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STATISTICAL PROPERTIES OF A NONLINEAR STRING

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In this report we present some results of an investigation of the qualitative behavior of a longitudinally vibrating nonlinear string with fixed ends, obeying the equation

$$\frac{\partial^2 x}{\partial t^2} = \frac{\partial^2 x}{\partial z^2}\left[1 + 3\beta\left(\frac{\partial x}{\partial z}\right)^2\right]. \tag{1}$$

This problem has been investigated in the work of Fermi, Pasta, and Ulam [1] by the method of numerical integration of the vibrations of a chain of nonlinear oscillators approximating a string and obeying the system of ordinary differential equations

$$\begin{aligned} \ddot{x}_l = (x_{l+1} + x_{l-1} - 2x_l)\{1 + \beta[(x_{l+1} - x_l)^2 \\ + (x_l - x_{l-1})^2 + (x_{l+1} - x_l)(x_l - x_{l-1})]\}, \\ l = 1, 2, \ldots, N-1, \quad a = 1; \quad L = N. \end{aligned} \tag{2}$$

The intent of [1] was to analyze the origin of statistical properties in such a relatively simple mechanical system with a large number of degrees of freedom. In the linear case ($\beta = 0$) the chain of oscillators may be represented in the form of N−1 completely independent modes (normal oscillators) and, consequently, does not have any statistical properties (there exists a complete set of N−1 single-valued integrals of motion). It has been assumed until recently that any nonlinearity will lead to the appearance of statistical properties, which we will characterize henceforth for brevity by the term stochasticity (ergodicity, mixing, finite entropy, according to A. N. Kolmogorov [2]). The negative result of [1] (motion with a clearly quasiperiodic character rather than stochasticity) therefore seemed surprising. However, later work of A. N. Kolmogorov and V. I. Arnol'd [3, 4] showed that such a result is, on the contrary, to be expected; for a sufficiently small perturbation the motion of a nonlinear system retains its quasiperiodic nature.[1] From the point of view of the modern theory of dynamical systems, one should expect the existence in general of some critical perturbation, where stochasticity sets in (see also [5]). The purpose of the present paper is to estimate the limits of stochasticity for a chain of oscillators [2].

Transforming to normal coordinates (for $\beta = 0$)

$$x_l = \sqrt{\frac{2}{N-1}} \sum_{k=1}^{N-1} Q_k \sin\frac{\pi k l}{N}, \tag{3}$$

we obtain the system of equations

$$\begin{aligned} \ddot{Q}_k + \omega_k^2 Q_k = -\frac{\beta}{8N}\Big\{ &\sum_{i+j=2}^{k-1} A_{ij}^{+} Q_{k-i-j}\,\omega_{k-i-j}^2 \\ &+ \sum_{i+j=N-k+1}^{2N-k-1} A_{ij}^{+} Q_{2N-i-j-k}\,\omega_{2N-i-j-k}^2 \\ &+ \sum_{i+j=2}^{N-k+1} A_{ij}^{+} Q_{i+j+k}\,\omega_{i+j+k}^2 \\ &- \sum_{i+j=N+k+1}^{2N-2} A_{ij}^{+} Q_{2N+k-i-j}\,\omega_{2N+k-i-j}^2 \\ &- \sum_{i+j=k+1}^{k+N-1} A_{ij}^{+} Q_{i+j-k}\,\omega_{i+j-k}^2 \\ &- \sum_{i+j=2N-k+1}^{2N-2} A_{ij}^{+} Q_{i+j+k-2N}\,\omega_{i+j+k-2N}^2 \\ &+ 2\sum_{i-j=k-1}^{k-N+1} A_{ij}^{-} Q_{k-i+j}\,\omega_{k-i+j}^2 \\ &+ 2\sum_{i-j=N-k+1}^{N-2} A_{ij}^{-} Q_{2N-k-i+j}\,\omega_{2N-k-i+j}^2 \\ &- 2\sum_{j-i=N-2}^{k+1} A_{ij}^{-} Q_{j-i-k}\,\omega_{j-i-k}^2 \Big\}, \end{aligned} \tag{4}$$

[1] The hypothesis of such a unique kind of stability of quasiperiodic motion is contained in [1]; we will refer to it henceforth as Kolmogorov stability.

where

$$i, j, k = 1, 2, \ldots, N-1, \quad \omega_k = 2 \sin \pi k / 2N, \qquad (5)$$

$$A_{ij}^{\pm} = Q_i Q_j \omega_i \omega_j [3\sqrt{(4-\omega_i^2)(4-\omega_j^2)} \pm \omega_i \omega_j]. \qquad (6)$$

Equations (4) are extremely cumbersome, hence we will investigate two limiting cases: $k \ll N$ and $(N-k) \ll N$. Assuming a small perturbation

$$\beta / 8N \ll 1, \qquad (7)$$

the solution (4) may be represented in the form

$$Q_n = C_n(t) \cos \theta_n(t), \quad \dot{\theta}_n = \omega_n'(t), \qquad (8)$$

where C_n, ω_n' are the slowly varying amplitudes and frequencies of the normal oscillators; the prime indicates that the frequency includes all corrections relating to the perturbation. Equations (4) may be written in the form

$$\ddot{Q}_k + \omega_k^2 Q_k \left\{1 - \frac{3\beta}{4N} \omega_k^2 (2 - \omega_k^2) Q_k^2\right\}$$

$$= \frac{\beta}{8N} \sum_m F_{km} \cos \theta_{km}, \quad \dot{\theta}_{km} = \omega_{km}'. \qquad (9)$$

These equations describe the motion of nonlinear oscillators under the influence of external forces with amplitudes $\beta F_{km}/8N$ and frequencies ω_{km}', i.e., the case of many resonances in a nonlinear system. For one resonance, applying the standard averaging technique [6], one can derive the so-called phase equation

$$\ddot{\Psi}_{km} = \frac{d\Omega_{km}}{dC_k} \frac{\beta F_{km}}{16\omega_k' N} \sin \psi_{km}, \quad \Omega_{km} = \omega_{km}' - \omega_k', \qquad (10)$$

from which we readily arrive at the size of the limiting curve $|\dot{\psi}_{km}|_{max}$ bounding the region of stable phase oscillations near resonance (see, e.g., [7]):

$$|\dot{\psi}_{km}|_{max} = \sqrt{\frac{\beta F_{km}}{4N\omega_k'} \frac{d\Omega_{km}}{dC_k}}. \qquad (11)$$

In the case of many resonances the nature of the motion depends substantially on the relation between the dimension of the limiting curve and the mean separation between resonances $\Delta\omega$. It was shown in [8, 9] that the limit stochasticity is defined by the condition

$$|\dot{\psi}_{km}|_{max} / |\Delta\omega| \sim 1. \qquad (12)$$

Let us express this condition in terms of the dimensionless characteristic of the nonlinear perturbation (2):

$$\beta[(x_{l+1} - x_l)^2 + (x_l - x_{l-1})^2$$

$$+ (x_{l+1} - x_l)(x_l - x_{l-1})] \approx 3\beta(\partial x / \partial z)^2, \qquad (13)$$

since $z = la$, $a = 1$ (2). The latter expression loses its validity for the very highest modes, but the equation remains correct in order of magnitude.

Carrying out the bulky computations on the right-hand side of Eq. (4), we obtain the following final estimates for the limit of stochasticity:

$$3\beta\left(\frac{\partial x}{\partial z}\right)_m^2 \sim \begin{cases} \frac{3}{k}, & k \ll N, \\ \frac{3\pi^2}{N^2}\left(\frac{k}{N}\right)^2, & N-k \ll N. \end{cases} \qquad (14)$$

Here $(\partial x / \partial z)_m$ is the maximum value of the derivative. It is supposed that initially only the one mode with index k is excited. Equation (14) is an expression of the first stochastic exchange between several neighboring modes. The main difference for the two limiting cases in (14) arises because of the mean spacing between resonances $\Delta\omega \sim 2\pi / N$ $(k \ll N)$, $\Delta\omega \sim \pi^2/2N^2$ $(N - k \ll N)$. For a continuous string one must use the first of the estimates in (14).

Equation (14) shows that stochasticity in the excitation of the lower modes is only possible for very large nonlinear perturbations. This explains the lack of success in [1]; a priori the excitation of the first mode was quite reasonable as an initial condition. For higher modes and large N, on the other hand, stochasticity sets in even with very little nonlinearity.

In Fig. 1 the two solid lines represent the limit of stochasticity on a log-log scale, the dashed curve represents an attempt to make a rough interpolation

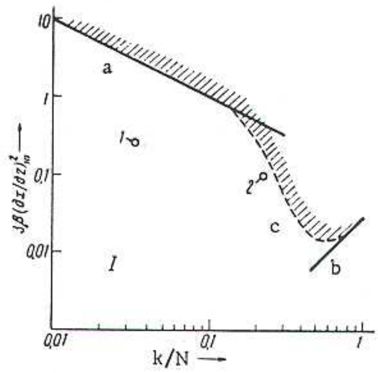

Fig. 1. I) Region of Kolmogorov stability; II) region of stochasticity; a) limit of stochasticity for $k \ll N$ (14); b) limit for $N - k \ll N$ (14); c) qualitative interpolation; the numerical values of the straight lines a, b are given for N = 32; 1) result of numerical calculation for N = 32, $x_m = 1$, k = 1, $\beta = 8$ [1]; 2) the same for k = 7, $\beta = 1/16$ [1].

between them; the circles are the results of a numerical calculation for two cases of cubic nonlinearity according to [1]. It is interesting to note that the first case falls well within the region of Kolmogorov stability, despite the large value of $\beta = 8$. The results of the numerical calculation [1] exhibit in this case a clearly pronounced quasiperiodicity. The second case lies near the limit of stochasticity; even though the value of $\beta = {}^1\!/_{16}$ is very small, the seventh harmonic is still excited. The model pattern in this case [1] bears little resemblance to quasiperiodic motion, being more nearly like undeveloped stochasticity.

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All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.