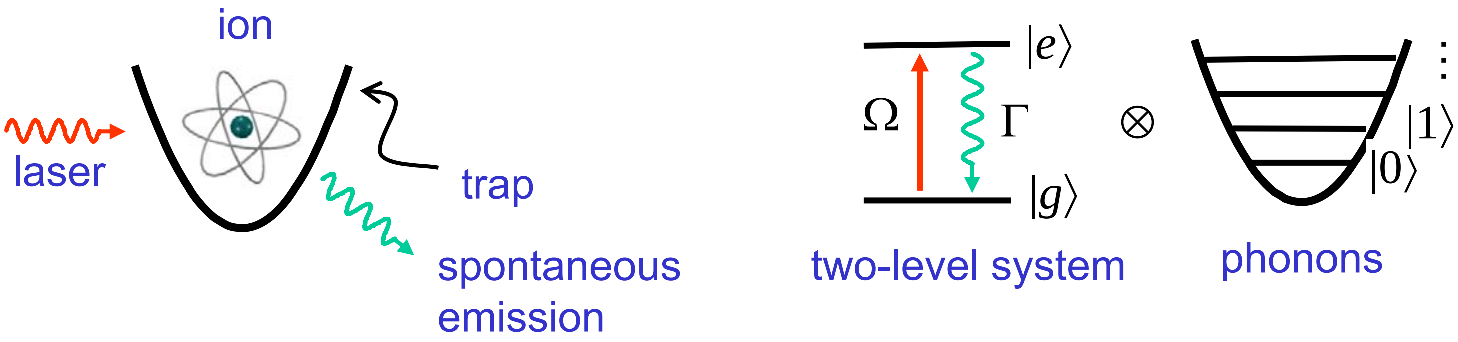
- including the center-of-mass motion

$$H = \frac{\vec{p}^2}{2M} + V_T(\vec{x}) + H_{0A} - \vec{\mu} \cdot \vec{E}(\vec{x},t)$$

coupling internal – external dynamics

atomic motion

- example 1: trapped ion



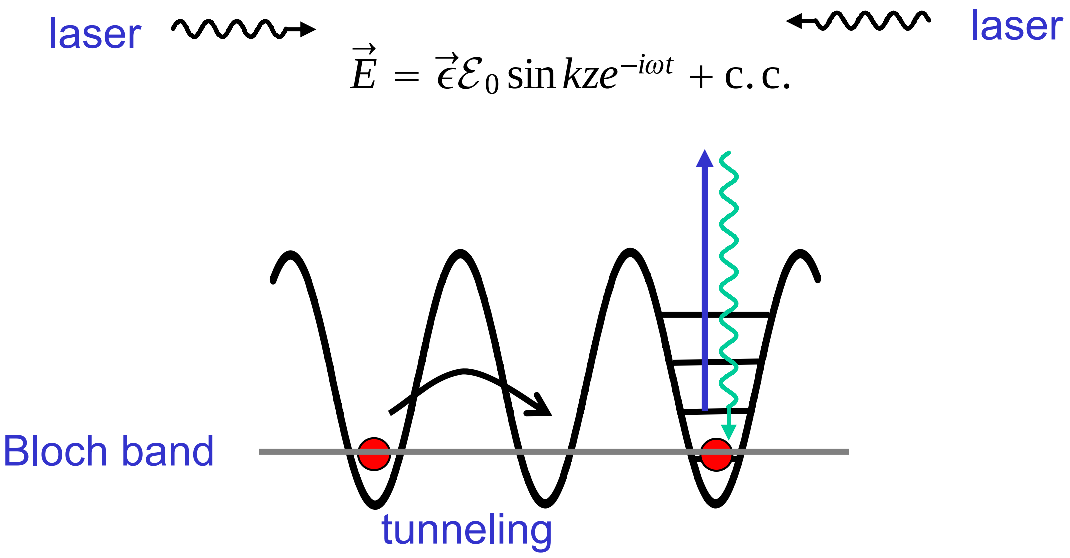
$$H = \frac{\hat{p}^2}{2M} + \frac{1}{2} M \nu^2 \hat{x}^2 + \hbar\omega_{eg}|e\rangle\langle e| - \hbar\left(\frac{1}{2}\Omega e^{ik\hat{x}-i\omega t}|e\rangle\langle g| + \mathrm{h.c.}\right)$$

trap

atom – laser coupling

atom – laser coupling

- example 2: atom in optical trap / lattice



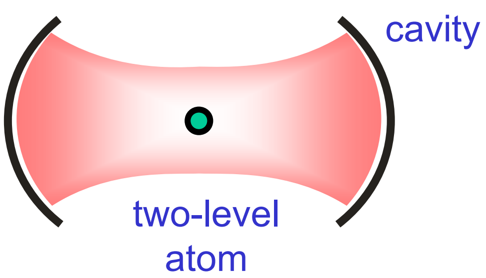
$$\vec{E} = \vec{\epsilon}\mathcal{E}_0 \sin kz e^{-i\omega t} + \mathrm{c.c.}$$

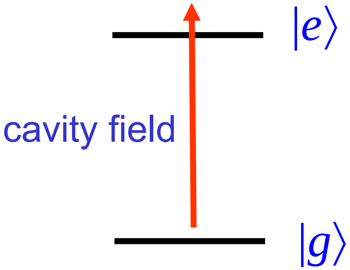
optical lattice as array of microtraps

$$V(x) = V_0 \sin^2 kx \qquad (k = \tfrac{2\pi}{\lambda})$$

mini-tutorial

- Cavity QED: Jaynes-Cummings model





$$H = \hbar\omega_{eg}|e\rangle\langle e| + \hbar\omega b^{\dagger}b - (\vec{\mu}\vec{g}\,\sigma_{+}b + \vec{\mu}^{*}\vec{g}^{*}\sigma_{-}b^{\dagger})$$

- dressed states