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B.V. Chirikov, V.V. Vecheslavov

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How Fast is the Arnold Diffusion?*)

B.V. Chirikov, V.V. Vecheslavov

Institute of Nuclear Physics
630090, Novosibirsk, USSR

ABSTRACT

A review of the current understanding of the Arnold diffusion — a universal instability of motion in many-dimensional nonlinear oscillator systems — is given with the special emphasis on the estimation of the diffusion rate. Two new phenomena of the fast Arnold diffusion in systems with strong and with weak nonlinearity are discussed.

*) A talk given at the International Workshop on Beam Dynamics, Novosibirsk, 1989.

1. INTRODUCTION

The main purpose of this talk is to discuss again the fascinating mechanism of the so-called Arnold diffusion, a universal instability of motion in many-dimensional Hamiltonian oscillator systems [1—5]. This fine phenomenon may play an important role in such diverse processes as the motion of asteroids in the solar system and the dynamics of a heavy particle in the storage ring [6]. The first example of such a universal instability was constructed and discussed by Arnold [1]. The diffusion nature of this instability was revealed and numerically confirmed in Refs [2, 4, 5] while Nekhoroshev imposed the rigorous upper bound on its rate [3].

To begin with, consider a many-dimensional Hamiltonian

$$H(I, \theta, t) = H_0(I) + \varepsilon \sum_{m,n} V_{mn}(I)\, e^{i(m\theta + n\Omega t)}, \tag{1.1}$$

where H_0 describes an unperturbed completely integrable system, and where small ($\varepsilon \to 0$) perturbation is represented by the Fourier series. The action-angle variables I, θ are N-dimensional vectors, and the explicit quasi-periodic dependence on time is characterized by M-dimensional frequency vector Ω; m, n are integer vectors of dimensions N and M, respectively (e. g. $m = (m_1, \ldots, m_N)$).

The long-term dynamics, we are interested in primarily, is controlled by the resonances, both coupling ($n=0$) and driving ($n \neq 0$). A first-order, or *primary, resonance* is defined by the relation

$$m\omega(I) + n\Omega = 0, \tag{1.2}$$

which determines *resonance surface* in the action space. Here $m\omega \equiv m_i \omega_i$; $n\Omega \equiv n_k \Omega_k$ are scalar products and unperturbed frequencies $\omega_i(I) = \partial H_0/\partial I_i$. The unperturbed oscillation is called nonlinear if frequencies $\omega_i(I)$ depend on the actions. Moreover, if the oscillation is nondegenerate, that is if the determinant

$$\left|\frac{\partial \omega_i}{\partial I_k}\right| = \left|\frac{\partial^2 H_0}{\partial I_i \partial I_k}\right| \neq 0 \qquad (1.3)$$

there is one-to-one correspondence between the action space and the frequency space. The latter is more convenient for a graphical picture of the resonance structure where each resonance surface (1.2) is simply a plane.

Under a weak perturbation ($\varepsilon \neq 0$) the nonlinear resonance acquires a finite width [4, 5]:

$$(\Delta\omega)_r = \frac{4}{|m|}\left|\frac{\varepsilon V_{mn}}{2\mu_m}\right|^{1/2} = \frac{4\tilde{\Omega}_r}{|m|} \sim \omega(\varepsilon\alpha v)^{1/2}. \qquad (1.4)$$

Here $1/\mu_m = m_i(\partial\omega_i/\partial I_k)m_k \sim |m|^2\alpha|\omega|/|I|$; $|m| = \sum_i |m_i|$; α is dimensionless nonlinearity parameter; and $\tilde{\Omega}_r$ is frequency of the small *phase oscillations* about the stable periodic trajectory at the resonance center. In the last rough estimate $v \sim V_{mn}/H_0$ for the largest perturbation harmonics.

For sufficiently small perturbation the *KAM theory* guarantees preservation of quasi-periodic motion on slightly deformed invariant tori for most (but not all!) initial conditions. The complementary set is just the region where the Arnold diffusion occurs in a many-dimensional system. Before turning to this main topic of the present talk we mention that for a relatively strong perturbation the *resonances overlap* produces a *global large-scale chaos* with only occasional small islets of regular motions embedded. The critical perturbation was roughly estimated in Refs [4, 8, 12] as

$$(\varepsilon\alpha)_{cr} \sim \left(\frac{\sigma}{5}\right)^{\frac{3}{2}Q+F} \sim v^3, \qquad (1.5)$$

where perturbation harmonics are assumed to decay exponentially $V_{mn} \sim vH_0 \exp[-\sigma(|m|+|n|)]$; $\sigma \ll 1$; $\langle V^2 \rangle = H_0$; $Q = N+M$ and $F=0$ from the simple resonance overlap criterion while the rigorous upper estimate inferred from Moser's results [8] is $F=4$.

Thus, for this particular problem both approaches are reasonably compatible so that estimate (1.5), being rather crude, seems to be not far from the truth, especially if $(N+M)$ is large enough. Unfortunately, this is not the case for Arnold diffusion which is the main topic of our discussion below. Typically, this diffusion persists for arbitrarily weak perturbation $\varepsilon \ll \varepsilon_{cr}$, and, in this sense, is *universal phenomenon* of many-dimensional nonlinear oscillations. The word «typically» means that there are exceptional systems, e. g. Toda lattice [9], which are completely integrable and whose motion is quasi-periodic for all initial conditions. Unlike those exceptional cases a typical Hamiltonian system is, for $\varepsilon \to 0$, only *KAM integrable* [10] that is up to the Arnold diffusion. Both the diffusion rate as well as the measure of chaotic component are very small in ε, and, hence, the KAM integrability is of a fairly good quality. It is as important as the approximate adiabatic invariance to which KAM integrability is closely related, namely, it may be called the *inverse adiabaticity* [11] (see also Section 2 below).

Arnold diffusion proceeds along the resonance surfaces (1.2), the whole set of which is everywhere dense in the phase space. If resonance surfaces intersect than any chaotic trajectory covers the whole invariant surface determined by the exact motion integrals, e. g. an energy surface of a conservative system ($\Omega=0$ in Eq. (1.1)). Moreover, chaotic trajectory comes arbitrarily close to any point on this surface. Yet, the motion is not ergodic because the measure of chaotic component is small! From simple geometrical considerations it is clear that resonances do intersect only if the number of freedoms $N>2$, i. e. only for $N\geqslant 3$ in a conservative system ($\Omega=0$) or for $N\geqslant 2$ and $\Omega\neq 0$. In this sense the Arnold diffusion, unlike the global chaos, is a many-dimensional phenomenon. An example of finite set of resonances is outlined in Fig. 1.

The main problem to be discussed below is the diffusion rate. Even though this rate is very low the diffusion may happen to be decisive in some long-term processes like the beam-beam interaction in storage rings [6]. We give a general review of our present understanding of the Arnold diffusion including various (and very different so far!) estimates as well as some results of our recent numerical experiments on a simple model. One new feature which has emerged from these studies is the existence of a wide perturbation range where the diffusion rate decreases as a power of perturbation ε only, i. e. relatively slow. Of course, asymptotically as $\varepsilon \to 0$ the rate drops exponentially in agreement with all previous results.

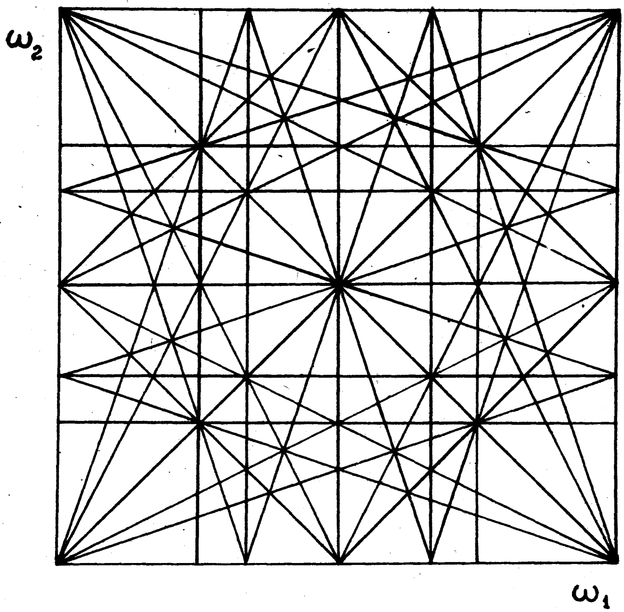

Fig. 1. A scheme of nonlinear resonances in frequency space for two freedoms ($N=2$) driven by an external perturbation ($\Omega \neq 0$) or for a conservative ($\Omega = 0$) system of $N > 2$ (projection onto energy surface).

2. STRONG NONLINEARITY, ESTIMATES

We shall call nonlinearity strong if the dimensionless parameter $\alpha \neq 0$ and does not depend on ε (see Eq. (1.4)). Then, for sufficiently weak perturbation the resonance width $(\Delta I)_r/I \sim \sim (\Delta\omega)_r/\alpha\omega \sim (\varepsilon v/\alpha)^{1/2}$ is relatively small. This substantially simplifies theoretical analysis of the Arnold diffusion.

The structure of nonlinear resonance in the phase space is shown in Fig. 2 where resonance phase $\psi = m^{(g)}\theta + n^{(g)}\tau$, and p is the conjugate momentum (a linear combination of I_k, for details see Ref. [4]). Superscript «g» indicates a particular, «*guiding*», *resonance* $m^{(g)}\omega + n^{(g)}\Omega = 0$ along which the diffusion goes on, and $\tau = \Omega t + \tau^{(0)}$ where $\tau^{(0)}$ is constant phase vector. The resonance domain of width $(\Delta p)_r \sim |I|\ (\varepsilon v/\alpha)^{1/2}$ is bounded by the unperturbed separatrix surface whose projection is shown in Fig. 2 by dashed line.

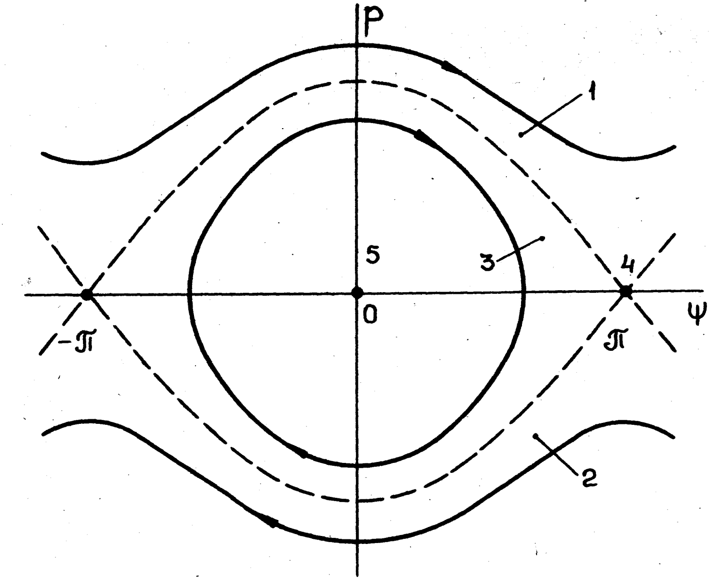

Fig. 2. The structure of nonlinear resonance with the chaotic layer between solid lines:

1, 2 are the domains of resonance phase ψ rotation in opposite senses; *3*—same of ψ oscillation; *4, 5* are the projections of unstable and stable periodic trajectories, respectively; arrows at layer edges indicate the direction of motion; unperturbed separatrix is shown by dashed line.

The perturbation destroys (splits) separatrix and forms a very narrow *chaotic layer* around. It is precisely this layer where the Arnold diffusion occurs. The rest of the resonance domain is filled with regular trajectories, also covering the invariant tori but of a different topology, as compared with the unperturbed ones. What is still more important, the motion near resonance acquires a new, very slow, frequency $\tilde{\Omega}_g \sim |\omega| (\varepsilon\alpha v)^{1/2}$, that of the ψ phase oscillation. Thus, a new set of resonances appears $l\tilde{\Omega}_g + m\omega + n\Omega = 0$ whose interaction results in both the formation of chaotic layer and the Arnold diffusion therein. Why the chaotic layer is always close, as $\varepsilon \to 0$, to the unperturbed separatrix? Because phase frequency $\tilde{\Omega}_g \to 0$ vanishes here which generally facilitates the chaos.

Consider the resonant Hamiltonian [4] (see Eq. (1.4) above):

$$H_g = \frac{p^2}{2\mu_g} - \mu_g \tilde{\Omega}_g^2 \cos\psi, \tag{2.1}$$

which describes a single (guiding) resonance. We are interested in the vicinity of separatrix ($H_g = \mu_g \tilde{\Omega}_g^2$) where the motion period T grows indefinitely:

$$T(w) = \frac{1}{\tilde{\Omega}_g} \ln \frac{32}{|w|}. \tag{2.2}$$

Here dimensionless quantity $w = (H_g/\mu_g\tilde{\Omega}_g^2) - 1$ characterizes the relative distance from the separatrix. The second term in resonant Hamiltonian (2.1) is one of perturbation terms in original Hamiltonian (1.1): $V_{mn}^{(g)} \equiv V_g$. This system is still completely integrable as it possesses integral $H_g = \text{const}$.

Now we include one more perturbation term with the phase

$$\varphi_{mn} = m\theta + n\tau \approx \xi_{mn}\psi + \omega_{mn}t + \beta_{mn}, \tag{2.3}$$

where factor $\xi_{mn} \sim 1$; β_{mn} is a constant phase, and perturbation frequency $\omega_{mn} = m\omega^{(g)} + n\Omega$ with resonant vector $\omega^{(g)}$ satisfying $m^{(g)}\omega^{(g)} + n\Omega = 0$. The interaction of two resonances ($m^{(g)}$ and m) breaks down the integrability and produces a chaotic layer around the unperturbed separatrix. The layer width can be estimated as [4],

$$w_s \approx \frac{\pi}{\Gamma(2|\xi_{mn}|)} \frac{\nu_1}{\xi_{mn}} \frac{V_{mn}}{V_g} (2\lambda)^{2|\xi_{mn}|+1} e^{-\pi\lambda/2}, \tag{2.4}$$

where $\Gamma(x)$ is the gamma function; $\nu_1 \sim 1$, and the basic parameter

$$\lambda = \frac{\omega_{mn}}{\tilde{\Omega}_g} \sim \left(\frac{\varepsilon_{cr}}{\varepsilon}\right)^{1/2} \frac{|m|}{|m^{(g)}|}. \tag{2.5}$$

The latter estimate is rather crude but it shows that in the region of Arnold diffusion ($\varepsilon \ll \varepsilon_{cr}$) the parameter $\lambda \gg 1$ is big, and chaotic layers are typically exponentially narrow.

With two resonances only the diffusion within the chaotic layer is restricted by its very small width and, hence, is of no importance for the global dynamics. To provide a long-range diffusion, at least one more resonance is required. A rough estimate for the diffusion rate in the actions is then [4]

$$D_I \sim \frac{\varepsilon^2 V_g^2}{T_a \tilde{\Omega}_g^2} \frac{w_s^2}{\lambda^4}, \tag{2.6}$$

where T_a is the averaged motion period within the chaotic layer: $\Lambda = T_a\tilde{\Omega}_g = \tilde{\Omega}_g T(\omega_s) + 1$, see Eq. (2.2). In terms of the reduced layer width and diffusion rate

$$\tilde{w}_s = w_s \frac{V_g}{V_{mn}}; \quad \tilde{D} = \frac{D_I}{D_I^0}; \quad D_I^0 = \frac{\varepsilon^2 V_{mn}^2}{\tilde{\Omega}_g}, \tag{2.7}$$

where the rate D_I^0 corresponds to the global chaos ($\varepsilon \gg \varepsilon_{cr}$), we arrive at the important relation

$$\tilde{D} \approx C \frac{\tilde{w}_s^2}{\Lambda \lambda^4} \tag{2.8}$$

between the rate of Arnold diffusion and the width of the corresponding chaotic layer. In the next section we are going to make extensive use of this remarkable relation. Fortunately, generally unknown factor C only weakly depends on system's parameters according to our experience [10]. All resonances essential for the Arnold diffusion but the guiding one are called *driving resonances.*

We see that both the width of chaotic layer as well as the rate of Arnold diffusion are exponentially small and are mainly controlled by the parameter λ (2.5), the ratio of perturbation frequency ω_{mn} to that of phase oscillation $\tilde{\Omega}_g$ in the guiding resonance. Thus, the Arnold diffusion is associated with a high-frequency perturbation $\lambda \gg 1$. This is just the opposite case as compared to a slow adiabatic perturbation. It is easy to see that the difference comes actually to which freedom is perturbing and which one is perturbed. For this reason we shall speak of the Arnold diffusion as *reverse adiabaticity*, and call λ the *adiabaticity parameter* [10, 11].

If only three primary resonances are operative, i. e. one can neglect all perturbation terms in Eq. (1.1) but three, the evaluation of both w_s as well as D_I can be performed quite accurately, within a factor of 2 at worst [4]. However, for sufficiently weak perturbation this is almost never the case because of the impact of higher-order resonances which may be not explicitly present in original Hamiltonian (1.1). Instead, they appear in higher approximations of perturbation theory. How to express these higher order effects in terms of the original Hamiltonian? A hint was given in Nekhoroshev's upper estimate [3] which can be represented in terms of Arnold diffusion as

$$\tilde{D} \sim D_0 \exp(-A\lambda_p^{1/E}) \tag{2.9}$$

with the most important parameter

$$E=E_N=\frac{(3N-1)\,N}{4}+2 \qquad (2.10)$$

for $M=0$. This upper estimate was later confirmed by many authors (see e. g., Ref. [13]). It is essential that λ_p in Eq. (2.9) is a formal adiabaticity parameter related to the primary driving resonances only which are explicitly present in the original Hamiltonian (1.1), i. e. λ_p is immediatly known (see next Section for an example).

A qualitative explanation of the dependence (2.9) and its relation to higher-order resonances was given in Ref. [4], yet the quantitative result turned out to be rather different, as compared to Eq. (2.10)

$$E=L\leqslant Q-1=N+M-1\,, \qquad (2.11)$$

where L is the number of linearly independent (incommensurable) unperturbed frequencies whose combinations determine the higher-order resonances. The maximal value of $L=Q-1$ is due to relation $m^{(g)}\omega+n^{(g)}\Omega=0$, at average, on the guiding resonance.

The difference between L and E_N is not necessarily a contradiction as E_N is the upper estimate. Yet, the problem is what is the true value of E, if any, i. e. if Eq. (2.9) is a good approximation at all? Numerical experiments to be discussed in the next Section seem to confirm the value of $E=L$. However, one is never sure that the perturbation is weak enough which is one of the conditions for applicability of Nekhoroshev's estimate.

Our recent numerical experiments [10] revealed another interesting feature of Arnold diffusion: even though E increases with L and the dependence $\tilde{D}(\lambda_p)$ becomes less steep, the factor A in Eq. (2.9) seems also to increase with L so that two curves $\tilde{D}(\lambda_p,\,L)$ for different L do intersect at some $\lambda_p(L)$. In other words, higher L works at big λ_p only. This is precisely the reason why a fairly simple 3-resonance approximation, which corresponds to $L=1$, is in a good agreement with numerical data for $\lambda_p\lesssim 4$ (see Refs [4, 14]).

A more accurate examination (than in Ref. [4]) of the estimate for factor A in Eq. (2.9) confirms the conjecture in Ref. [10] that actually $A=BL$ where now B weakly depends on the parameters. Particularly, this implies that the diffusion rate for some $\tilde{L}<L$ may happen to be bigger than for the actual L. Such an enhanced diffu-

sion can be caused by the driving resonances formed by a fewer number ($\tilde{L}$) of unperturbed frequencies. Therefore, one should find such $\tilde{L}(\lambda_p) \leqslant L$, for each λ_p, which provides the highest diffusion rate. Thus we arrive at the dependence

$$\tilde{D}(\lambda_p) \sim D_0 \exp(-B\tilde{L}\lambda_p^{1/\tilde{L}}), \tag{2.12}$$

where $\tilde{L}(\lambda_p) \leqslant L$. As our estimates are rather crude we may smooth over such a broken line. To this end we consider dependence $\tilde{L}(\lambda_p)$ as a continuous one, and derive it from the local condititon $\partial\tilde{D}(\lambda_p, \tilde{L})/\partial\tilde{L}=0$ whence $L = \ln\lambda_p$. Substituting this into Eq. (2.12) we obtain a fairly simple estimate

$$\tilde{D} \sim D_0 \lambda_p^{-Be}, \tag{2.13}$$

where $e=2.7182...$ We shall call this surprising regime the *poor adiabaticity*. It persists while $\tilde{L} < L$, i. e. for $\lambda_p < e^L$. Subsequently, the exponential dependence is recovered. Thus, our final estimate for the diffusion rate becomes

$$\frac{\tilde{D}}{D_0} \sim \begin{cases} \lambda_p^{-Be} & \lambda_p \leqslant e^L \\ \exp(-BL\lambda_p^{1/L}) & \lambda_p \geqslant e^L \end{cases} \tag{2.14}$$

Notice, that both curves $\tilde{D}(\lambda_p)$ are tangent at $\lambda_p = e^L$.

According to our numerical experiments (see next Section) $B \approx 2.8$ which is close to $B = \pi$ for $L=1$ (see Eqs (2.4), (2.6)) and we assume the latter value in what follows. The exponent in the first Eq. (2.14) then becomes: $Be \approx \pi e \approx 8.5$.

The quantity λ_p in Eq. (2.14) is some formal parameter related to the primary resonances. Instead, we may introduce a new, «true», adiabaticity parameter λ by the relation

$$\lambda = L\lambda_p^{1/L}. \tag{2.15}$$

This provides an alternative description for the impact of higher-order resonances. Notice, that λ in Eq. (2.4) is the true one.

The estimate (2.13) makes sense if Q is big which is satisfied, however, for a few freedoms also if the spectrum of the external quasiperiodic perturbation is reach enough ($M \gg 1$).

3. STRONG NONLINEARITY, NUMERICAL EXPERIMENTS

As in earlier studies [2, 4, 14] we made use of a simple model specified by the Hamiltonian

$$H=\frac{p_1^2+p_2^2}{2}+\frac{x_1^4+x_2^4}{4}-\mu x_1x_2-\varepsilon x_1 f(t)\,, \tag{3.1}$$

where $f(t)$ was some periodic or quasiperiodic function (for details see Ref. [10]). A peculiar feature of the unperturbed system ($\mu=\varepsilon=0$) is in that the motion is almost harmonic $x_i\approx a_i\cos\theta_i$ in spite of strong nonlinearity: $\alpha=(I/\omega)\,d\omega/dI=4/3$; $\dot\theta=\omega=\beta a$; $\beta\approx 0.85$. For the periodic driving perturbation of basic frequency Ω, and guiding resonance $\omega_1=\omega_2$, the formal (primary) adiabaticity parameter $\lambda_p=|\omega_{1n}|/\tilde\Omega_g\approx\Omega/2\beta\sqrt{\mu}$ $(\omega_{1n}=\omega_1-n\Omega;\ \tilde\Omega_g=\beta\sqrt{\mu})$.

Fig. 3, taken from Ref. [14], clearly shows the transition from a

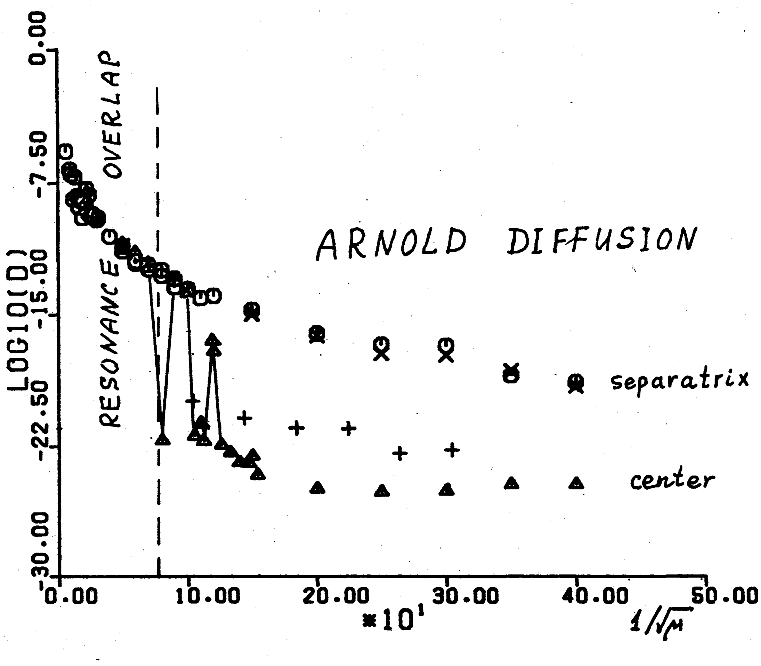

Fig. 3. Diffusion rate vs adiabaticity parameter $\lambda_p\approx 0.02/\sqrt{\mu}$ $(\varepsilon/\mu=0.01)$ at the guiding resonance center, and inside the chaotic layer. The vertical dashed line marks the transititon from resonance overlap to the Arnold diffusion [14].

global chaos due to a strong resonance overlap to Arnold diffusion within a narrow chaotic layer. The latter is explicitly seen in Fig. 4 [14]. The results of direct measurements of the diffusion rate in Ref. [14] confirm estimate (2.9) with the following fitted values of the parameters: $E=2$; $D_0=26$; $A=7.9$ or $B\approx 4$ (see Eq. (2.14)).

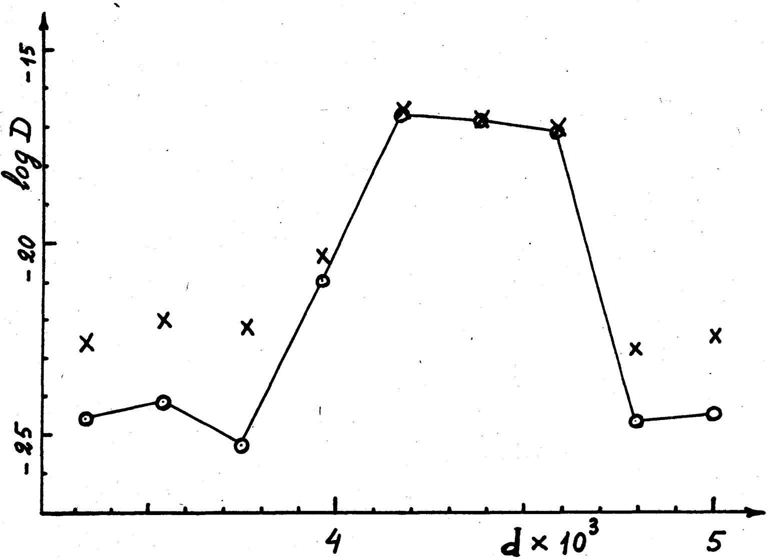

Fig. 4. Diffusion rate vs initial conditions: $\lambda_p\simeq 4$; $\varepsilon/\mu=0.1$; $2d=x_1(0)-x_2(0)$; $p_1(0)\simeq p_2(0)=0$; resonance center corresponds to $d=0$ [14].

In Ref. [14] the value of $\lambda_p\approx 10$ only has been reached because of a fast decay of the diffusion rate. To study much weaker perturbation we turned in Ref. [10] to the measurement of the chaotic layer width, evaluating the diffusion rate from Eq. (2.8). Actually, the only quantity to be measured was the dependence $T(w)$ (see Eq. (2.2)) from which both w_s (minimal T) as well as the true $\lambda=w_s/\Delta w_m$ (Δw_m the maximal single change in w) were calculated. In this way we managed to proceed as far as up to $\lambda_p\approx 50$ (but to $\lambda\approx 14$ only, see Eq. (2.15), $L=2$ and below).

First of all we checked Eq. (2.8) and calculated unknown factor $C\approx 3.5$ from the direct measurement of diffusion rate in Ref. [14] within the interval $\lambda_p=3.7-8$. The main numerical data of Ref. [10] are presented in Fig. 5 by crosses. They are well fitted by Eq. (2.9) with the parameters $E=2$; $D_0=2.0$; $A=5.60$ hence,

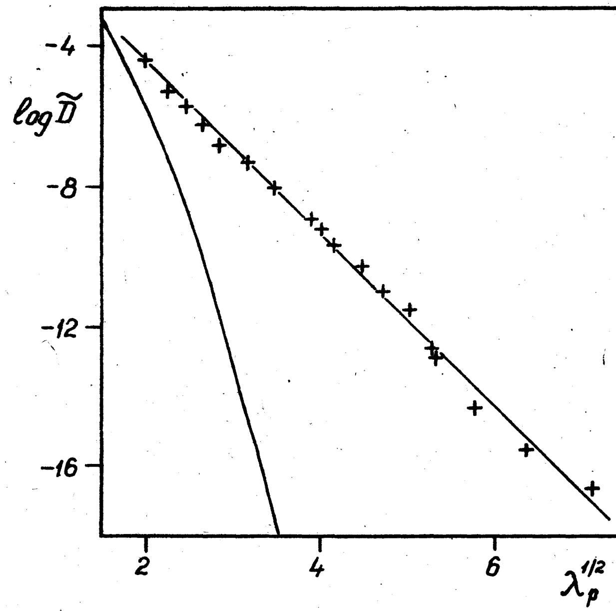

Fig. 5. Arnold diffusion rate vs λ_p: crosses are numerical data; straight line is estimate (2.9) with $E=2$; $A=5.60$; $D_0=2.0$; curve is the same estimate with $E=1$; $A=\pi$ (primary resonances).

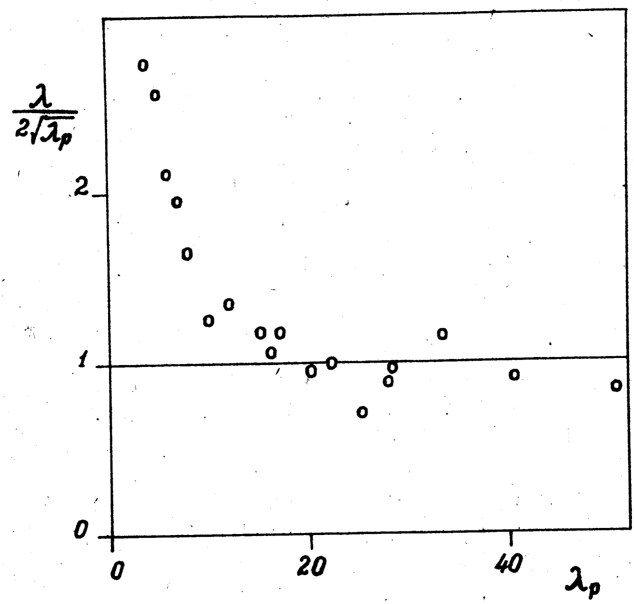

Fig. 6. Relation between true (λ) and primary (λ_p) adiabaticity parameters for the data in Fig. 5. Horizontal line is Eq. (2.15) with $L=2$.

$B \approx 2.8$. The latter is reasonably close to $B = \pi$ for primary resonances (see Eqs (2.4) and (2.6)). Fig. 6 demonstrates the building-up of asymptotic value for the ratio $\lambda/2\sqrt{\lambda_p} \to 1$ (see Eq. (2.25)).

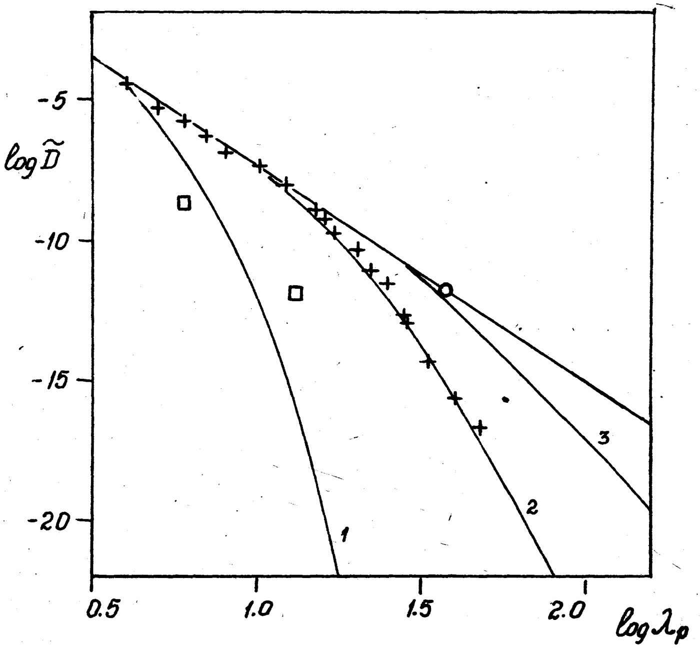

Fig. 7. Is there a poor adiabaticity? Straight line is power law, and curves the exponentials in Eq. (2.14) with $L=1, 2, 3$ as indicated. Crosses are the data from Fig. 5 ($L=2$); squares for $\omega/\Omega = 11/2$ ($L=1$); circle for independent Ω_1, Ω_2 ($L=3$).

Finally, in Fig. 7 we present preliminary data which suggest existence of a region of poor adiabaticity according to Eq. (2.14). The power law is shown by the upper straight line while the curves correspond to the asymptotic exponentials for different L values. The data from Fig. 5 (crosses) fit the combined dependense (2.14) as well as the asymptotic law only. Not a very conclusive result! However, the rest of data in Fig. 7 give, in our opinion, some preliminary indication in favor of the poor adiabaticity. They include: (i) linearly dependent frequencies $\omega/\Omega = 11/2$ which reduces both the value of $L \to 1$ and the diffusion rate (squares); and (ii) two independent driving frequencies $\Omega_2/\Omega_1 = 1.2381966...$ which increases $L \to 3$ and the rate (circle). Of course, the study of this new phenomenon — the poor adiabaticity — needs to be continued.

4. WEAK INSTABILITY, NONRESONANT CASE

If unperturbed Hamiltonian

$$H_0 = \omega^{(0)} I \tag{4.1}$$

is linear in the actions with a constant frequency vector $\omega^{(0)}$ we call the nonlinearity weak. In such a case all the nonlinearity comes from a weak perturbation $\varepsilon V(I, \theta)$ only. We mention that this situation is typical for the beam-beam interaction in a storage ring [6]. A generalized KAM theory [7] is still applicable which guarantees the motion stability for most initial conditions provided the linear frequencies $\omega_m^{(0)}$ are incommensurable:

$$m\omega^{(0)} + n\Omega \neq 0 \tag{4.2}$$

for any integer vectors m and n. Actually, for most vectors $\omega^{(0)}$ and Ω the following lower estimate holds (see Ref. [7])

$$|m\omega^{(0)} + n\Omega| > \frac{G\,|m|}{q^{Q+\nu}} \tag{4.3}$$

with some constant G and any $\nu > 0$; $Q = N + M$; $q = |m| + |n|$.

If, however, perturbation is a power series in $x_k = (2I_k/\omega_k)\cos\theta_k$ and conjugated momenta p_k it is sufficient for Eq. (4.2) to hold for $|m| \leqslant 4$ only (see Ref. [7]), as, generally, the stabilizing nonlinear frequency shift $|\delta\omega| \sim H_0$. This is just the case near the center of a typical nonlinear resonance ($\varepsilon \ll \varepsilon_{cr}$) which is, thus, stable (Section 2). Notice, that the beam-beam interaction is generally not of this type.

Coming back to the general case of weak nonlinearity consider, first, a single resonance perturbation $\sim \varepsilon v$, and the nonlinear frequency shift $\delta\omega \sim \varepsilon v$. The system remains integrable but, in contrast to the strong nonlinearity, the resonance width $\Delta I \sim I(\varepsilon/\alpha)^{1/2} \sim I$ would be much bigger while the phase frequency $\tilde{\Omega} \sim \omega(\varepsilon\alpha v)^{1/2} \sim \varepsilon\omega v$ much lower as $\alpha \sim \varepsilon v$ (cf. Eq. (1.4)). Other estimates remain essentially unchanged and we may use the results of Section 2 with adiabaticity parameter $\lambda_p \sim 1/\varepsilon$ rather than $\lambda_p \sim 1/\sqrt{\varepsilon}$ for strong nonlinearity (see Eq. (2.5)). Particularly, we expect the diffusion rate to be given by Eq. (2.14). This may be compared to a recent estimate in Ref. [15] where the principal parameter $L = N + 3$ which is reasonably close to our $L = N - 1$ (see Eq. (2.11), $M = 0$).

However, this agreement is not well justified. The essential difference is in that we consider one resonant term in the perturbation, i. e. a violation of eq. (4.2) for one couple of vectors m, n while in Ref. [15] all resonances are excluded. For a strong nonlinearity it makes no difference as resonances depend on initial conditions. In the case of weak nonlinearity such a dependence takes place for high-order resonances only: $q \gtrsim \varepsilon^{-1/Q} \equiv q_\varepsilon$ (see Eq. (4.3)). Only these resonances can work as guiding ones with a very big (true) adiabaticity parameter

$$\lambda \sim q_\varepsilon e^{q_\varepsilon/Q} \sim -\ln D, \tag{4.4}$$

which corresponds to an enormously slow diffusion. Here we assumed the phase frequency $\Omega_{q_\varepsilon} \sim \varepsilon V_{q_\varepsilon}^{1/2} \sim \varepsilon e^{-q_\varepsilon}$, and made use of Eq. (2.15): $\lambda \sim \lambda_p^{1/Q} \sim \tilde{\Omega}_{q_\varepsilon}^{-1/Q}$. Estimate (4.4) is very rough, of course, but it gives an idea of the crucial dependence on the guiding resonance.

In Ref. [13] the effect of considerable decrease in the diffusion rate has been predicted, for the weak nonlinearity, namely:

$$-\ln D \gtrsim q_\xi N \ln q_\xi, \tag{4.5}$$

where $\ln q_\xi \equiv -\ln \varepsilon/4(N-1) \sim \ln q_\varepsilon$ and $M=0$. This rate is much less than for the strong nonlinearity (see Eq. (2.9)) but greatly in excess of Eq. (4.4). Again, there is no contradiction but a big difference, even bigger than for strong nonlinearity (cf. Eqs (2.10), (2.11)).

To conclude, the weak nonlinearity is more difficult to study but it provides much better stability in the nonresonant case.

5. WEAK NONLINEARITY, RESONANT CASE

In the previous section we have already mentioned the effect of a single resonance which is very similar to that for the strong nonlinearity. That is not the case at all for two or more linear resonances (4.2). To see this we may change variables in such a way to remove the unperturbed Hamiltonian (4.1) (a simple example of this procedure will be given below). Then, the small perturbation parameter ε does no longer affect the motion structure, the degree of chaos, for instance, but determines the motion time scale only. To put it in other way, with a new time $\tau = \varepsilon t$ there is no more any small parameter in the problem, and hence the motion would be generally strongly chaotic provided the number of freedoms is, at least, two. The latter is equal to the number of independent resonance phases $\psi_k = m^{(k)}\theta + n^{(k)}\tau$ with different vectors $m^{(k)}$ in Eq. (4.2) (see Ref. [4]). This interesting nonlinear phenomenon had been discovered in Ref. [16] and further studied in Ref. [17].

Until recently the Arnold diffusion has been understood as the diffusion along nonlinear resonances (chaotic layers) in a many-dimensional phase space. In case of weak nonlinearity, however, a qualitatively different resonance structure is possible which has been discovered and studied in detail by Sagdeev, Zaslavsky and coworkers [18]. We consider here a simple example following Ref. [18]. The Hamiltonian is

$$H = \frac{p^2 + \omega_0^2 x^2}{2} + \varepsilon \cos(x - \Omega t) = \omega_0 I + \varepsilon \cos[a \cos\theta - \Omega t], \qquad (5.1)$$

where $a = (2I/\omega_0)^{1/2}$ is the amplitude of unperturbed oscillation, and I, θ are the action-angle variables.

This model has been widely used in the studies of plasma heating via the particle-wave interaction. We wonder if it has any relevance to the particle dynamics in accelerators, and, particularly, to the beam-beam interaction.

If we put $\omega_0 = 0$ the model describes a single nonlinear resonance and is completely integrable with no trace of chaos. The quantity x is phase variable, and p is the action while nonlinearity is strong. Yet, for any $\omega_0 \neq 0$ (in particular, arbitrarily small) the nonlinearity becomes weak, and the motion drastically changes. To remove linear term in Hamiltonian (5.1) we transform to new phase $\varphi = \theta - \omega_0 t$:

$$\frac{H}{\varepsilon} = \cos[a \cos(\varphi + \omega_0 t) - \Omega t] = J_n(a) \cos\left(n\varphi + \frac{\pi n}{2}\right) +$$

$$+ \sum_{\substack{k \neq n \\ k = -\infty}}^{\infty} J_k(a) \cos\left[k\varphi - (\Omega - k\omega_0)\, t + \frac{\pi k}{2}\right], \qquad (5.2)$$

where $J_n(a)$ is the Bessel function, and we assume the resonance condition: $\Omega = n\omega_0$. The sum represents a high-frequency perturbation as $\varepsilon \to 0$ since $\dot{\varphi} \sim \varepsilon$. Neglecting this perturbation we arrive at the time averaged Hamiltonian

$$\bar{H}=\varepsilon J_n(a)\cos\left(n\varphi+\frac{\pi n}{2}\right)\rightarrow\varepsilon\sqrt{\frac{2}{\pi a}}\cos A\,\cos\Phi\,. \qquad (5.3)$$

where $A=a-\pi n/2-\pi/4$; $\Phi=n\varphi+\pi n/2$ and the last simplified expression holds for big $a\gg 1$.

The most remarkable peculiarity of this model is in that a *single* resonance ($n\omega_0=\Omega$) generates an infinite grid of stable ($\sin A=\sin\Phi=0$) and unstable ($\cos A=\cos\Phi=0$) fixed points on the surface (A, Φ) as outlined in Fig. 8. This should be contrasted

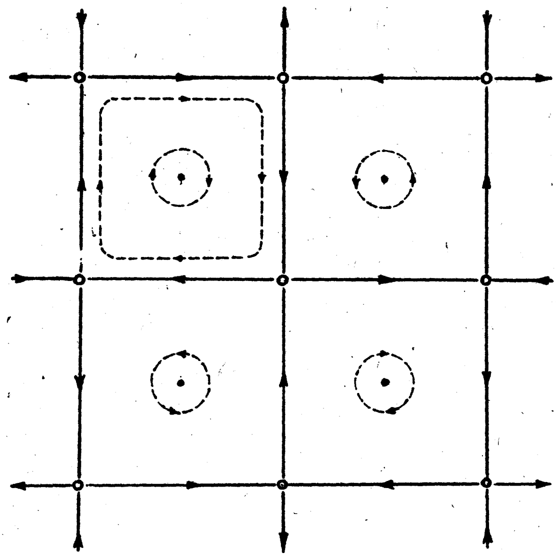

Fig. 8. An example of the single resonance grid for weak nonlinearity in model (5.1) on plane (A, Φ): $A\gg 1$; unstable fixed points are connected by separatrices (straight lines). Arrows show the direction of motion.

with the resonance picture for strong nonlinearity when, as is well known, there is an one-dimensional chain of fixed points only, which extends in the direction of the phase variable. This difference drastically changes the motion. While strong nonlinearity absolutely bounds the oscillation in action, the weak nonlinearity allows, in this model, indefinite motion over the resonance grid. It is true, in approximation (5.3) the oscillation is confined within an individual cell of the grid. However, the high-frequency perturbation produces a connected chaotic web along separatrices which allows a trajectory to wander indefinitely. The size of a resonance cell ($\Delta A=\pi$; $(\Delta I)_c=\pi\omega_0 a$) does not depend on ε but the oscillation frequency in the cell does:

$$\tilde{\Omega}_0 = \sqrt{\frac{2}{\pi}}\,\frac{\varepsilon n}{\omega_0 a^{3/2}} \tag{5.4}$$

near the center, and

$$\tilde{\Omega}_s \sim \frac{\tilde{\Omega}_0}{|\ln w_s|} \sim \frac{\tilde{\Omega}_0}{\lambda} = \frac{\tilde{\Omega}_0^2}{\omega_0} \tag{5.5}$$

in chaotic layer of width w_s ($\ln w_s \sim -\lambda$) where adiabaticity parameter $\lambda = \omega_0/\tilde{\Omega}_0$ (see Section 2).

The diffusion is caused by transition of the trajectory from one cell to a next one when it crosses the central line of a chaotic layer (the unperturbed separatrix) which is the border between neighbouring cells. The average time between successive crossings of the central line, or the recurrence time to this line, was evaluated in Ref. [19] and is equal to

$$T_R \approx 3\lambda \frac{2\pi}{\tilde{\Omega}_s} \sim \frac{\omega_0^2}{\tilde{\Omega}_0^3}. \tag{5.6}$$

Hence, the diffusion rate in chaotic web of a single resonance is

$$D_I \equiv \frac{\langle (\Delta I)^2 \rangle}{t} = \frac{(\Delta I)_c^2}{T_R} \equiv \frac{D_1}{I^{5/4}} \sim \frac{(\varepsilon n)^3}{\omega_0^3 a^{5/2}}. \tag{5.7}$$

The rate drops only as the cube of perturbation, so the Arnold diffusion in this case is fairly fast. A similar result for another model of the resonance grid was derived and numerically confirmed recently in Ref. [20]. Curiously, the poor adiabaticity in a strongly nonlinear system, considered in Section 2 above, is also a power law with close exponent: $D \sim \varepsilon^{\pi e/2} \approx \varepsilon^4$ (see Eq. (2.14)).

The discovery of the resonance grid and the fast Arnold diffusion was really dramatic. Model (5.1) has been studied by many plasma phisicists as early as in 1977 (see References in [18]). But it took about 10 years to understand the phenomenon. Moreover, in book [5] the resonance grid is now (!) obvious in Fig. 2.11 (cf. with our Fig. 8) but was missed by the authors as well as by the translators of this book into Russian.

As the diffusion rate decreases with I (5.7) the average action grows slower as compared to homogeneous diffusion. Generally, if $D_I = D_1 I^{-\beta}$ the diffusion equation

$$\frac{\partial f}{\partial t}=\frac{1}{2}\frac{\partial}{\partial I}D_I\frac{\partial f}{\partial I}, \qquad (5.8)$$

where $f(I)$ is the distribution function gives:

$$\langle I^{2+\beta}\rangle=\left(1+\frac{\beta}{2}\right)D_1t; \quad I\sim(D_1t)^{1/(2+\beta)}$$

$$\langle I^{13/4}\rangle=\frac{13}{8}D_1t; \quad I\sim(D_1t)^{4/13}. \qquad (5.9)$$

The latter relations are for $\beta=5/4$ in Eq. (5.7).

The formation of infinite resonance grid in the model under consideration is due to an oscillatory dependence of the averaged Hamiltonian (5.3) on the action I. This holds only if the resonance is maintained to accuracy [18]:

$$|\delta\dot{\varphi}|\lesssim\tilde{\Omega}_0. \qquad (5.10)$$

If, for example, $\delta\dot{\varphi}=\delta\Omega/n$ ($\delta\Omega=\Omega-n\omega_0$) the chaotic web is restricted to the region $a^{3/2}\lesssim\varepsilon n^2/\omega_0\delta\Omega$ provided $\varepsilon\gtrsim\omega_0\delta\Omega/\sqrt{n}$. If, instead, we would add to the averaged Hamitonian (5.3) a monotonic term like $\varepsilon V_0(I)$ the condition (5.10) were independent of ε. Suppose $V_0=I^2$, then the diffusion over the web is restricted by $I\lesssim n^{4/7}\omega_0^{-1/7}$.

We mention, that in a many-dimensional system of weak nonlinearity the resonance grid takes shape of something like honeycomb. In any dimension the diffusion over such a structure is always fast as compared to the case of strong nonlinearity ($\varepsilon\to0$). Another important distinction of the weak-nonlinearity diffusion is in that it can occur in the minimal dimension $N>1$ when the chaos is possible at all [18].

6. CONCLUDING REMARKS

For sufficiently small perturbation a typical nonlinear oscillator system (1.1) is KAM integrable, i. e. its motion is regular and stable for most initial conditions (Section 2). The Arnold diffusion violating complete integrabilily, is not only very slow but, moreover, is confined within narrow chaotic layers of a negligible measure. Hence, at first glance, it seems to be of no importance. This is the case, indeed, in a purely dynamical system. However, the presence of any additional, external, noise drastically changes the motion as the Arnold diffusion may greatly enhance the effect of noise independent of the initial conditions. The enhancement is the bigger the weaker is the noise as it is described in some detail in Ref. [4]. The average rate of Arnold diffusion drops to

$$\langle \tilde{D} \rangle_A \sim \tilde{D} \frac{\tilde{w}_s}{\lambda} \sim \frac{\tilde{w}_s^3}{\Lambda \lambda^5}, \tag{6.1}$$

as compared to the noise-free diffusion (2.8).

With the presence of dissipation (the radiation damping of electrons, for example) a qualitatively new mechanism of particle transport along resonances comes into play. It had been predicted by Tennyson [21] and studied in detail by Gerasimov [22].

Still another instability of motion — the modulational diffusion — occurs under a low frequency modulation, external or internal [23, 24]. It leads, for a not-too-weak perturbation ($\varepsilon \gtrsim \varepsilon_m$, $\varepsilon_m \ll \varepsilon_{cr}$), to the formation of the modulational layers of relatively large width $\Delta\omega_m$. Within the layer the motion is chaotic due to the overlap of close resonances in a modulational multiplet. Critical ε_m decreases with modulation frequency but remains finite unlike the Arnold diffusion. Width $\Delta\omega_m$ depends on the modulation depth (amplitude) and is equal approximately to the width of the motion Fourier spectrum. In a chaotic layer around separatrix the latter is of the order of phase oscillation frequency $\tilde{\Omega}_g$. Hence, it is no surprise that estimate (2.14) for the rate of Arnold diffusion can be applied, roughly, also to the modulational diffusion upon substituting $\Delta\omega_m$ for $\tilde{\Omega}_g$. More accurate evaluations are presented in Ref. [23] but for the primary resonances only ($L=1$). The effect of high-order resonances on modulational diffusion was apparently observed in numerical experiments in Ref. [25] (see Fig. 3 there).

The most important difference from the Arnold diffusion is in the measure of the chaotic component which is bigger by the factor $\sim w_s^{-1}$ for the modulational diffusion. Particularly, the average diffusion rate $\langle \tilde{D} \rangle_m \sim \tilde{D}_m/\lambda \gg \langle \tilde{D} \rangle_A$ is also much bigger (cf. Eq. (6.1)).

Our final remark if of a different nature, very important though. It concerns the problem of error estimation for computation, in general, and in numerical simulation, particularly. If the equations in question are Hamiltonian and the numerical algorithm is conservative, or canonical one [26], which seems to be most effective, we put forward the following conjecture: the growth of computational errors is determined by the artificial Arnold diffusion due to numerical discretization in time; moreover an external, although artificial, noise is present as a result of round-off and other numerical errors.

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B.V. Chirikov, V.V. Vecheslavov

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В.В. Вечеславов, Б.В. Чириков

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