The mapping is determined in unit square, or, better to say, on unit torus, and has generation function :

$$H(\bar{\varphi}, \psi) = \frac{\bar{\varphi}^2}{2} + \bar{\varphi}\psi + k\,F(\psi) \;\; ; \;\; F'(\psi) = -f(\psi) \qquad (1.4)$$

One the one hand, the simplicity of this mapping allows to follow over 10^8 iterations by the present-day computers. On the other hand, the system (1.3) proved out to be unexpectedly diverse to simulate a number of real physical processes (see [6, 7]). We note only that a physicist can take ψ as an oscillator's co-ordinate (the phase angle), and φ as its momentum (the action). In this interpretation model (1.3) describes the motion of a nonlinear oscillator under periodical perturbation (kicks).

We have mostly used as the perturbation function $f(\psi)$ a simple nonlinear expression :

$$f(\psi) = \psi^2 - \psi + 1/6 \qquad (1.5)$$

The "linear" perturbation(*)

$$f(\psi) = \psi - 1/2 \qquad (1.6)$$

is a trivial one in the sence that contemporary ergodic theory proves the stochasticity in this case rigorously under the condition :

$$k < -4 \;\;, \text{ or } : \;\; k > 0 \qquad (1.7)$$

This was shown by Sinai and Oseledets (see [17]) whose theory can be also extended to a more general case of limited derivative : $|f'(\psi)| \geqslant C > 0$. The latter condition inevitably leads to $f(\psi)$ discontinuity, due to its periodicity. As to the continuous perturbation which is of interest for applications the present-day ergodic theory is helpless.

In what follows we present some results of numerical experiments concerning the stochastic component of the described mapping [11, 12, 6].

§ 2. TECHNIQUES OF NUMERICAL EXPERIMENTATION(**)

The computation of mapping (1.3) has been run on the BESM-6 at the Computing Centre of the Siberian Division of the USSR Academy of

(*) To avoid misunderstanding we emphasize that in the latter case mapping remains nonlinear as well due to the fractional part taken. By precisely the same reason the perturbation (1.5) is a singular one (derivative discontinuity).

(**) We will say just experimentation, or experiments for the sake of brevity.

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SOME NUMERICAL EXPERIMENTS WITH A NONLINEAR MAPPING : STOCHASTIC COMPONENT

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RESUME

Une méthode et les résultats d'expériences numériques avec une transformation canonique simple sont décrits. La transformation simule le mouvement d'un oscillateur non linéaire soumis à une perturbation périodique. L'existence d'une composante stochastique est montrée et sa structure générale est étudiée. Quelques données concernant la composante stable sont fournies à la fois dans la région stochastique et dans la zone transitoire. L'ensemble de la surface de la composante stable décroît exponentiellement avec le paramètre de stochasticité.

L'effet d'une faible dissipation sur la composante stochastique a été étudié. Il a été trouvé que le mouvement stochastique est en général détruit par dégénérescence en un mouvement périodique. En présence d'une dissipation suffisamment forte une transformation des figures dans le plan de phase en lamelles a été observée, ce qui retarde considérablement la destruction de la composante stochastique.

ABSTRACT

Method and results of numerical experiments with a simple canonical mapping are described. The mapping simulates the motion of a nonlinear oscillator under periodic perturbation. The existence of stochastic component is shown and its general structure is demonstrated. Some data is given concerning the stable component in both stochastic region and transitional zone. Overall area of the stable component decreases exponentially with the stochasticity parameter.

The impact of a weak dissipation on stochastic component has been studied. It was found that stochastic motion is generally destroyed, degenerating into a periodical one. Under sufficiently strong dissipation a foliation of the phase plane was observed which delays considerably the destruction of the stochastic component.

§ 1. INTRODUCTION

The study of mappings is most interesting, to our mind, in connection with simulation, by them, various dynamical processes. Analitical investigation of a mapping is often considerably simpler than that of the corresponding differential equations. It is still more true for the numerical simulation by computer, especially if we are interested in long range processes. This kind of simulation, or, as one uses to say, numerical experimentation makes it possible to study some general regularities of dynamical system behavior. A very interesting one is so-called *stochastic component* (see, for instance, [1-3]). We will designate by this term irregular motion which is practically indistinguishable from the "real" random process. To be precise, we speak about the mixing motion with positive KS-entropy (see § 3).

The following mapping is an elementary example :

$$\bar{\psi} = \{k\psi\} \tag{1.1}$$

where curly brackets designate the fractional part. This mapping has been studied a good deal for the real numbers (see, for instance, [4]), stochasticity corresponding to the condition $k > 1$. A similar mapping for integers :

$$\bar{r} \equiv kr \quad (\text{mod } 2^p) \tag{1.2}$$

was proposed by Lehmer [5] as a so-called pseudo-random number generator playing an important role in many kinds of computation. We believe this is the best way for simulation of random sequence [5]. Yet the theory of such a generator is still at its very outset.

As to the simulation of physical processes they are, according to the Liouville's theorem, measure preserving, or even canonical ones, the latter meaning they are described by the Hamilton equations.

The mapping (1.1) cannot serve for such a simulation since only the reverse mapping preserves the measure. This kind of mapping is called the endomorphism in the theory of dynamical systems. An appropriate model for physical processes would be the automorphism, or, still better, the canonical mapping.

Our experience of many years shows that a simple canonical mapping (and its variants) is a very interesting model which potentialities are still far from exhausted, namely :

$$\begin{aligned} \bar{\varphi} &= \{\varphi + k f(\psi)\} \\ \bar{\psi} &= \{\psi + \bar{\varphi}\} \end{aligned} \tag{1.3}$$

Sciences. We have used hand coded program to achieve the maximum computing speed. The computation time was about 20 μsec per iteration.

The main output data was in the form of distribution function of the trajectory over the phase plane, i.e. the number of trajectory crossing each of the cells of the phase square. The distribution function is to bulky to be exposed here. Instead, we reduced it to a much more compact phase map (Fig. 3) which records only the fact of crossing, or missing, each of the cells by the trajectory.

The finest subdivision of the phase plane for distribution function was $128 \times 128 = 16\,384$ cells. For a phase map it is unnecessary to occupy the whole word of computer's memory for each cell, instead one binary digit is sufficient (the logical variable). This makes it possible to increase the number of cells up to $512 \times 1\,024 = 524\,288$. With such a number of cells the output of even a phase map becomes impossible, so one has to restrict oneself to counting missed and crossed cells and to output of peculiar sections of the phase plane map (Fig. 3). Let us note that all array dimensions for distribution function and phase map have been taken equal to some power of 2. This considerably simplifies the programming as well as increases computing speed.

The recent experiments has been done in the regime of man-machine interaction by means of display "The Screen" on-line with computer. The display's hardware as well as software, intended for a man-machine interaction, were designed and realized in the Institute of Automation and Electrometry of the Siberian Division of the USSR Academy of Sciences (*) The display was put into operation with BESM-6 at the Computing Centre here [14]. The man-machine interaction has been performed by means of light pen and light buttons while the picture of motion has been followed on the screen of display (Fig. 6). Each still on the screen contained 400 successive phase points. The photographic camera could accumate several successive stills on film which gives the possibility to see very many iterations at once (Fig. 1). Our computer code provided also "enlargement" of some sections of the phase plane.

For aesthetic reasons we shifted all the picture on phase plane by 1/2 along the φ – axis, so the mapping had a form :

$$\begin{aligned} \bar{\varphi} &= \{\varphi + k f(\psi)\} \\ \bar{\psi} &= \{\psi + \bar{\varphi} - 1/2\} \end{aligned} \tag{2.1}$$

(*) We seize the opportunity to express our sincere gratitude to Mrs. E.G. Babat and Mr. B.S. Dolgovesov of this Institute for their great assistance in taming the display at the earliest stage of its operation.

Thus approximate value of the entropy can be avaluated from :

$$h \approx < \ln \lambda^+ (\psi_0) > \qquad (3.10)$$

where one should put $\lambda^+ = 1$ in the turning interval. We assume, further, the hypothesis that stochastic component occupies practically all the phase square. Then one can average in (3.10) approximately over the phase ψ_0.

For the perturbation (1.5) the integral (3.10) can be calculated exactly to give :

$$kh = \mathrm{H}\left(\frac{k}{2} + 1\right) + \mathrm{H}\left(\frac{k}{2} - 1\right) \approx k\,(\ln k) - 1]$$

$$\mathrm{H}\,(x) = x \ln\,(x + \sqrt{x^2 - 1}) - \sqrt{x^2 - 1} \qquad (3.11)$$

The last expression is an approximation for $k >> 1$.

In some experiments we have used an analitical perturbation :

$$f\,(\psi) = \frac{\sin 2\pi\,\psi}{2\pi} \qquad (3.12)$$

Approximation $k >> 1$ gives in this case for $\lambda^+ \approx |kf'| + 2 + 1/kf'|$ and for the entropy :

$$h \approx \int_0^1 d\psi_0 \quad \ln |k \quad \mathrm{Cos}\, 2\pi\,\psi_0| = \ln \frac{k}{2} \qquad (3.13)$$

For the experimental determination of the entropy Sinai's formula has been used [10] :

$$h = \overline{\ln\,(\bar{l}/l)} \qquad (3.14)$$

where the upper bar indicates averaging along the trajectory (in "time"). Expressions (3. 6, 14) are equivalent, due to the ergodicity of motion. We chose $l = 10^{-7}$, so the maximum value of $\bar{l} \sim 10^{-4}$ for the largest $k = 10^3$ has been still very small. The computation of mapping (1.3) has been done for two trajectories which initial points were by the transversal vector $\vec{l}$ apart. After each iteration the length of this vector was brought to the initial value without changing its direction.

A summary of theoretical and experimental data concerning the entropy is given in Table 1. As already noted above the initial distance between trajectories was chosen as $l = 10^{-7}$. Its increase up to 10^{-3} changes the experimental value from 3.615 to 3.72 ($k = 100.2$; $\bar{l} \sim 0.1$; singular perturbation). Standart number of iterations in measuring the entropy has been 10^4. A decrease of this figure to 10^3 leads to entropy change from 4.234 to 4.242 ($k = 142.0$; analytical perturbation). This

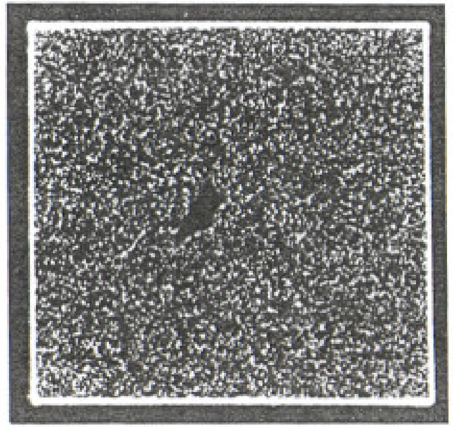


Fig. 1 a

Fig. 1 b

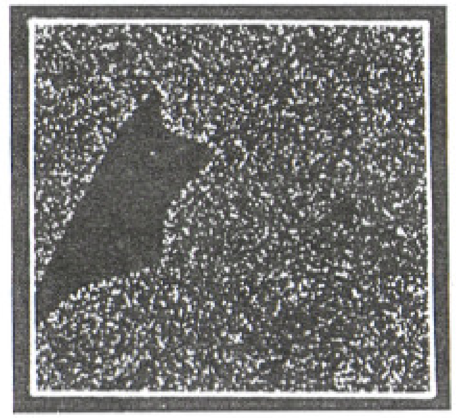


Fig. 1. – A picture of motion on the phase plane of mapping (2.1) with singular perturbation (1.5); k = 12.9282032; 2×10^4 iterations: a - the whole phase square; b - a section with magnification $\times 4$.(display).

§ 3. KS-ENTROPY

The most strong property of stochastic motion is positive (non-zero) KS-entropy (Krylov – Kolmogorov – Sinai entropy [8-10])(*). It characterizes the average rate of local instability, the latter developping exponentially (see, for instance, [7]). Exponential local instability is the ultimate cause of mixing as well as the other statistical properties of dynamical system.

(*) Don't confuse it with the thermodynamical entropy, see [6, 7]. We will call the former just entropy for the sake of brevity.

To investigate the local stability we take an arbitrary-phase trajectory : $\varphi_0(n)$, $\psi_0(n)$ where n stands for the serial number of iteration and consider the motion of two initially close phase points :

$$\xi = \varphi - \varphi_0 \quad ; \quad \eta = \psi - \psi_0 \tag{3.1}$$

Linearizing mapping (1.3) around the chosen trajectory we obtain :

$$\begin{aligned} \bar{\xi} &= \xi + k f'(\psi_0) \cdot \eta \\ \bar{\eta} &= \xi + [1 + k f'(\psi_0)] \cdot \eta \end{aligned} \tag{3.2}$$

The eigenvalues of the latter system are equal to :

$$\lambda = 1 + \frac{k f'}{2} \pm \sqrt{k f' \left(1 + \frac{k f'}{4}\right)} \tag{3.3}$$

and its eigenvectors make angle θ with the ψ – axis :

$$\mathrm{tg}\, \theta = \frac{k f'(\psi_0)}{\lambda - 1} \tag{3.4}$$

In the ψ_0 interval where :

$$-4 < k f'(\psi_0) < 0 \tag{3.5}$$

$|\lambda| = 1$, i.e. the transversal vector $\vec{l}(\xi,\eta)$ turns : $l(\xi,\eta) = \bar{l}(\bar{\xi},\bar{\eta})$. We shall call this interval (or intervals) the *turning interval*. The complementary set of ψ_0 we are going to term the *extension interval* since one of eigenvalues here $\lambda^+ > 1$ ($\lambda^+ \ \lambda^- = 1$).

According to [10] the entropy can be expressed by :

$$h = \langle \lim_{l \to 0} \ln (\bar{l}/l) \rangle \tag{3.6}$$

where the averaging extends over all the stochastic component. For a large k :

$$\bar{l}/l \approx \lambda^+ \tag{3.7}$$

almost everywhere. Indeed, the direction of extension eigenvector is almost fixed :

$$\mathrm{tg}\, \theta^+ \approx 1 - \frac{1}{k f'(\psi_0)} \tag{3.8}$$

excluding a narrow phase region near the turning interval. The situation is similar for the contraction eigenvector :

$$\mathrm{tg} \left(\theta^- - \frac{\pi}{2}\right) \approx \frac{1}{k f'(\psi_0)} \tag{3.9}$$

Table 1

KS-entropy

k	Singular perturbation (1.5)			Analytical perturbation (3.12)	
	Exper.	Theory (3.11)	Simplified theory : $h \approx (\ln k) - 1$	Exper.	Simplified theory : $h \approx \ln k/2$
6.2	0.958	0.909	0.826	1.157	1.133
14.0	1.654	1.655	1.639	1.949	1.946
25.0	2.241	2.225	2.219	2.537	2.526
50.0	2.914	2.913	2.912	3.227	3.219
100.2	3.615	3.608	3.607	3.914	3.914
142	3.938	3.955	3.956	4.234	4.263
200	4.308	4.298	4.298	4.603	4.605
1 000	5.926	5.908	5.908	6.206	6.215

gives an idea of experimental accuracy for the entropy. The agreement with theory is unexpectedly good, especially for small k. This result proves, in our opinion, that we have got, indeed, the stochastic mapping, in accordance with the above hypothesis the stochastic component occupying practically all the phase square.

An additional check of ergodicity consisted of investigating the uniformity of stochastic trajectory distribution over the phase plane. For this the phase square was subdivided into $N_1 = 128 \times 128 = 16\,384$ cells, and the number of trajectory crossing each of the cells (n_i) was counted. A criterion of uniformity used was the variance :

$$D = \langle (n_i - M)^2 \rangle \quad \text{where} \quad M = \langle n_i \rangle = n/N_1$$

is the mean number of crossing ; n stands for the number of iterations and averaging was carried out over all the cells. The expected value of D is : $D/M = 1 \pm \sqrt{2/N_1} = 1 \pm 0.011$, the last term giving root-mean-square deviation. Experimental value for the singular perturbation (1.5) at $k = 16$ and $n = 10^7$ is : $D/M = 1.017$. The probability of such a deviation is about 12 %.

Finally the stochasticity was further checked by spying the process of crossing the phase plane cells by the trajectory. At the beginning of random motion there is a number of missed cells (N_0) to remain which can be avaluated according to standart Poisson distribution :

$$N_0 = N_2\, e^{-M} \pm \sqrt{N_0} \tag{3.15}$$

sequence of motion. Two islets lie in the turning interval (1.2), one of them (1) being strongly prolate along the extension eigenvector (Fig. 2,b). Another islet (3) lies in the extension interval and is strongly prolate along the contraction eigenvector (Fig. 2,c).

In Tables 3,4 the data on the total area of stable component is given. The area's sharp dying out with k corresponds, roughly, to the estimate [11] :

$$S(k) \sim \left(\ln\frac{k}{2}\right)^{-2} \exp\left[-3\left(\ln\frac{k}{2}\right)\left(\sqrt{\frac{\pi k}{2}}-1\right)\right] \tag{4.6}$$

Table 3

Stable component (missed cells) for singular perturbation (1.5)

k	4	8	16
Number of missed cells	42 038	60	0
Area	0.08	1.1×10^{-4}	0

Total number of phase plane cells : 512 x 1 024 = 524 288.

Table 4

Stable component for analytical perturbation (3.12)

k	3.67	4.78	5.98	8.64	10.5
Number of missed cells	48 958	10 292	1 681	24	0
Area	0.093	0.02	0.0032	4.6×10^{-5}	0

Total number of phase plane cells : 512 x 1024 = 524 288.

The latter is based on Sinai's evaluation [15] for the number of periodical trajectories of a stochastic system :

$$\nu(T) \sim e^{h(T-1)} \tag{4.7}$$

Most of them are unstable, of course. However, some of the trajectories happen to cross the turning interval and may become stable (Fig. 2).

All the periodical trajectories generally form an everywhere dense set. It is not excluded that the set of stable regions is also everywhere dense (Sinai's hypothesis). Nevertheless, their total measure might be small (for $k \gg 1$). To clarify the question additional experiments are needed.

where $N_2 = 512 \times 1\,024 = 524\,288$ is the total number of phase plane cells. Results of this experiment are presented in Table 2.

Table 2

Filling up the phase plane by stochastic trajectory

n	10^7	2×10^7	3×10^7	4×10^7	5×10^7
N_0, experiment	11 531	258	6	1	0
N_0, Poisson distribution (3.15)	11 500 ± 107	251 ± 16	5.8 ±2.4	0.12 ± 0.35	3×10^{-3} ± 0.05

Trajectory processing was done in 5 iterations ; k = 16.

§ 4. ISLETS OF STABILITY

Now we are going to discuss the question : does the stochastic component occupy the whole phase square ? Apparently no, at least, not always.

First of all, however large the parameter k may be there are always such special k values for which stable regions, or, as we use to say, "islets" of stability, exist [11]. The largest "islets" correspond to fixed points, i.e. to the periodical motion with period T = 1.

Let us consider the latter case in more detail [11]. For a fixed point (φ_0, ψ_0) of the mapping (1.3) we have :

$$\varphi_0 = 0 \ ; \ k f(\psi_0) = r \tag{4.1}$$

where r is any integer. The fixed point is stable if (3.5) :

$$-4 < k f'(\psi_0) < 0 \tag{4.2}$$

The special (stable) values of k are determined by the compatibility (4.1) and (4.2). The latter takes place within the interval :

$$\Delta k \approx \frac{8}{k f(\psi_1),\ f''(\psi_1)} \ ; \ f'(\psi_1) = 0 \tag{4.3}$$

around the k value corresponding to the centre of the turning interval (4.2) and being equal to :

$$k_r = \frac{r}{f(\psi_1)} - \frac{2}{r f''(\psi_1)} \tag{4.4}$$

One can easily see that for a large k $\psi_0 \approx \psi_1$ and the size of stable region :

$$\Delta\varphi \sim \Delta\psi \sim 4/k f'' \qquad (4.5)$$

A more general case of stability was investigated also by Sinai and Dunskaya (see Supplement in [13]).

In Figure 1 a picture of motion for a special $k = 12.928203$ is shown. One can clearly see the stable region which the stochastic component cannot penetrate into.

There are possible, however, more intricate stable regions leaving the turning interval. An example of such a region is given in Figure 2. One can clearly see two islets of stability (1.3) on the phase map of the whole phase square (Fig. 2,a). An additional investigation has shown that there is one more islet denoted in Figure 2,a by dashed-line (2). It is narrower than a phase plane cell (1/32 x 1/32) and therefore remains unseen. The period of motion in this case is T = 3, and the figures on the phase map show the

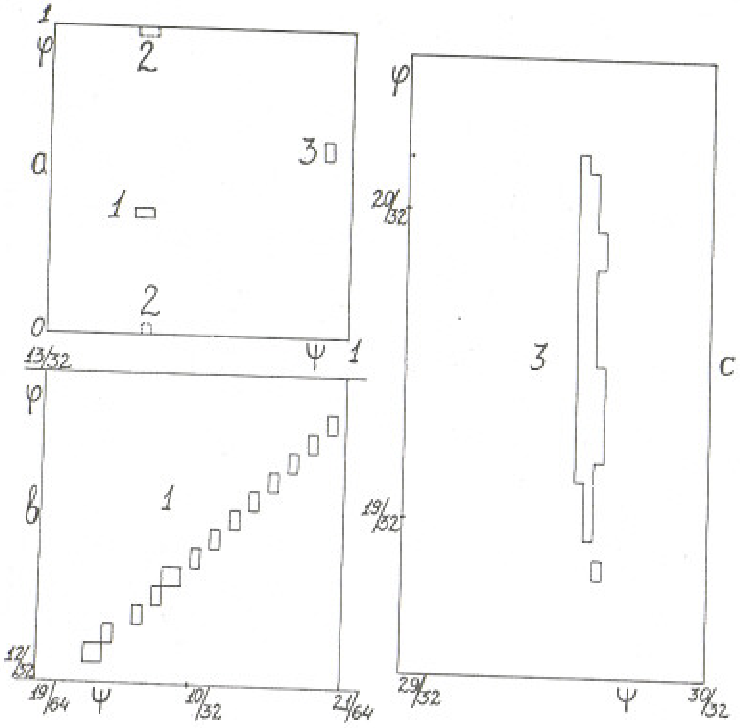


Fig. 2. – Phase map of the mapping in Fig. 1 with k = 8 : a - the whole phase square (32 × 32 subdivision); b, c - some sections of the phase plane (512 × 1024 subdivision).

We still have some doubt as to whether the stable component may not be so "fine" to escape observation (like one of islets in Figure 2). The question may be put another way : would the visible area of the stable component grow with decreasing the size of the phase plane cell ? We think that in any event the stable component cannot occupy considerable area at $k \gg 1$. This is indicated by the following gross experiment. We have got in all about 100 runs with $k \gg 1$, in no one of them entering a stable region.

§ 5. THE BORDER OF STOCHASTICITY

We use this term for both a curve separating stable and stochastic regions on the phase plane, and the critical values of system parameters at which the motion becomes stochastic. The latter can be obtained by investigating the local stability of motion (see, for example, [8, 10, 6, 7]). For mapping (1.3) and $f'(\psi) \sim 1$ we find from (3.5) :

$$k_s \sim 4 \qquad (5.1)$$

in resonable agreement with experimental data [6, 7].

Attempts to follow the border of stochasticity on the phase plane showed immediately that a very complicated transitional zone does exist [11, 12]. This is of little surprise since the border-line of stochastic region is to be itself a stochastic curve. The complexity of transitional zone structure is demonstrated in Figure 3 where a number of trajectories with different initial conditions is shown. Some of them are stable while other fill up broad stochastic layers. The most peculiar feature of the motion is, to our mind, overlapping stochastic layers (cross-shaded areas) related to different trajectories. Let us note that the size of a phase plane cell in Figure 3 is about $3.10^{-5} \times 6.10^{-5}$.

For a detailed study of transitional zone the use was made of the canonical mapping :

$$\begin{aligned} \bar{\varphi} &= \varphi - \psi^3 \\ \bar{\psi} &= \psi + \bar{\varphi} \end{aligned} \qquad (5.2)$$

which simulates an islet of stability around the origin, the size of the islet being :

$$\varphi_s \sim \psi_s \sim 1 \qquad (5.3)$$

These estimates are readily derived like (5.1) from the condition of local stability.

In computation of (5.2) we managed to squeeze the main code loop into the fast buffer memory of BESM-6 processor. Besides, the fixed-point

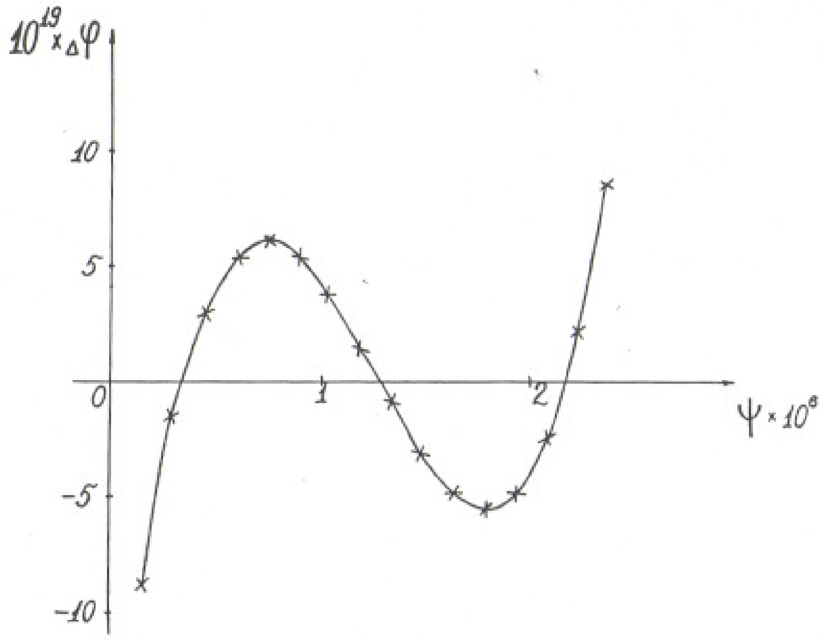


Fig. 4. – A piece of phase trajectory at the limit of stable region; mapping (5.2); 10^7 iterations.

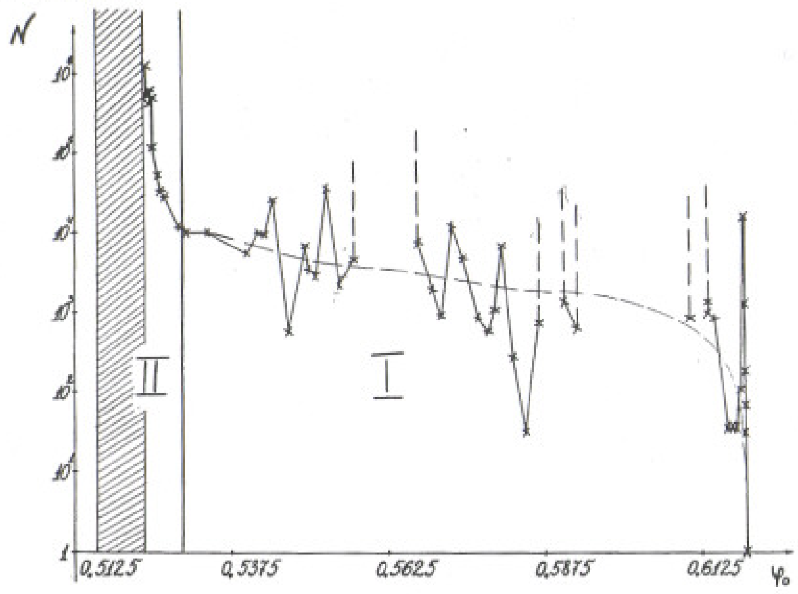


Fig. 5. – Dependence of the «lifetime» N (number of iterations) in transitional zone on initial φ_0 ($\psi_0 = 0$): I - transitional zone main region - the «plateau»; II - a barrier protecting the stable region; interruptions in the curve $N(\varphi_0)$ correspond to the internal region of nonlinear resonances; the hatched strip characterizes experimental uncertainty in the position of the border of eternal stability (5.4); mapping (5.2).

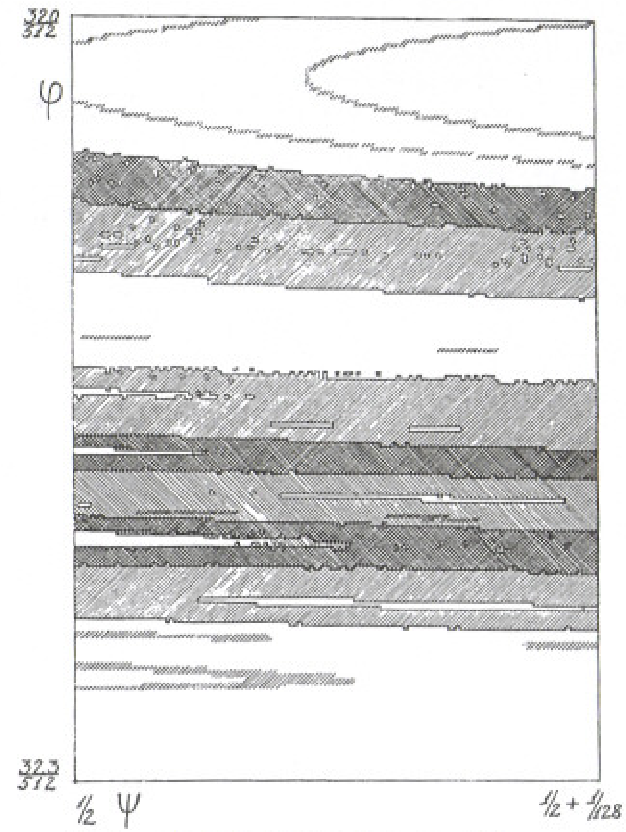


Fig. 3. – Transitional zone phase map of the mapping in Fig. 1 for various initial conditions; 10^7 iterations. The size of section shown is $3/512 \times 1/128$, it is subdivided into 374×128 cells.

arithmetic was used. As a result the computation time was cut down to 7 μsec per iteration.

The main objective in study of transitional zone was to locate the so-called border of eternal stability which had been predicted by the KAM-theory (the theory by Kolmogorov – Arnold – Moser [16]). Our experimental results show that this border coincides, in order of magnitude, with the border of stochasticity (5.3). According to [12], the border of eternal stability for mapping (5.2) lies somewhere in the interval :

$$0.517 < \varphi_0 < 0.524 \; ; \; \psi_0 = 0$$
$$0.77 < \psi_0 < 0.78 \; ; \; \varphi_0 = 0 \qquad (5.4)$$

Already the first experiments showed that in immediate vicinity of this border motion is stable to the accuracy of round-off errors (*). For example, at the trajectory with initial conditions $\varphi_0 = 0$; $\psi_0 = 0.67$ (comp. (5.4)) only a weak diffusion was observed with the rate

$$D_{exp} \approx 3 \times 10^{-27}$$

It is interesting to mention that for random round-off errors the diffusion rate would be a 100 times more : $D_r \approx 3 \times 10^{-25}$. This proves once more that, generally speaking, one should not consider round-off errors as a random perturbation since they are determined completely by a definite computer algorithm.

We have proceeded with the experiments using double precision computation to decrease a single round-off error down to $\sim 10^{-24}$ [12]. This was achieved at the expense of computation time which rised up to about 200 μsec per iteration.

Experimental diffusion rate at a stable trajectory dropped down to $D_{exp} \approx 2 \times 10^{-51}$ which was again about 150 times less than for random runding-off.

Figure 4 shows a tiny little piece of a stable trajectory running in an immediate vicinity of stochastic region (initial conditions : $\varphi_0 = 0.516$; $\psi_0 = 0$). The difference $\Delta\varphi$ between the trajectory and the interpolation quadratic curve has been computed and plotted (Fig. 4) to reduce trajectory oscillations. We see that all 16 experimental points (of total number 10^7 !) are fitted to a smooth curve –stable phase trajectory– with the accuracy $\sim 10^{-20}$.

Inspite of complicated structure the transitional zone has rather sharp boundaries as seen from Figure 5. Here the number of iterations until leaving the transitional zone for stochastic region is plotted versus the initial φ_0 ($\psi_0 = 0$) [12].

Similar transitional zones have been observed also around the islets of stability for mapping (2.1). An example of motion in transitional zone of the latter mapping is shown in Figure 6 while Figure 7 gives a section of the same zone with magnification x 16.

§ 6. A WEAK DISSIPATION

In this section we will give some preliminary results of recent experiments concerning the impact of a weak dissipation on stochastic component.

(*) A single round-off error was $\sim 10^{-12}$.

We mean the dissipation of both signs, that is to say, oscillation damping as well as autooscillations. A model mapping was chosen (comp. (1.3)) :

$$\begin{aligned} \overline{\varphi} &= \{\varphi + k f(\psi) - \epsilon(\varphi - \varphi_0)\} \\ \overline{\psi} &= \{\psi + \overline{\varphi} - 1/2\} \end{aligned} \qquad (6.1)$$

New parameter ϵ characterizes the rate (inverse number of iterations) of exponential "damping" to $\varphi = \varphi_0$ or to a sufficiently "strong" nonlinear resonance around periodical trajectory.

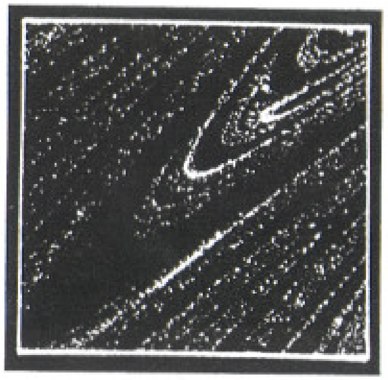
The latter process may lead to a "cupture" of stochastic trajectory into an islet of stability, and, hence, to its degeneration into a periodical trajectory. Our experiments show that such a cupture does use to happen, indeed. However, the "lifetime" of stochastic component has proved to be fairly long, and it rises with decreasing ϵ and, especially, with increasing k. For example, at special $k \approx 3.46$ and $\epsilon = 10^{-4}$ the cupture occurs in 7×10^4 iterations (in average, with very big fluctuations due to the initial conditions). If $\epsilon = 5 \times 10^{-4}$ the latter number drops to 1.5×10^4, i.e. about inversely proportional to ϵ. The stable area in this case is equal to 0.12. For $k = 12.9$ and $\epsilon = 6 \times 10^{-4}$ (the stable area is about 10^{-2}) the capture occurs in 1.4×10^5 iterations, i.e. about inversely proportional to the stable area. If ϵ rises 4 times as much stochasticity "lifetime" goes down about 5 times, again nearly in proportion to $1/\epsilon$.

For a small dissipation the time of capture is much longer than that of getting into the transitional zone of a stable region. Spying the motion on display we have clearly seen the system coming many times to an islet of stability, penetrating into the transitional zone, staying there hundreds iterations, and then coming back to the stochastic region. An example of motion inside a narrow transitional zone is given in Figure 6.

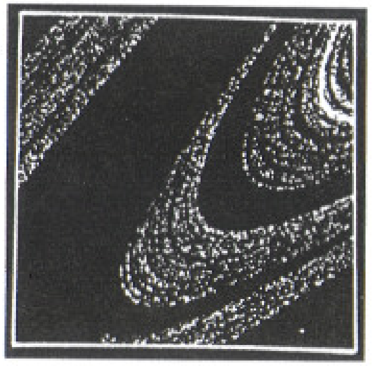
For $k \approx 3.46$ the system returns to the transitional zone in about 10^4 iterations independently of ϵ and with much less fluctuations than for the "lifetime". If $k \approx 12.9$ this number increases to 15×10^3.

The ergodicity of motion implies that the longer system stays inside the transitional zone the more seldom does it come back into the corresponding region. Hence, a kind of barrier arises at the internal edge of transitional zone (see Fig. 5) which considerably delays the penetration into the stable region, provided the dissipation is small enough.

Let us mention in conclusion an interesting observation : under a fairly strong dissipation ($\epsilon \sim 0.1$) a kind of "foliation" of the phase plane occurs (Fig. 8), that is to say, the formation of a set of strips which the phase point keeps off. It is very likely that such a foliation, more and more fine though, takes place for any dissipation whatever small it may be. Could this "reduction" of stochastic component prevent it from being captured into islets of stability and provide, thus, the stationary stochasticity ? The strong dissipation certainly improves the "stability" of stochastic



8 a



8 b

Fig. 8. – A picture of motion for the mapping in Fig. 6; k = 9.76; ϵ = 0.3; 2 $\times 10^4$ iterations: a - the whole phase square; b - a section with magnification $\times 4$.

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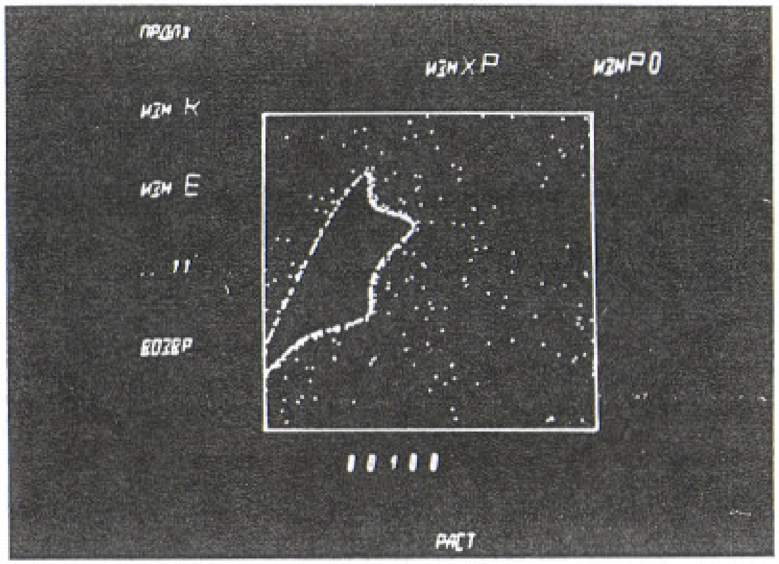

Fig. 6. – A picture of motion in the transitional zone of mapping (2.1) with singular perturbation: k = 3.46; $\epsilon = 10^{-4}$; 400 iterations (display, light buttons are seen around the phase square).

component against the capture, we have been observed this, But does it provide, under certain conditions, the eternal stochasticity in spite of dissipation ?

That is the Question !

We are deeply indebted to the Nature for inexhaustible diversity of Its simplest Revelations.

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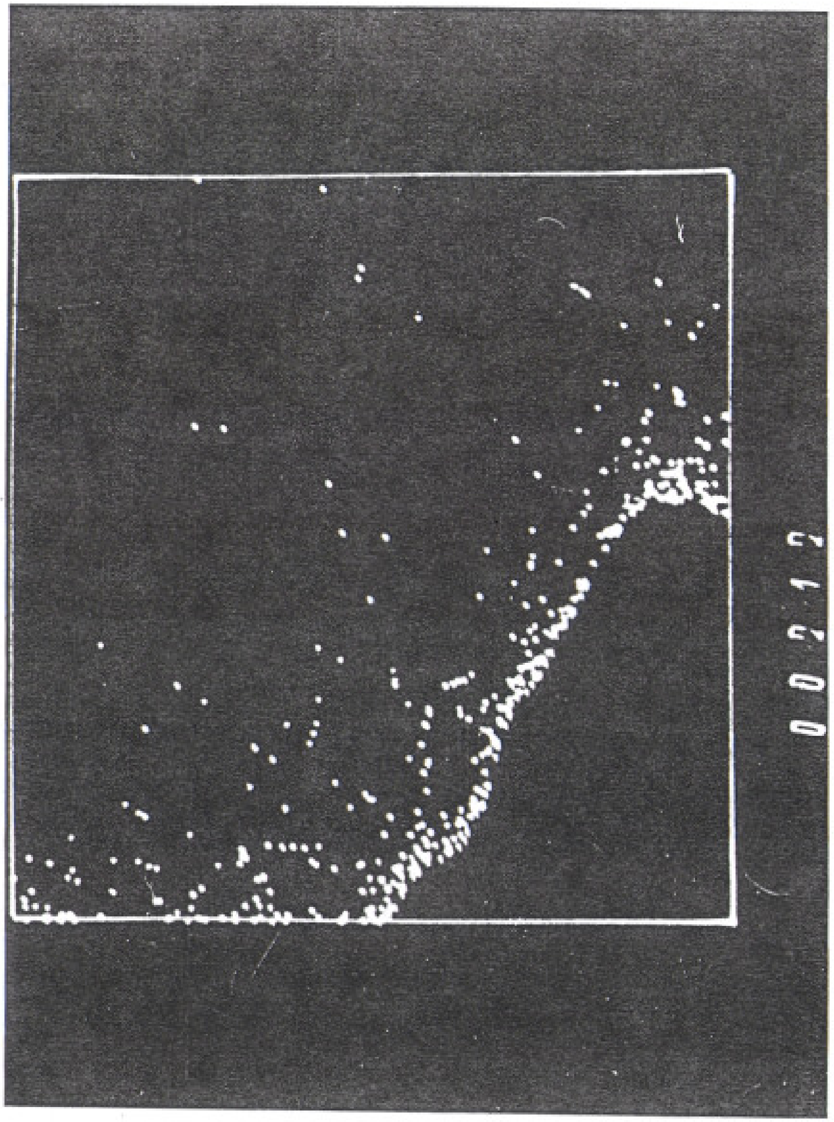

Fig. 7. – A section of the transitional zone in Fig. 6 with magnification x 16.

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