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KAM INTEGRABILITY

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In 1892, almost a hundred years ago, Poincaré has published his famous theorem /1/ on nonexistence of isolating analytical motion integrals (except energy) in a generic conservative Hamiltonian system (in external static field, for an isolated system - except the integrals of Poincare's group). This theorem, being formally correct, nevertheless made a lot of confusion, at least among physicists who payed no attention to the importance of the term "analytical" (integrals). It seemed obvious - why should any singularities appear in a simple mechanical motion? In 1923 young Fermi has even published a paper /2/ where he ostensibly proved that the Poincaré theorem implied the ergodicity of motion (on energy surface). However, as a physicist he had apparently never believed his own formal result, and by the end of life he decided to check it via numerical simulation on one of the first computers /3/. The numerical experiment did not confirm the Fermi "theorem". The surprise of the authors was so great that they did pay no attention to the clear signs of ergodicity in some runs (see, e.g. Fig. 3 in Ref./3/). Thus they had missed the phenomenon which has been termed later on the dynamical chaos (see Ref. /4/). Instead they have discovered a remarkable stability of nonlinear oscillations which gave a strong impetus to the future development of powerful mathematical methods for "constructing" the whole families of completely integrable nonlinear equations (see, e.g., Ref./5/)**). The integrals in question are analytical, indeed, so all these completely integrable systems are exceptional in accordance with the Poincaré theorem.

The mystery of this theorem has been finally resolved in the fascinating KAM theory, one of whose creator was Professor Jürgen Moser. His decisive contribution to the theory was in the studies of nonanalytical perturbations, mappings including.

*) This work proposed for publication in the collected volume dedicated to Professor Moser's sixtieth birthday, Zurich, Switzerland (1988).

**) In the development of those methods the integrability of another model - the Toda lattice - which had been discovered also in numerical experiments /6/, played an essential role.

He devoted many papers to the development of this theory as well as to its various applications (see, e.g., Ref./7/).

According to the KAM theory a sufficiently weak perturbation of a nonlinear system preserves the full set of its motion integrals for most of initial conditions. The measure of the complementary set with unstable trajectories goes down to zero with perturbation, yet this set is everywhere dense. It consists of narrow chaotic layers along destroyed separatrices of the nonlinear resonances. A detailed description of such a structure is given, for instance, in Ref./8/.

Even though the problem is obviously improper the theory guarantees, in the case of two freedoms ($N = 2$), the eternal stability of motion in the sense of small variations of the unperturbed integrals over indefinite time interval (chaotic layers including!). This is because chaotic layers are isolated in this case from each other by invariant tori while the motion instability within a layer is sharply restricted by its negligible width.

The situation drastically changes in a many-dimensional system ($N > 2$) where chaotic layers form a single connected set, the everywhere dense network, or "web", comprising the whole energy surface. A chaotic trajectory within this set comes arbitrarily close to any point of energy surface, yet it is not ergodic as it remains always on the set of a small measure! A priori, such an intricate structure of motion appears to be completely unlikely, at least for physicists. The KAM theory did help them to considerably develop imagination, and now the above generic picture seems already to be quite natural and comprehensive in terms of nonlinear resonances and their interaction. The most important implication of this picture is a slow motion over the web /9/ which proved to be chaotic and was termed the <u>Arnold diffusion</u> /8/.

However, the problem remains essentially improper. One way to regularize it is imposing an external weak noise whose effect would be amplified by Arnold's diffusion for arbitrary initial conditions, the more so the weaker is the noise /8/. Another way is in restriction of the motion time which converts the everywhere dense web into a finite-mesh grid.

In this paper we consider a different problem: what is the accuracy of approximate motion integrals in the KAM theory for arbitrary initial conditions? Following this approach we introduce a new concept of approximate integrability which we shall term the KAM integrability /10/.

Any motion integrals, if only approximate, are of the primary importance in physics. A classical example is the adiabatic invariants. It turns out that adiabaticity is closely related to KAM integrability /11/. As is well known by now (see also below) the chaotic layers are formed by a high-frequency perturbation, while the adiabatic invariance holds in case of a low-frequency one. Clearly, the both are different only in which of interacting freedoms is treated as perturbing, and which one as perturbed. Hence, the KAM integrability may be called the inverse adiabaticity.

The variation of unperturbed motion integrals is proportional to perturbation and generally is not very small for any initial conditions. However, such relatively big perturbations do not accumulate and can be calculated, theoretically, to a high accuracy. The accumulating variations, on the other hand, are caused by the diffusion only which is very slow and which does place indeed, the principal limit to the accuracy of KAM integrals. Another important characteristic of this accuracy is the width of chaotic layers to which the diffusion is confined without external noise. Both characteristics are interrelated: loosely speaking, the diffusion rate is proportional to the square of layer width (to the cube in presence of noise /12/). Precisely this dependence is going to be used below for evaluating the diffusion rate at a very weak perturbation. In this case the diffusion depends on high-order resonances in a very complicated way. Nevertheless, a fairly simple, very rough though, estimate for the diffusion rate was obtained in Ref./12/. Our main objective below is the extension of this estimate onto a considerably weaker perturbation, and its comparison with the rigorous upper estimate in Ref./13/.

We confirm the exponential dependence on perturbation in the limit of sufficiently weak perturbations for both the diffusion rate and layer width. On the other hand, we have found some preliminary indications of the existence of a rather broad (in perturbation) domain with only a power-law decay for the diffusion rate. This interesting phenomenon requires further studies.

In any event, the diffusion falloff is fairly sharp which proves once more a high precision and quality of the KAM integrals, and hence, their importance in physics.

1. Model

We make use of the same model as in Ref./12/ which is specified by the Hamiltonian

$$H(x_i,p_i) = \frac{p_1^2+p_2^2}{2} + \frac{x_1^4+x_2^4}{4} - \mu x_1 x_2 - \varepsilon x_1 f(t). \tag{1}$$

Introduce small dimensionless perturbation parameters

$$\tilde{\mu} = \frac{\mu}{a^2}; \quad \tilde{\varepsilon} = \frac{\varepsilon f}{a^3}, \quad \tilde{\gamma} = \frac{\tilde{\varepsilon}}{\tilde{\mu}} = \frac{\varepsilon f}{\mu a}, \tag{2}$$

where a is oscillation amplitude ($x_i \approx a_i \cos\theta_i$) related to the frequency $\dot{\theta}_i = \omega_i \approx \beta a$; $\beta = 0.8472...$ (see Ref./8/). The driving periodic force is chosen in the form

$$f(t) = \frac{\cos(\Omega t)}{1 - A_0 \cos(\Omega t)} \approx \sum_m \frac{2}{\sigma} e^{-\sigma m} \cos(m\Omega t). \tag{3}$$

The latter expression holds for $\sigma = (1-A_0^2)^{1/2} \ll 1$. If, moreover, $\Omega/\omega \ll 1$ the driving resonances $\omega = n\Omega$ form a dense net which increases slow diffusion.

The main coupling resonance $\omega_1 = \omega_2$ is taken as the guiding resonance along which the diffusion proceeds. At $\varepsilon = 0$ the motion near resonance has a form of the phase oscilltion, i.e. the oscillation of resonance phase $\psi = \theta_1 - \theta_2$ as well as of amplitudes a_i . Approximately, it is described by the "pendulum" Hamiltonian /8/

$$H_1(\psi,P) \approx \frac{\beta^2}{a_0^2}P^2 - \frac{\mu a_0^2}{2}\cos\psi . \tag{4}$$

Here P is momentum conjugated to ψ and $a_0 = \overline{a}_1 = \overline{a}_2 = const$. The stable equilibrium at $\psi = 0$ $(H_1 = -\mu a_0^2/2)$ corresponds to the stable periodic trajectory at resonance center. In its vicinity the small phase oscillation of frequency $\Omega_\mu = \beta\mu^{1/2}$ is harmonic. Smallness of this frequency as $\mu \to 0$ does determine the inverse adiabaticity of the driving perturbation with parameter ε.

The unstable equilibrium at $\psi = \pi$ $(H_1 = \mu a_0^2/2)$ corresponds to the unstable periodic trajectory which is crossed by a separatrix surface (separatrix) - the boundary of a nonlinear resonance in the phase space. Any, arbitrarily weak, perturbation destroys ("splits up") the separatrix. In its vicinity a chaotic layer arises (Fig. 1) along which, i.e. in the direction perpendicular to the figure plane, the diffusion goes on if $\varepsilon \neq 0$.

We make use of a canonical mapping $(x_i, p_i) \to (\overline{x}_i, \overline{p}_i)$ generated by the function

$$G(x_i, \overline{p}_i) = x_1\overline{p}_1 + x_2\overline{p}_2 + H(x_i, \overline{p}_i) \tag{5}$$

as the numerical algorithm. Its accuracy grows with the number of steps per oscillation period $N_1 = 2\pi/\beta a$, and it proves to be sufficiently high provided $N_1 \gtrsim 30$ $(a \leqslant 1/4)$ /8,12/.

2. Primary resonances

Arnold diffusion is only possible if there are at least three resonances - a guiding one, and two driving. With one driving resonance the diffusion goes at some angle to the layer, and thus is restricted by the same mechanism as the layer width.

If the perturbation is moderate, i.e. is not too weak, it suffices to take account of primary, or first-order resonances only. In model (1) these are the guiding resonance $\omega_1 = \omega_2$, and a couple of most close driving resonances $\omega_1 = m\Omega$.

On the other hand, the perturbation should not be too strong to avoid global chaos due to resonance overlap. The driving resonances alone do overlap under condition /8/: $\varepsilon \gtrsim a\Omega^2/2\pi\beta^2 f_m$, or

$$\lambda \equiv \frac{|\delta\omega|}{\Omega_\mu} \lesssim \pi\left(\frac{\tilde{\nu}_m}{2}\right)^{1/2} = \lambda_1 .$$

Here λ is the <u>adiabaticity parameter</u> which plays the principal role in the problem of KAM integrability; $|\delta\omega| \approx \Omega/2$ is the maximal detuning in respect to the neighbouring driving resonances; $f_m \approx 2\exp(-\sigma m)/\sigma$ are amplitudes of driving force (3), and $\tilde{\nu}_m = \varepsilon f_m/\mu\tau$ (see Eq.(2)). At $\lambda \ll \lambda_1$ the diffusion rate (in energy) grows up to (see, e.g., Ref./15/):

$$D_1 \equiv \frac{\overline{(\Delta H)^2}}{t} \approx \frac{\pi}{4}\,\frac{(\varepsilon f_m a\omega)^2}{\Omega} . \qquad (6)$$

Taking account of the coupling resonance, the overlap border increases to

$$\lambda_2 \approx \pi\left[\left(\frac{\tilde{\nu}_m}{2}\right)^{1/2} + \frac{1}{2}\right] = \lambda_1 + \frac{\pi}{2} . \qquad (7)$$

Only if $\lambda \gtrsim \lambda_2$ the diffusion is confined within a narrow chaotic layer of the coupling resonance. Introduce a dimensionless layers width $w = (2H_1/\mu a_0^2) - 1$. On the unperturbed separatrix $w = 0$ while in the resonance center $w = -2$. According to Ref./18/ the half-width of chaotic layer is

$$w_s \approx 4\pi\tilde{\nu}_m \lambda^2 e^{-\pi\lambda/2} \qquad (8)$$

provided $\tilde{\nu}_m \lesssim 1$. In the middle between two driving resonances ($|\delta\omega| \approx \Omega/2$) all three domains of chaotic layer (1,2,3, Fig. 1) have equal width. Otherwise, one of the external domains (1 or 2, ψ rotation) is much more narrow. On the contrary, the width of internal domain (3, ψ oscillation) would be twice as much. The explanation is as follows /8/. Changes in w by the driving perturbation occur around $\psi \approx 0$ that is they follow with a period

$$T(w) \approx \frac{1}{\Omega_\mu} \ln \frac{32}{|w|} , \tag{9}$$

which is the period of ψ rotation and the half-period of ψ oscillation. In asymmetric case ($|\delta\omega| \ll \Omega/2$) the perturbation is operative only on one half-period of oscillation, and only for one direction of rotation. Hence, the perturbation period for oscillation doubles, and this increases the layer width /8/.

Chaotic layer is exponentially narrow (8) that mainly depends on the adiabaticity parameter $\lambda \gg 1$. In turn, λ is related to that part of perturbation only which determines the guiding resonance (parameter μ).

A simple equation

$$w_s \approx \lambda W \tag{10}$$

relates the layer width to separatrix splitting $W = (\Delta w)_{max}$ which, in turn is equal to the maximal change in w over period T (9).

The splitting of separatrix, and the formation of an intricate homoclinic structure was known already to Poincaré who even obtained the first exponential estimate for W /1/ (Sections 226 and 397). Subsequently, this problem was studied in many papers (see, e.g., Refs./7,16,17/). The concept of chaotic layer near separatrix was first developed in Ref./18/ and than thoroughly investigated in Refs./8,19,20/. The results of numerical simulations in the latter papers (especially in Ref./19/) are well in agreement with a simple estimate like Eq.(8) provided λ is not too big.

The diffusion rate along the coupling resonance (in energy) is also exponentially small /8/ (cf. Eq.(6)):

$$D_H \approx \frac{\pi^2}{3} \frac{(\varepsilon f_m a\omega)^2}{\Omega_\mu^2 T_a} e^{-\pi\lambda} . \tag{11}$$

Here T_a is the average period of motion in a chaotic layer (see Eq.(9)):

$$\Lambda = \Omega_\mu T_a = \ln \frac{32e}{w_s} = \Omega_\mu T(w_s) + 1 . \tag{12}$$

Comparing Eqs.(11) and (8), we arrive at our basic relation:

$$\begin{aligned} \widetilde{D} &\equiv \frac{D_H \Omega_\mu}{(\varepsilon f_m a\omega)^2} \approx C \frac{\widetilde{w}_s^{\,2}}{\Lambda \lambda^4} ; \\ \widetilde{w}_s &= \frac{w_s}{\widetilde{\nu}_m} , \qquad C = \frac{1}{48} ; \end{aligned} \tag{13}$$

between dimensionless diffusion rate $\widetilde{D}$ and reduced layer width $\widetilde{w}_s$. Notice that Eq.(13) actually holds around $\delta\omega \approx \Omega/2$ only. Otherwise, w_s depends on the more close driving resonance (on smaller $|\delta\omega|$) while D_H does so on larger $|\delta\omega|$.

3. High-order resonances

Arnold diffusion was observed first in numerical experiments /21/, and then was studied in detail in Refs./8,22/. As was noticed already in Ref./8/ the diffusion rate considerably exceeded simple estimate (11) when $\lambda \gtrsim 5$. Qualitatively, this was explained by the effect of high-order resonances, the higher the bigger was λ. Even though their amplitudes are very small they form a much more dense net as compared to primary driving resonances. This leads to a decrease in detuning: $\delta\omega \to \widetilde{\delta\omega} < \delta\omega$, and, hence, to a poor adiabaticity: $\lambda \to \tilde{\lambda} < \lambda$. Clearly, the motion structure in this region is extremely complicated, so that any analytical theory can, at best, provide a very rough order-of-magnitude estimate only. One was obtained in Ref./8/ (see also Ref./23/), namely:

$$\tilde{D} \sim D_0 \exp\left(-A\lambda^{1/M}\right). \qquad (14)$$

Here M is the number of linearly independent (incommensurable) unperturbed frequencies which form the high-order resonances; D_0 and A assumed to be constant. Unlike Eq.(11) we take now for $\lambda = \Omega/2\Omega_\mu$ its maximal value on primary resonances assuming $\widetilde{\delta\omega}$ to weakly depend on the original $\delta\omega$. According to Eq.(14) an effective adiabaticity parameter (cf. Eq.(11))

$$\tilde{\lambda} \approx \frac{\widetilde{\delta\omega}}{\Omega_\mu} \approx \frac{A}{\pi}\lambda^{1/M} \lesssim \lambda, \qquad (15)$$

the latter inequality being the condition for applicability of estimates (14,15).

The theory, developed in Ref./8/, shows that the basic relation (13) does hold for arbitrary resonance set. However, constant C generally depends on system's parameters, and may considerably change. Eqs.(13-15) imply then

$$\tilde{w}_s \sim \left(\frac{\Lambda D_0}{C}\right)^{1/2} \tilde{\lambda}^2 e^{-\pi\lambda/2}. \qquad (16)$$

Earlier, in paper /13/, a rigorous upper estimate has been obtained which can be transformed to type (14) with the parameter /8/

$$M = M_N = \frac{(3N-1)N}{4} + 2 > N \,, \tag{17}$$

where N is the number of freedoms for a conservative Hamiltonian system. Even for $N = 2$ parameter $M_N = 4.5$ considerably exceeds the value $M = 2$ found in Ref./12/. This is in no contradiction with the upper bound (17), of course. Yet, a more effective estimate is desirable.

In Ref./12/ the diffusion rate D_H was directly measured on a supercomputer CRAY-1. Even though, it was possible to reach $\lambda \approx 10$ only due to a rapid decrease in the diffusion rate. In the present work a different technique is used, namely, we measure the width of a chaotic layer w_s , and then calculate D_H from Eq.(13). As a result we have reached $\lambda \approx 50$ on a personal computer! The back side of the coin will be discussed in Section 6.

4. Numerical experiments

We employ the numerical algorithm, described in Section 1, and the following values of parameters (in the main series of experiments):

$$\varepsilon/\mu = 0.01\,; \quad \Omega = 0.0346\,, \quad A_0 = 0.995 \quad \left(f_m \approx 11.5 \right);$$

$$a = 0.225\,; \quad \omega/\Omega \approx 5.5\,; \quad |\delta\omega| \approx \Omega/2\,; \quad \tilde{\nu}_m \approx 0.5$$

Variables μ and λ run over a broad range:

$$200 \leqslant \frac{1}{\sqrt{\mu}} \leqslant 2500\,; \qquad 4 \leqslant \lambda \leqslant 50 \tag{18}$$

For $\lambda > 20$ the double precision (20 decimal places) is used. The initial conditions are chosen within the chaotic layer.

In the method employed everything is calculated from the only measured quantity, the period $T(w)$ (9). To suppress big fluctuations in chaotic motion and, thus, to minimize errors in T, a special averaging of resonance phase ψ is applied /14/.

Given $T(w)$ the quantity w is calculated from Eq.(9). The difference in successive w values is satisfactorily described by a simple relation /8/:

$$\Delta w \approx W \sin\varphi; \qquad W^2 \approx 2\,\overline{(\Delta w)^2}, \tag{19}$$

where φ is some high-order-resonance phase, random in a chaotic layer. The actual width of chaotic layer w_m is calculated from the minimal period $T(w_m)$ which satisfactorily agrees with average period T_a (12): $\langle \Omega_\mu (T_a - T(w_m))\rangle = 1.14$. To find the full width w_s a correction is introduced according to Eq.(4.49) in Ref./8/, namely:

$$\frac{w_s}{w_m} \approx 1 + \left(\frac{100}{n}\right)^{0.4} \approx 2 \ . \tag{20}$$

Here $n = t/T_a$ is the mean number of periods over the total motion time t; typically, $n \approx 100$ is chosen. All obtained quantities are averaged over 10 trajectories to suppress big fluctuations which are characteristic for the chaotic motion with chaos border (at layer edges) /24/.

The diffusion rate is calculated from Eq.(13) where $\tilde{\lambda} = w_s/W$ is substituted for λ. Parameter C, assumed to be constant, is found from the same Eq.(13) using the data of Ref./12/ in the range $\lambda \approx 3.7 \div 8$. Averaging over 11 points provides

$$\langle \log C \rangle = 0.55 \pm 0.27; \qquad C \approx 3.6 \tag{21}$$

where the dispersion of decimal logarithm values is given. No systematic variation of C with λ is observed. Notice that C value exceeds that in Eq.(13) by a quite big factor of about 200 which is not completely clear (see below).

The results of the main series of measuremenets with $a = 0.225$ ($\delta\omega = 0.0156$) are shown in Fig. 2. The straight line represents dependence (14) with parameters $D_0 \approx 2.0$, and $A \approx 5.60$ obtained by the least square fit of numerical data. According to Ref./12/, $D_0 \approx 26$ and $A = 7.9$ which gives the idea as to the accuracy of estimate (14). However, the scattering of points in Fig. 2 appears to be surprisingly small. This may be related to the fact that only one parameter μ is varying.

In Fig. 3 the dependence of diffusion rate on another parameter - reduced detuning $\delta = 1-(2|\delta\omega|/\Omega)$ - is shown. The value $\delta = 0$ corresponds to a halfway between the two driving resonances, while $\delta = 1$ falls just on one of them. Horizontal line represents Eq.(14) with the parameters fitted in Fig. 2.

In Fig. 3 the data for two values of $1/\sqrt{\mu} = 600;\ 300$ are shown. In the first case, besides the main force (3), two other types of driving perturbation are represented: i) two-frequency force with $\Omega_1 = 5\Omega$ and $\Omega_2 = 6\Omega$; $\Omega_2/\Omega_1 = 6/5$, and with the same amplitudes as in Eq.(3); ii) the force with two independent frequencies $\Omega_2/\Omega_1 = 1.2381966...$. All three versions are in a good agreement. They reveal a significant dependence of diffusion rate on detuning $\delta\omega$. This is most striking in a narrow interval $\delta < 10^{-3}$ which covers a low-order resonance $2\omega = \Omega_1 + \Omega_2$. A similar drop in $\widetilde{D}$ does occur also at $\delta \approx 1$. The rest of dependence shows the accuracy of estimate (14) with a constant A .

Low diffusion rate at $\delta \approx 0$ puts al[illegible] the upper bound for a possible background which turns ou[illegible] be about 4 orders of magnitude below the diffusion, independent of λ . We mention that for $\varepsilon = 0$ the fictitions diffusion rate calculated from Eq.(13) drops by 11 orders of magnitude. Notice that a finite width of chaotic layer in this case is real due to the interaction with other coupling resonances $m_1\omega_1 = m_2\omega_2$, $m_1 \neq m_2$. On the other hand, the "real" additional diffusion caused by the discreteness of numerical algorithm (5) is completely negligible. According to estimate (14) its $\widetilde{D} \sim \sim 10^{-70}$ (!) owing to a very big value of adiabaticity parameter $\lambda = \pi / \beta\sqrt{\mu} \approx 800$ at $1/\sqrt{\mu} = 500$.

In Fig. 4 the dependence of effective adiabaticity parameter $\widetilde{\lambda} = w_s/W = w_s (2\overline{(\Delta w)^2})^{-1/2}$ on $\lambda = \Omega / \varepsilon \Omega_\mu$ is depicted. The asymptotic relation (15) - the horizontal line - is reached, within fluctuations, at $\lambda \gtrsim 15$. Notice that in the whole range $\lambda \lesssim 10$, studied in Ref./12/, the relation (15) is satisfied poorly. This may be a reason for quite big C value as obtained from data in Ref./12/.

Our results confirm the value $M = 2$ for the main parameter in estimate (14). It was obtained in Ref./12/ and explained there by the presence of two independent frequencies, Ω and ω (in coupling resonance $\omega_1 = \omega_2 = \omega$). This explanation is further confirmed by a sharp decrease in diffusion rate at $\delta = 0$ (Fig. 3) when the two frequencies become commensurable ($\omega/\Omega = 11/2$) and $M = 1$.

Thus, the theoretical value of M seems to be confirmed. If so, one would expect a considerable increase in $\widetilde{D}$ for two independent frequencies Ω_1, Ω_2 of the driving perturbation as $M = 3$ in this case. However, this is not observed (Fig. 3, □). In the next Section we attempt to resolve this contradiction.

5. A weak adiabaticity?

We assume that parameter A in estimate (14), surmised to be constant, is actually growing with M . Suppose that it grows linearly, i.e. $A = BM$ where B is constant now. Then, at $\lambda = 12$ (Fig. 3, □) the diffusion rate, calculated for $M = 2$ and $M = 3$, is nearly the same. Moreover, the data in Fig. 2 imply the value $B \approx 2.80$ which is close to $B = \pi$ for $M = 1$ (see Eq.(11)).

In case of modified estimate $\widetilde{D} \sim D_0 \exp(-BM\lambda^{1/M})$ the curves $\widetilde{D}(\lambda)$ for different M do intersect. Particularly, this implies that the diffusion rate for some $\widetilde{M} < M$ may happen to be bigger than for the actual M . Such an enhanced diffusion can be caused by the driving resonances formed by a fewer number $(\widetilde{M})$ of unperturbed frequencies. For example, at sufficiently small λ the diffusion is always drived by primary resonances (11), and hence $\widetilde{M} = 1$ for any M. Therefore, one should find such $\widetilde{M}(\lambda) < M$, for each λ, which provides the highest diffusion rate. In this way we arrive at the dependence $\widetilde{D}(\lambda)/D_0$ in the form of successive functions $exp(-B\widetilde{M}\lambda^{1/\widetilde{M}})$ with different $\widetilde{M} \leqslant M$. As our estimates are fairly rough we may smooth over that broken curve. To this end we consider dependence $\widetilde{M}(\lambda)$ to be continuous, and derive it from the local condition $\partial\widetilde{D}(\lambda,\widetilde{M})/\partial\widetilde{M} = 0$, whence $\widetilde{M} = \ln\lambda$. Substituting the latter relation into $\widetilde{D}(\lambda,\widetilde{M})$ we arrive at a fairly simple estimate

$$\widetilde{D} \sim D_0\lambda^{-Be} \tag{22}$$

where $e = 2.71\ldots$. That slow decay of the diffusion rate - the weak adiabaticity - persists, however, while $\widetilde{M} < M$ only, i.e. for $\lambda < \lambda_m = e^M$. Subsequently, the exponential dependence is recovered.

Thus, if our hypothesis is true, the final estimate for the diffusion rate becomes

$$\widetilde{D} \sim \begin{cases} D_0\lambda^{-Be}; & \lambda \lesssim e^M \\ D_0 exp(-BM\lambda^{1/M}); & \lambda \gtrsim e^M \end{cases} \tag{23}$$

Notice that both curves $\widetilde{D}(\lambda)$ are tangent to each other at $\lambda = e^M$.

In Fig. 5 the data of Fig. 2 (+) are represented in $\log$ - $\log$ scale. The straight line is power law (23) with fitted parameters $D_0 \approx 1.6$; $B \approx 2.84$. The latter value is very close to $B \approx 2.80$ previously obtained. The curves in Fig. 5 show exponential dependence (23) for $M = 1; 2; 3$. Squares correspond to minimal detunings $\delta < 10^{-3}$ (see Fig. 3) with expected $M = 1$ (Section 4).

We also measured the diffusion rate for two independent frequencies Ω_1, Ω_2 (expected $M = 3$) and $\lambda = |\Omega_1 - \Omega_2|/2\Omega_\mu = 35.7$ $(1/\sqrt{\mu} = 1500)$. At this $\lambda > e^3 \approx 20$ estimates (23) with $M = 2$ and $M = 3$ differ by 3 orders of magnitude. Indeed, the measured values of $-\log\tilde{D}$ lie all in the interval 13.8 ÷ 9.6 (at average, $-\langle\log\tilde{D}\rangle = 11.9$) while Eq.(23) gives 14.3 ($M = 2$) or 11.8 ($M = 3$). A considerable dispersion in $\tilde{D}$ is apparently related to the dependence on detuning $\delta = 0.00084 \div 0.085$ (cf. Fig. 3). At $\delta = 0$ the diffusion rate drops down to $-\log\tilde{D} = 16.8$ which is comparable to the estimate for $M = 2$.

We understand, of course, that the consideration and data given above are but preliminary indications toward the existence of a domain of weak adiabaticity (22). Nor the hypothetical relation $A = BM$ follows directly from a simple theory in Ref./8/. Instead, it requires a more accurate evaluation of the density of high-order resonances. To summarize, this interesting question remains as yet open.

6. Concluding remarks

The main result of our studies is the confirmation of exponential estimate (14) in a broad range of adiabaticity parameter λ (Fig. 2). In our opinion, the most important problem to be solved would be the relation between simple empirical estimate (14) and the rigorous estimate from above in Ref./13/ (see Eq.(17)). Why do they differ? Could it be related to the fact that fairly large values of λ studied in this paper are still not big enough? Generally, we may also ask if the applicability domain of estimate (14) is restricted in λ from above?

On the other hand, we certainly know that this domain is bounded from below, and not only by the resonance overlap (Section 2). Already in paper /12/ the deviation from dependence (14) at $\lambda \lesssim 4$ was observed. This could be roughly explained by a change in M value from 2 ($\lambda \gtrsim 4$) to 1 ($\lambda \lesssim 4$). In this paper we put forward the hypothesis which extents such a behaviour on arbitrary M (Section 5). Surprisingly, this leads to the conclusion on existence of the domain of "weak adiabaticity" where the diffusion rate falls off as a some power of the adiabaticity parameter only (23). The domain width in λ rapidly grows with M, and it certainly deserves further thorough studies.

Penetration into the region of large λ in this work proved to be possible due to a new method for evaluation of the diffusion rate via the width of the chaotic layer. How powerful it might seem, the method has its own limitations. Particularly, it fails (in the present form at least) as soon as a modulational chaotic layer appears whose width is irrelevant to the diffusion rate /14,25/. More precisely, the applicability of the new method is restricted by the condition $\lambda > \lambda_{mod}$ where λ_{mod} is some critical value at which a modulational layer is formed. Another disadvantage of the method is in its poor accuracy due to unknown factor C which may considerably change (cf. Eqs.(21) and (13)).

Even in the domain of weak adiabaticity the long-term variation of motion integrals rapidly drops with perturbation which confirms a high accuracy of the KAM integrals, and emphasizes again their importance in physics. At the same time, the estimates obtained may happen to be helpful in those special applications where the diffusion, no matter how slow, turns out to be significant as, for example, in the dynamics of an asteroid or a heavy particle in the storage ring (see, e.g., Ref. /26/).

We are happy to acknowledge Professor Moser's great contribution to the international collaboration which is so vital in this, as well as in other , fields of research.

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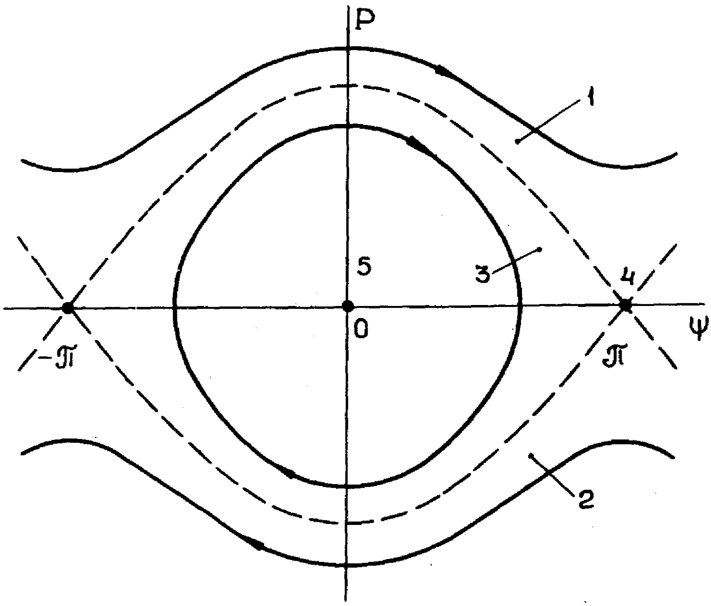

Fig. 1. Outline of a chaotic layer (see Eq.(4)): 1, 2 are the domains of resonance phase ψ rotation in opposite senses; 3 same of ψ oscillation; 4, 5 are unstable and stable periodic trajectories, respectively; arrows at layer edges indicate the direction of motion; unperturbed separatrix is shown by dashed curve.

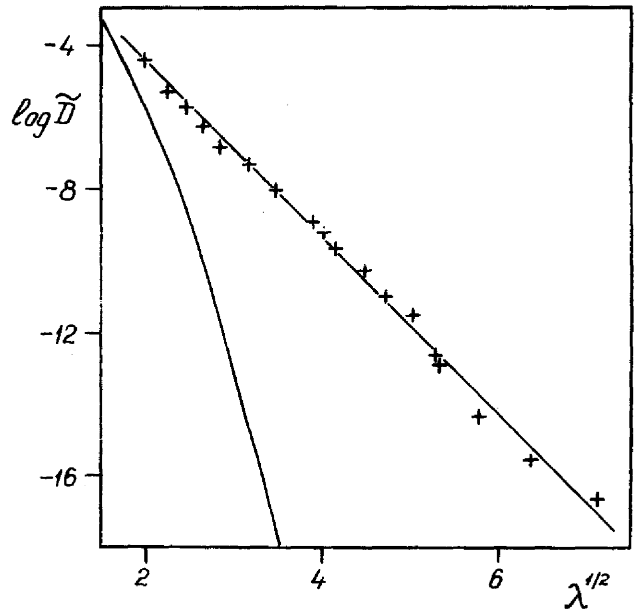

Fig. 2. Main results of numerical experiments (+) for model (1): $\lambda = \Omega / 2\Omega_\mu$; straight line is estimate (14) with $M = 2$; curve shows the effect of primary resonances (11); logarithm is decimal.

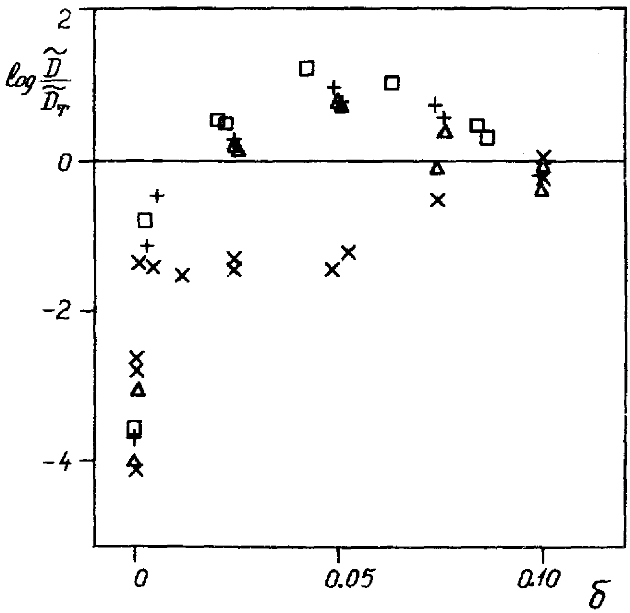

Fig. 3. Diffusion rate $\widetilde{D}$ (13) vs. reduced detuning $\delta = 1 - 2|\delta\omega|/\Omega$, $\widetilde{D}_T$ is estimate (14) with $\delta\omega = \Omega/2$. $1/\sqrt{\mu} = 300$: ($\times$) force (3). $1/\sqrt{\mu} = 600$: (+) force (3); ($\triangle$) $\Omega_2/\Omega_1 = 6/5$; ($\square$) independent Ω_1, Ω_2 .

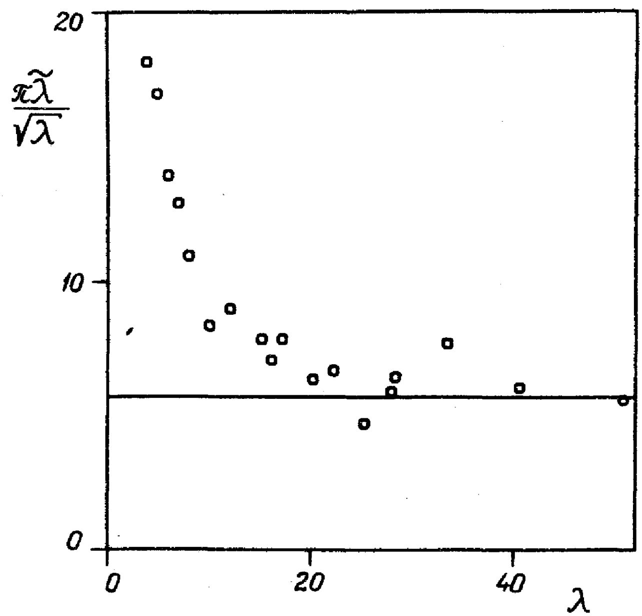

Fig. 4. Effective adiabaticity parameter $\tilde{\lambda}$ vs. λ for the main data (Fig. 2). Horizontal line is Eq.(15) with $M = 2$ and fitted $A = 5.60$ from Fig. 2.

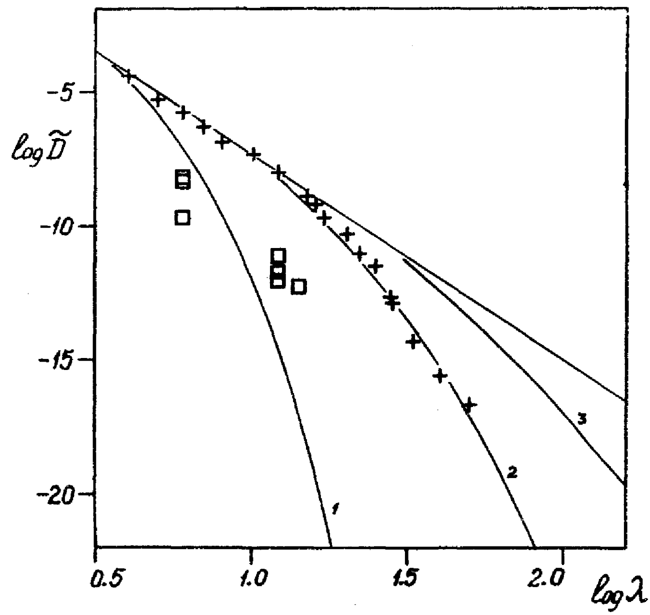

Fig. 5. Conjectured weak adiabaticity (23): straight line is power law; curves correspond to exponentials with $N = 1, 2, 3$ as indicated; (+) data from Fig. 2 ($N = 2$); (□) $\delta = 0$ (Fig. 3, $N = 1$).

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