Joseph Ford

STATISTICAL PROPERTIES OF NONLINEAR STRING

By F. M. Izrailev and B. V. Chirikov

Institute of Nuclear Physics of the Siberian Branch
of the Acad. of Science, Novosibirsk

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The qualitative behavior of the longitudinal oscillations of a nonlinear string with fixed ends (Fig. 1) and satisfying the equation

$$\frac{\partial^2 x}{\partial t^2} = \frac{\partial^2 x}{\partial z^2}\left[1 + 3\beta\left(\frac{\partial x}{\partial z}\right)^2\right] \qquad (1)$$

is investigated in the present work. This problem was examined for the first time in a work of Fermi, Pasta and Ulam [1] by the method of numerical integration for a chain of nonlinear oscillators (Fig. 1) approximately representing a string and satisfying the following system of ordinary differential equations:

$$\ddot{x}_l = (x_{l+1} + x_{l-1} - 2x_l)\left\{1 + \beta\left[(x_{l+1} - x_l)^2 + (x_l - x_{l-1})^2 + (x_{l+1} - x_l)(x_l - x_{l-1})\right]\right\} \qquad (2)$$

where $l = 1, 2, \ldots, N - 1$; Q = L, and L = N. The purpose of [1] was to trace the emergence of statistical properties on that relatively simple example of a mechanical system with a large number of degrees of freedom. In the linear case ($\beta = 0$), the chain of oscillators can be represented in the form of the set of N - 1 completely independent modes (normal modes) and, consequently, does not have any statistical properties. Until recently it was assumed (see, for example, [2]) that any nonlinearity, no matter how small, was adequate for the emergence of statistical properties. Therefore the authors of [1] expected that the initial energy of system (2), concentrated only in the first (lowest mode) would be distributed in the course of time approximately uniformly among all the modes. However, the numerical solution of (2) led the directly opposite picture: first, the energy exchange occurred only among the first few modes; secondly, and this is still more essential, the energy exchange had a quasi-periodic character, so that in a definite, relatively short, interval of time all the energy was collected once more in the first mode with a precision of several per cent. Further calculations made by T~~ak~~uck showed that for considerably longer intervals of time the quasi-periodic character of the motion becomes still more definite; in particular, the return of all the energy to the first mode occurs with considerably greater precision . [The authors are grateful to Professor Ulam for the communication of these interesting details.]

These results caused the authors of [1] to express a hypothesis of the existence sui generis of new normal coordinates (quasi-modes) for nonlinear systems (see also [3]).

At approximately the same time, preservation of the quasi-periodic character of the motion of the mechanical system during fairly small, but finite, excitation was proved analytically in the works of Kolmogorov and Arnol'd (see, for example, [4,5]). It is true that the Kolmogorov-Arnol'd theory is inapplicable directly to system (2), because that system is linear in zero approximation ($\beta = 0$), but a corresponding generalization can be made. Thus, from the present-day point of view, the result of [1] is reasonable.

On the other hand, when the nonlinear excitation is great enough and N in system (2) is large enough, statistical properties evidently should appear sooner or later, as follows from the well known fact of the applicability of statistical mechanics to any system with a large number of degrees of freedom (see also [6]). Consequently, some critical value of the excitation (β_{cr}) should exist, corresponding to the boundary between the region of quasi-periodic motion and the region of stochasticity. We will use the latter term to designate the essential statistical properties (ergodicity, intense mixing and, most important of all, positive Kolmogorov entropy [7], which designates physically the exponential decrease of time correlations in the system. The region of quasi-periodic motion will henceforth be called the region of Kolomogorov stability, since from the standpoint of a mechanical system the motion in that region is maximally stable--there is a complete set of N-1 integrals of the motion. In the region of stochasticity, however, there is a total of one motion integral--the energy,--so that it is the region of maximal instability. Thus, the critical value of excitation β_{cr} determines the boundary of the so-called stochastic instability, which is the most essential for a nonlinear system [6,10].

The purpose of the present study is to estimate the boundary of stochastic instability for the system of oscillators (2).

1. Fundamental Relations

First we change to the normal coordinates (for the linear case) in Eqs. (2) with the formulas:

$$x_\ell = \sqrt{\frac{2}{N-1}} \sum_{k=1}^{N-1} Q_k \sin\frac{\pi k \ell}{N} \qquad (I.1)$$

After complicated calculations we obtain:

$$\ddot{Q}_k + \omega_k^2 Q_k = -\frac{\beta}{8N}\Big\{ \sum_{i+j=2}^{k-1} A^+_{ij} Q_{k-i-j}\,\omega^2_{k-i-j} + \sum_{i+j=N-k+1}^{2N-k-1} A^+_{ij} Q_{2N-i-j-k}\,\omega^2_{2N-i-j-k} +$$

$$+ \sum_{i+j=2}^{N-k+1} A^+_{ij} Q_{i+j+k}\,\omega^2_{i+j+k} - \sum_{i+j=N+k+1}^{2N-2} A^+_{ij} Q_{2N+k-i-j}\,\omega^2_{2N+k-i-j} - \sum_{i+j=k+1}^{k+N-1} A^+_{ij} Q_{i+j-k}\,\omega^2_{i+j-k} - \qquad (I.2)$$

$$- \sum_{i+j=2N-k+1}^{2N-2} A^+_{ij} Q_{i+j+k-2N}\,\omega^2_{i+j+k-2N} + 2\sum_{i-j=k-1}^{k-N+1} A^-_{ij} Q_{k-i+j}\,\omega^2_{k-i+j} +$$

$$+ 2\sum_{i-j=N-k+1}^{N-2} A^-_{ij} Q_{2N-k-i+j}\,\omega^2_{2N-k-i+j} - 2\sum_{j-i=k+1}^{N-2} A^-_{ij} Q_{j-i-k}\,\omega^2_{j-i-k} \Big\}$$

where где

$$i,j,k = 1,2,\dots,N-1$$

$$\omega_k = 2\sin\frac{\pi k}{2N} \qquad (I.3)$$

and the values of $A^{\pm}_{ij}$ are given by the expressions

$$A_{ij}^{\pm}=Q_iQ_j\omega_i\omega_j\left[3\sqrt{(4-\omega_i^2)(4-\omega_j^2)}\pm\omega_i\omega_j\right] \qquad (I.4)$$

Equations (1.2) can be presented in the form

$$\ddot{Q}_k+\omega_k^2Q_k=-\frac{\beta}{8N}\sum_{p,q,s=1}^{N-1}B_{pqs}Q_pQ_qQ_s \qquad (I.5)$$

Then we separate from that sum the terms $Q_kQ_i^2$ and transfer them to the left side:

$$\ddot{Q}_k+\omega_k^2Q_k\left(1+\frac{\beta}{8N\omega_k^2}\sum_{i=1}^{N-1}D_{ki}Q_i^2\right)=-\frac{\beta}{8N}\sum_{(pqs)}B_{pqs}Q_pQ_qQ_s \qquad (I.6)$$

where D_{ki} is certain new coefficients (see below) and (pqs) designates the sum of the remaining terms. The sense of the separation of the sum in (1.5) consists in the different effect of the two sides. The terms in the left side of (1.6) lead to the dependence of the frequency of normal oscillators both on their amplitude and on the amplitude of other oscillators. However, the remaining terms in the right side of (1.6) have the character of external forces with different frequencies. Thus the problem is reduced to studying the motion of a nonlinear oscillator close to many resonances.

We seek the solution in the form:

$$Q_n=C_n(t)\cos\theta_n(t); \qquad \dot{\theta}_n=\omega'_n(t) \qquad (I.7)$$

where C and ω'_n are the slowly changing -- under the conditions of (2.12) -- amplitudes and frequencies of the normal modes; the prime indicates that the frequency is not equal to its value in linear approximation (1.3) but includes all the corrections connected with the excitation.

By using (1.7) it is possible to represent the right side of (1.6) in the form:

$$\sum_{(pqs)} B_{pqs} Q_p Q_q Q_s = \sum_m F_{km} \cos\theta_{km} \qquad (I.8)$$
$$\dot{\theta}_{km} = \omega'_{km}$$

where ω'_{km} is the frequencies of the external forces acting on the oscillator k.

The sum on the right side is:

$$\frac{1}{\omega_k^2}\sum_{i=1}^{N-1} D_{ki} Q_i^2 = 12\sum_{i=1}^{N-1} Q_i^2 \omega_i^2 (2-\omega_i^2) - 6 Q_k^2 \omega_k^2 (2-\omega_k^2) + \ldots$$
$$+ 12 Q_{N-k}^2 \omega_{N-k}^2 + 6 Q_{N-k}^2 \frac{\omega_{N-k}^2}{\omega_k} \sqrt{(4-\omega_k^2)(4-\omega_{N-k}^2)} \qquad (I.9)$$

Since henceforth we will be interested mainly in the boundary of stochasticity when a small number of modes has been excited, it is possible to ignore the contribution of the further modes ($Q_{N-K} \simeq 0$). Let us note further that the sum in (1.9) is identical for all the oscillators (does not depend on k) and consequently it leads to an identical shift of all the frequencies and in first approximation does not influence the resonances.

Finally, equation (1.6) is transformed to the following form:

$$\ddot{Q}_k + \omega_k^2 Q_k \left\{ 1 - \frac{3\beta}{4N} \omega_k^2 (2-\omega_k^2) Q_k^2 \right\} =$$
$$= \frac{\beta}{8N} \sum_m F_{km} \cos\theta_{km} \qquad (I.10)$$

Let us examine at first the influence of one resonance harmonic on the right side of (1.10). By using the method of averaging [8] we obtain the following equations in slow variables:

$$\dot{C}_k = \frac{\beta F_{km}}{16\omega_k' N} \sin\Psi_{km}; \qquad \Psi_{km} = \theta_{km} - \theta_k$$

$$\dot{\Psi}_{km} = \omega_{km}' - \omega_k' + \frac{\beta F_{km}}{16 C_k \omega_k' N} \cos\Psi_{km} \tag{I.II}$$

Let us assume from the start that the term with F_{km} in the second equation can be neglected (the criterion of this will be given below); if we differentiate with respect to time and use the first equation, we obtain the so-called phase equation:

$$\ddot{\Psi}_{km} = \frac{d\Omega_{km}}{dC_k} \cdot \frac{\beta F_{km}}{16\omega_k' N} \sin\Psi_{km}, \quad \Omega_{km} = \omega_{km}' - \omega_k' \tag{I.I2}$$

From that equation we find the amplitude of the oscillations Ψ_{km} defining the size of the separatrix on the phase plane (Ψ_{km}, $\dot{\Psi}_{km}$) (see, for example [9]):

$$|\dot{\Psi}_{km}|_{max} = \sqrt{\frac{\beta F_{km}}{4N\omega_k'} \cdot \frac{d\Omega_{km}}{dC_k}} \tag{I.I3}$$

The term in (1.11) that was discarded characterizes the width of the resonance region, and therefore approximate equation (1.12) corresponds to the case where the size of the separatrix is much larger than the width of the resonance, that is:

$$\xi \equiv \frac{\beta F_{km}}{8N\omega_k' C_k^2 \; d\Omega_{km}/dC_k} \ll 1 \tag{I.I4}$$

Actually there are many resonances of (1.10). The motion of the oscillator in that case depends essentially on the relation between the size of the separatrix and the average distance between resonances

$$\Delta\omega = \omega'_{km} - \omega'_{k,m\pm 1} \tag{I.15}$$

If

$$\chi \equiv \frac{|\dot{\psi}_{km}|^2_{max}}{(\Delta\omega)^2} \ll 1 \tag{I.16}$$

it is possible to neglect in first approximation the influence of all the resonances except the closest. Then, as is evident from phase equation (1.12), the motion has the character of stable oscillations and the energy exchange between the modes is negligibly small

$$\left(\Delta C_k / C_k\right) \sim \left(\frac{\beta F_{km}}{8N\omega'_k C_k^2 \; d\Omega_{km}/dC_k}\right)^{1/2} = \xi^{1/2} \ll 1 \tag{I.17}$$

by virtue of (1.14). This also is Kolmogorov stability.

In the reverse limiting case ($\chi \gg 1$) the motion is stochastic [10,11] This conclusion cannot be considered rigorously proved; rather it is a distinctive heuristic hypothesis based on qualitative physical concepts and confirmed on particular examples [10,11]. The boundary of stochasticity of interest to us is determined with the estimate:

$$\chi = \frac{\beta F_{km}}{4N\omega'_k(\Delta\omega)^2} \cdot \frac{d\Omega_{km}}{dC_k} \sim 1 \tag{I.18}$$

Let us emphasize that in reality no sharp boundary exists between the two regions, which should rather be considered as limiting cases. Actually, there is an entire transitional region ($\chi \sim 1$), in which the motion has a very complex character and depends essentially on the initial conditions [10,11].

If condition (1.14) is not fulfilled ($\xi \gtrsim 1$), the simple phase equation (1.12) ceases to be valid. Nevertheless it can be shown [10,11] that the boundary of stochasticity is determined as before by estimate (1.18). This is connected with the fact that the larger term $(\beta F_{km} 16 C_k \omega_k' N) \cos \psi_{km}$ expresses the linear properties of the system and therefore cannot in itself lead to stochasticity. Stochasticity can be connected only with the nonlinearity $(d\Omega_{km}/dC_k) \neq 0$. As a result of that nonlinearity a change of the frequency $\sim (d\Omega_{km}/dC_k)\dot{C}_k t$ occurs, where t is the characteristic time of interaction. The shift of frequency leads to an additional shift of phase by the value $\Delta\psi_{km} \sim (d\Omega_{km}/dC_k)\dot{C}_k t$, which depends on the initial phase. At $\Delta\psi_{km} \gtrsim 1$ the time correlation between the phases is disrupted, and this also leads to stochasticity. If we assume $t \sim (\Delta\omega)^{-1}$ (1.15), that is, that it is of the order of the quasi-period of excitation, we get estimate (1.18). Let us note that in the case $\xi \gtrsim 1$ the energy exchange between the modes already is not small (1.17). Nevertheless, if the nonlinearity is small enough ($\chi \gg 1$), the motion in that case too will have a quasi-periodic character. Strong energy exchange between the modes means that the "true" natural modes differ substantially from the unexcited. It was precisely this situation which took place in [1].

2. The Lower Modes ($k \ll N$)

In this limiting case, by replacing the summation in (1.2) by integration, it is possible to obtain for F_{km} the estimate:

$$F_{km} \sim 16\pi^4 (\Delta k)\left(\frac{k}{N}\right)^4 \langle C^3 \rangle \tag{2.I}$$

where the value $\langle C^3 \rangle$ designates a certain cubic combination of the amplitudes of those modes which participate in the formation of the frequency ω_{km}; Δk is the interval of the excited modes. The correction to the frequency of the k-th mode on account of nonlinearity is obtained by applying to equation (1.10) the standard technique of averaging [8] and is:

$$\omega_k \to \omega_k - \frac{9}{32}\frac{\beta}{N}\omega_k^3(2-\omega_k^2)C_k^2 \tag{2.2}$$

Hence, we obtain for Ω_{km} the estimate[1]:

$$\Omega_{km} \sim \frac{9}{16}\frac{\beta}{N}\left(\frac{\pi k}{N}\right)^3 \langle C^2\rangle; \quad \frac{d\Omega_{km}}{dC_k} \sim \pi^3 \frac{\beta}{N}\left(\frac{k}{N}\right)^3 \langle C\rangle \tag{2.3}$$

The resonance frequencies are

$$\omega_{km} = 2\sin\frac{\pi(k+2m)}{2N} \qquad \pm m = 0,1,2,\ldots \tag{2.4}$$

The multiplier 2 before the m is connected with the fact that in the case of cubic nonlinearity (2) which preserves the symmetry of interaction with respect to the sign of the shift, only the modes of identical parity (every other one). The average distance between adjacent resonances, from (2.4), is

$$\Delta\omega \sim \frac{2\pi}{N} \tag{2.5}$$

Let us emphasize that this is precisely the average distance, since the frequencies (2.4) are substantially shifted on account of nonlinear effects; therefore, depending on the initial conditions, the local distances between the resonances can change. If we substitute all these expressions in (1.18), we get

$$\sqrt{\chi} \sim 20\beta\langle C^2\rangle\sqrt{\Delta k}\left(\frac{k}{N}\right)^3 \sim 1 \tag{2.6}$$

We will express the latter condition through the dimensionless characteristic of the nonlinear excitation (2):

$$\beta\left[(x_{l+1}-x_l)^2+(x_l-x_{l-1})^2+(x_{l+1}-x_l)(x_l-x_{l-1})\right]\approx 3\beta\left(\frac{\partial x}{\partial z}\right)^2 \quad (2.7$$

since z = la = 1 (2). The latter expression ceases to be valid for very high modes; however, the equality in order of magnitude is preserved. It remains to us now to express the amplitude of the excited modes through the shift of the string (I. 1.7):

$$x_l=\sqrt{\frac{2}{N-1}}\sum_k C_k \cos\theta_k \cdot \sin\frac{\pi k l}{N} \quad (2.8)$$

If we square and average with respect to $\theta_k(t)$ and l, we get:

$$\overline{x^2}=\frac{\sum_k C_k^2}{2(N-1)} \quad (2.9)$$

The maximum value of X_m can be estimated with the formula[2]:

$$x_m^2\approx 4\overline{x^2}\approx\frac{2\langle C^2\rangle\Delta K}{N-1} \quad (2.10)$$

If we use (2.6) and (2.10) and calculate $(\partial x/\partial z)_m = X_m\frac{\pi k}{N}$, we obtain the final estimate for the boundary of stochasticity:

$$3\beta_{kp}\left(\frac{\partial x}{\partial z}\right)_m^2\sim 3\frac{\sqrt{\Delta K}}{K} \quad (2.11)$$

In that form the estimate is also acceptable for a homogeneous string (does not depend on N).

The entire examination of resonances in a nonlinear system is valid under the condition of smallness of the excitation (1.11)

$$\varepsilon\equiv\frac{\beta F_{km}}{16NC_k\omega_k'^2}\sim\beta\left(\frac{\partial x}{\partial z}\right)_m^2\ll 1 \quad (2.12)$$

3. The Higher Modes ($k \approx N$)

Analogously to the preceding case, we obtain the following estimates for the resonance excitation:

$$F_{km} \sim 80 \langle C^3 \rangle \tag{3.I}$$

for the nonlinearity (2.2):

$$\frac{d\Omega_{km}}{dC_k} \sim \frac{9\beta\langle C \rangle}{N} \tag{3.2}$$

for the resonance frequencies:

$$\omega_{km} = 2 \cos\frac{\bar{u}}{2N}\sqrt{(k-N)^2 + 2m} \tag{3.3}$$

$$\pm m = 0, I, 2, \ldots$$

and for the average distance between resonances:

$$\Delta\omega \sim \frac{\bar{h}^2}{2N^2} \tag{3.4}$$

Relation (2.10) does not change, and therefore we obtain the estimate for the boundary of stochasticity:

$$3\beta_{kp}\left(\frac{\partial x}{\partial z}\right)^2_m \sim \frac{3\pi^2 \Delta k}{N^2}\left(\frac{k}{N}\right)^2 \tag{3.5}$$

Let only one mode with the number k be excited at the initial moment. Let us examine the very start of the development of stochasticity, when the energy exchange takes place only between certain adjacent modes. Then it can be assumed that $\Delta k \sim 1$ and we obtain the picture presented in Fig. 2. The solid straight lines depict the boundary of stochasticity (on a log-log scale) and the broken curve represents an attempt at rough interpolation between them; the circles are the results of a numerical calculation for two cases of cubic nonlinearity according to [1]. It is of interest to note that the first case lies far in the region of Kolmogorov stability in spite of the large value $\beta = 8$, since at the start of the excitation the lowest mode (k = 1) is excited. The results of the numerical calculation show in this case a clearly expressed quasi-periodicity (Fig. 4, taken from [1]). The second case lies close to the boundary of stochasticity, although the value $\beta = 1/16$ is very small, but in return the seventh mode is excited (only once!). The picture of the oscillations in that case resembles quasi-periodic motion very little and rather reminds one of under-developed stochasticity (Fig. 5, taken from [1]). The quasi-period in that case must have amounted to 8000 cycles (5.1).

4. <u>The Distribution of the Stochasticity</u>

As was noted above, estimates (2.11) and (3.5) indicate the boundary of the origination of stochasticity. Let us examine now the distribution of it on adjacent modes. We will limit ourselves to the case $k \ll N$ (continuous string). We will utilize relation (2.6) for the estimate. If the interval of the stochastic modes Δk is large enough, the amplitudes of the normal modes C_k already cannot be considered identical. By virtue of the stochasticity the energies of the modes must be identical (on the average):

$$W_k \doteq \frac{C_k^2 \omega_k^2}{2} \sim \frac{W_0}{\Delta k} \qquad (4.1)$$

where W_0 is the total energy of the oscillations. Criterion (2.6) is rewritten in the form

$$\frac{40\beta}{\pi^2} \cdot \frac{W_0}{N} \cdot \frac{k}{\sqrt{\Delta k}} \sim 1. \qquad (4.2)$$

Since $k/\sqrt{\Delta k}$ rises with increase of k, the right edge k_{max} of the interval Δk always will be in the region of stochasticity and, consequently, the stochasticity will always be distributed on still higher modes. However, the left edge (k_{min}) in the end turns out to be on the boundary of stochasticity and then goes into the region of Kolmogorov stability. How will the energy exchange between the modes close to k_{min} take place after that? Evidently the energy will, on the average, go over to the higher modes. The mechanism of this process can be as follows: as a result of energy exchange the modes $\sim$ k_{min} can transfer their energy to the higher modes, but the reverse transfer will be difficult because the boundary of stochasticity (k_{min}) is, for the diffusion of energy over the modes, a distinctive reflecting wall [11].

The estimate (4.2) can be rewritten for the left edge of the interval in the form:

$$\frac{k_{min}}{\sqrt{k_{max}-k_{min}}} \sim k_0 \qquad (4.3)$$

where k_0 is the number of the excited mode corresponding to the boundary of the onset of stochasticity ($\Delta k \sim 1$).

If the stochasticity has just set in, then $k_{min} \sim k_0$ and the interval $\Delta k \sim 1$ remains narrow until k_{min} increases substantially, that is, until the interval Δk is shifted far to the side of the higher modes. From (4.3) we have

$$\Delta k \sim \left(\frac{k_{min}}{k_0}\right)^2 \qquad (4.4)$$

When $k_{min} \gg k_0$ the interval is substantially expanded, its right edge being moved considerable more rapidly than the left (4.3):

$$k_{min} \sim k_0 \sqrt{k_{max}} \qquad (4.5)$$

5. Concluding Comments

The estimates made above (## 2 and 3) can serve, in our opinion, as a basis for explanation of the results of work [1]. A numerical verification of the position of the boundary of stochasticity is desirable. It would be especially interesting to examine the expected displacement of stochasticity to the side of the higher modes (#4). The alternative explanation of the results of [1] advanced by Ford [12] and based on the arithmetic properties of unexcited frequencies (1.3) (on their incommensurability, that is, the impossibility of the equality $\sum_k \omega_k n_k$ for whole values of n_k not simultaneously equal to zero) is, in our opinion, incorrect. The inadequacy of such an explanation has also been pointed out in [19][2]. All the more so is it impossible to agree with far-going proposal by Ford to renounce in general the requirement of ergodicity in the statistical mechanism, replacing it by a specific (and often special) selection of the initial conditions. In particular, in [12] an "equipartition" of energy was obtained in a system of linear oscillators. It is not difficult to see that "equipartition" was completely contained in the especially selected initial conditions, because in a linear case the modes are completely independent. In general, all the statistical properties of such a system can be assigned only in the form of initial conditions, and the motion itself of the system has no statistical properties.

We will not turn our attention to the fact that stochasticity, which is the subject of discussion in this work, is not exactly ordinary from the point of view of the theory of dynamic systems. The fact is that the boundary of stochasticity depends not only on the parameters of the excitation but also on the position of the system in phase space, on the surface of constant energy. According to the

..timates of this work, only part of the phase space (region II, Fig. 2) is stochastic, whereas the remaining part is a region of Kolmogorov stability (I, Fig. 2.). From the point of view of the theory of dynamic systems (see, for example, [13]), such a motion is even ergodic (the trajectory does not embrace all the energy surface). Nevertheless, from the physical point of view it seems to us to be compltetly justified to speak of the statistical properties of the system in some part of the phase space. Evidently, a generalization of the usual concepts of the theory of dynamic systems and statistical mechanics to similar systems with a separated phase space is required. One of the possible ways is limitation of the time of motion in such a way that the trajectory of the system cannot reach the boundary of stochasticity [14]. In the presence of such a limitation, a system has the usual statistical properties. Another way, more convenient from the practical point of view, is to consider the boundary of stochasticity as a reflecting wall in the phase space for the distribution function characterizing the behavior of the system [11]. The results of [11] show that this is possible with a certain degree of precision. It is not excluded, however, that precise boundary conditions will not be successfully formulated, if only because, instead of a sharp boundary of stochasticity, there exists an entire transitional region with a very complex character of the motion.

It is possible, however, that systems with a separated phase space are a unique exception. Thus, for example, by increasing the nonlinearity it is possible, evidently to attain such a position that the stochasticity is distributed over all the modes (2.11), at least for an infinite number of degrees of freedom. But the same result can be achieved if hard collisions are introduced among the masses the chain (Fig. 1), as was shown in [15] by means of a numerical integration of the equations of motion. Such collisions play the role of brief but very strong nonlinearity.

In [11], another example of a dynamic system with a separated phase space was investigated -- the motion of a light particle between oscillating heavy walls. The separation of the phase space was essentially connected in that case with uniformity of the motion. Upon transition to a larger number of measurements in analogous systems the stochasticity is distributed over the entire energy surface [16].

Finally, some words about Kolmogorov stability. The preservation of the quasi-periodic character of the motion, detected in [4,5], means a preservation of the complete set of single-valued integrals of the motion, in spite of the excitation. Those integrals, however, are not analytical. Hence it follows, in particular, that Poincare's theorem [21] of the absence in a nonlinear system, under certain (adequately broad) conditions, of single-valued analytical integrals of motion except energy does not necessarily indicate the ergodicity of such a system. It should be borne in mind, however, that the absolute, or perpetual (for any t) stability was demonstrated in [4,5] only for systems with two degrees of freedom. In the presence of a larger number of degrees of freedom the existence of single-valued integrals of motion of the excited system depends on the arithmetical properties of the unexcited frequencies. Since from the physical point of view the arithmetical properties of frequencies are indefinite (for example, the difference between rational and irrational frequencies has no meaning), the question of the absolute stability of systems with a large number of degrees of freedom remains open. There is an example of the instability of such a system [17]. In [18] a possible mechanism of such instability was pointed out and an estimate of the time of its development was given. For the problem considered in the present work, that time can be estimated very roughly as

$$\tau \sim T e^{1/\varepsilon} \sim 10^{11} T \qquad (5.I)$$

$$T \sim \frac{4\pi}{\varepsilon\omega_k} \sim 9000 \text{ циклов cycles}$$

Here T is the quasi-period of motion (Fig. 4) and ε is the parameter of smallness of excitation according to (2.12); the numerical values of τ and T are given for the case in Fig. 4. Thus, if such an instability exists, it is extremely weak and can play a role only in exceptional cases. In particular, the stochasticity which follows from that leads to a collision term in a kinetic equations of an essentially different type (and of far less value) from the usual. The last-mentioned is connected with the "rough" stochasticity, of the type considered in this work. The presence of weak instability possibly indicates the validity of Fermi's theorem [22] of ergodicity, since it relates to systems with a number of degrees of freedom larger than two[3].

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Footnotes

1. The disappearance of $d\Omega_{km}/dC_k$ at $\omega_k = 2$ $(k = N/2)$ is a result of the neglect in (1.9) of the terms with Q_{N-k}, which is not allowable at $k \approx N/2$; when those terms are taken into consideration the estimate for $d\Omega_{km}/dC_k$ does not change in order of magnitude.

2. It is assumed that all values of C_k are of the same order; see #4.

3. We will note however, that the arithmetic properties of the frequencies can play a certain role in the region of Kolmogorov stability [4,5] (see also below).

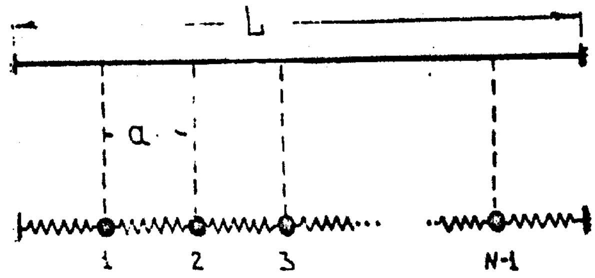

Fig. 1. String with fixed ends. L is the total length of the string; a - is the size of the section modelled by any oscillator of the chain.

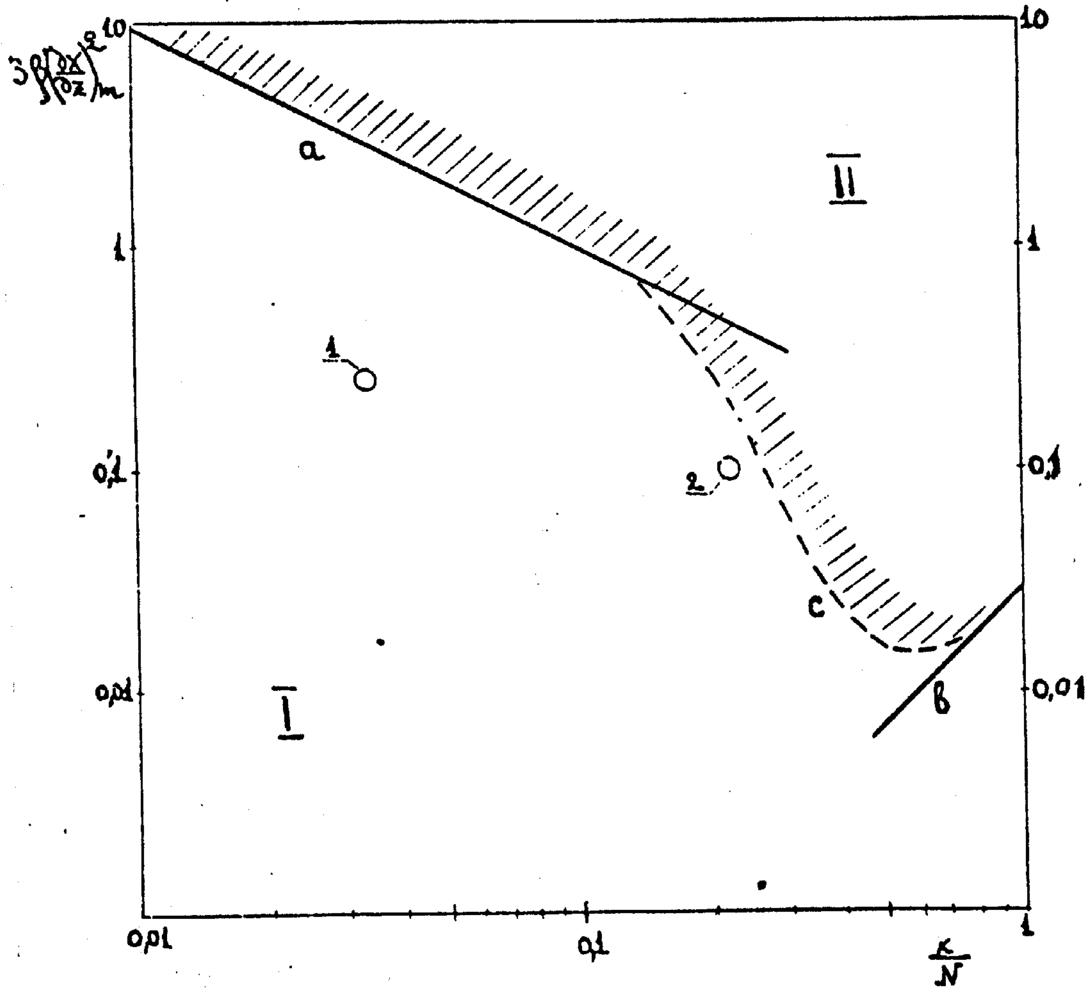

Fig. 2. I-Region of Kolmogorov stability; II - Region of stochasticity. a - boundary of stochasticity for $k \ll N$ (2.11); b - boundary for $k \approx N$ (3.5); c - qualitative interpolation; numerical values of straight lines a and b given for N = 32; 1 - result of numerical calculation for N = 32; $X_m = 1$; k = i, $\beta = 8$ [1]; 2 - the same for k = 7; $\beta = 1/6$ [1].

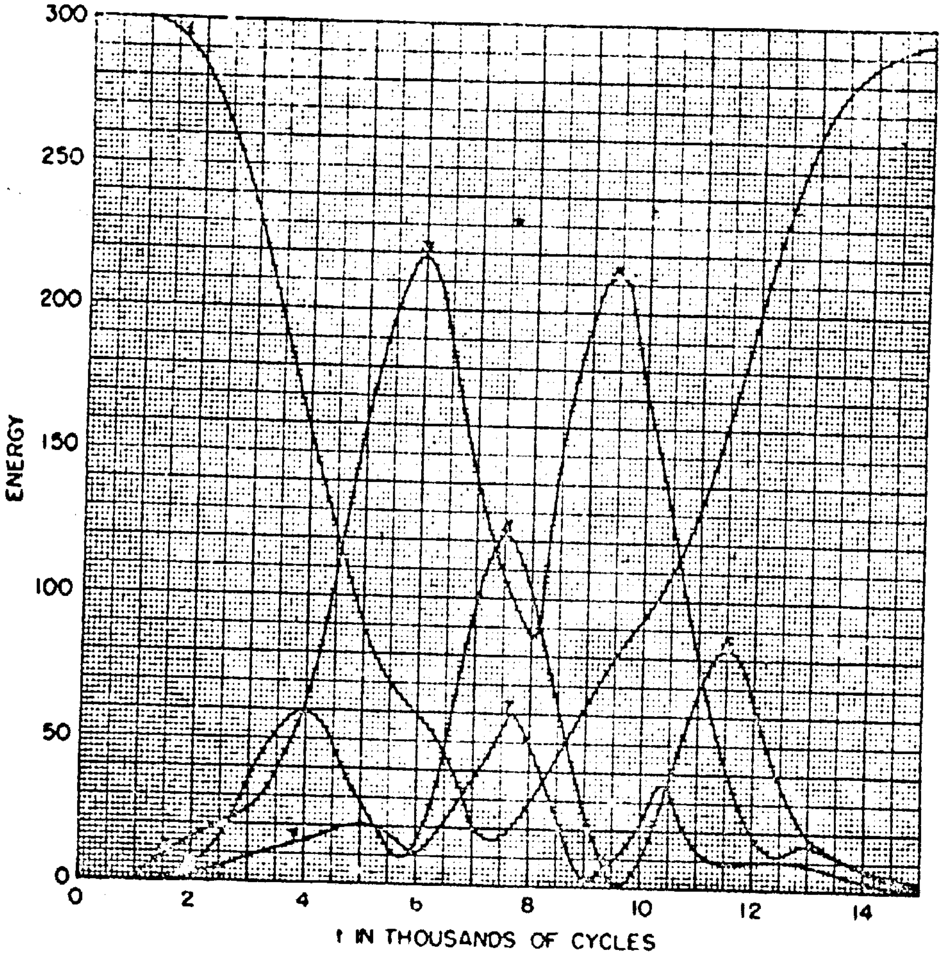

Fig. 4. The initial configuration assumed was a single sine wave; the force had a cubic term with $\beta = 8$ and $\delta t^2 = 1/8$. Since a cubic force acts symmetrically (in contrast to a quadratic force), the string will forever keep its symmetry and the effective number of particles for the computation $N = 16$. The even modes will have energy 0.

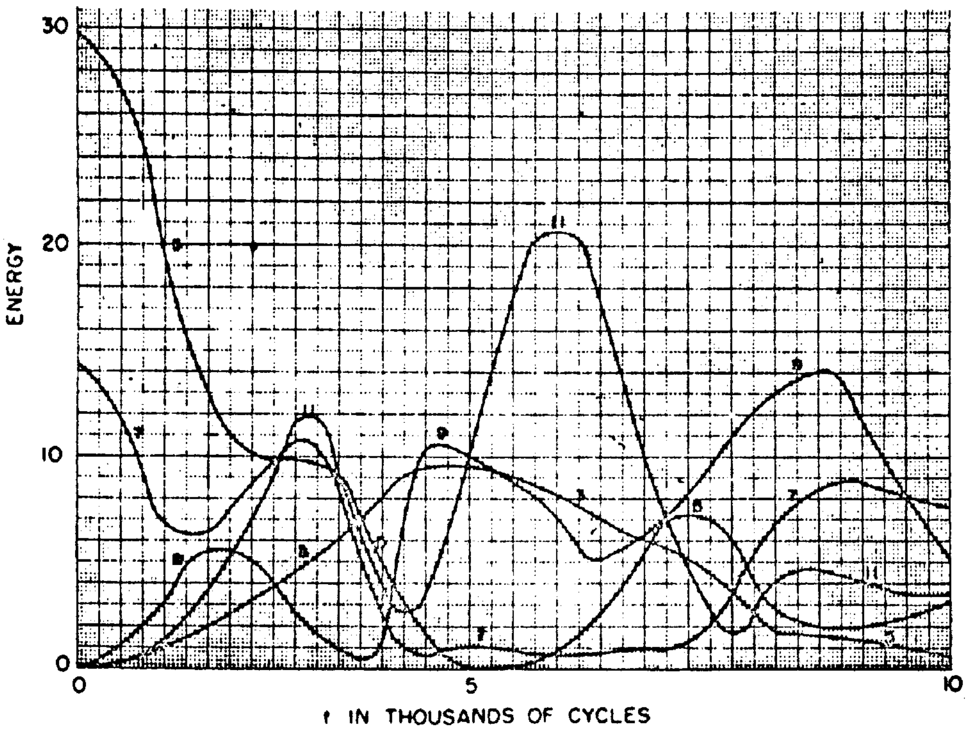

Fig. 5. $N = 32$; $\delta t^2 = 1/64$; $\beta = 1/16$. The initial configuration was a combination of 2 modes. The initial energy was chosen to be 2/3 in mode 5 and 1/3 in mode 7.