# Stability of the motion of a charged particle in a magnetic confinement system

→ "Diffusion" Model for nonadiabatic loss in a magnetic mirror.

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Recent work on the stability of the motion of a single charged particle in an axisymmetric mirror system is reviewed. A semiempirical condition for the overlap of nonlinear resonances leads to a reasonably accurate estimate of the boundary for the stochastic instability and for the rate of the diffusive change in the magnetic moment of the particle in the instability region. These estimates are compared with numerical calculations of the trajectories of charged particles in magnetic confinement systems.

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One of the basic directions in research on controlled fusion is the confinement of hot plasmas in "open" magnetic systems. In the Soviet Union, work in this direction began with the papers by G. I. Budker, who suggested confining a plasma in a confinement system with magnetic mirrors ("corks") (Ref. 1; see also Ref. 2). A necessary but – very unfortunately – far from sufficient condition for successful plasma confinement in magnetic systems is stability of the motion of an individual charged particle. The longitudinal motion of the particle is of course governed by the effective potential energy

$$U_\perp = \mu B(s), \tag{1}$$

where B is the magnetic field, s is the coordinate along a line of force,

$$\mu = \frac{v_\perp^2}{2B} = \frac{v_\perp^2}{2\omega} = \frac{v^2 \sin^2 \beta}{2\omega} \tag{2}$$

is the orbital magnetic moment of the particle, $\beta$ is the angle between the velocity **v** and the field **B**, and $\omega$ is the gyrofrequency. We are using a system of units in which $\gamma m = e = c = 1$, where $\gamma = (1 - v^2/c^2)^{-1/2}$. Written in this form, all the equations in this paper are valid for any velocity of the charged particle.

If the magnetic field has a maximum along a line of force, and $\mu$ = const, the particle executes stable longitudinal oscillations. The magnetic moment of the particle, however, which is proportional to the action variable for Larmor rotation, is not an exact integral of motion, but simply a "adabatic invariant," which is approximately conserved under certain special conditions. Before work on magnetic confinement systems was carried out, the accuracy of the conservation of the adiabatic invariants was not actually studied, and even the qualitative conditions for the conservation of such invariants were formulated in an extremely hazy manner. The basic condition was believed to be that the properties of the system vary "slowly" in comparison with the characteristic frequencies of the system. For this particular problem, this condition means that the change in the gyrofrequency $\omega(s)$ due to the longitudinal oscillations of the particles must be slow. We now know that this condition is generally not correct, even in a qualitative sense. Credit must be given to the insight and physical intuition of G. I. Budker, who recognized that the problem of the conservation of the magnetic moment of a particle is an extremely subtle theoretical problem. Wisely, he resorted to experiment, formulating a beautiful experiment with tritium in a magnetic mirror confinement system. This experiment, carried out by Rodionov,[3] and others, carried out at Livermore,[3] showed that the magnetic moment is conserved well enough that certain minimal requirements of the magnetic confinement system could be satisfied. The many plasma instabilities which were subsequently discovered drew attention away from the problem of the stability of the motion of an individual charged particle. More recently, however, interest has rekindled in this problem (see, for example, Ref. 4), for two reasons: First, much progress has now been made in suppressing the plasma instabilities. Second, we would like a more accurate estimate of the minimum necessary magnetic fields and dimensions of the confinement systems, since these systems are now becoming extremely complicated and expensive engineering

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projects. In addition, a new field of application of the theory of the particle motion in a magnetic mirror system has arisen: the van Allen belts. For the very energetic protons in these belts, the accuracy with which the magnetic moment is conserved is apparently a decisive factor governing the motion (see, for example, Ref. 5).

In principle, the problem of the stability of the motion of a charged particle in an axisymmetric magnetic mirror confinement system was solved by Arnol'd.[6] Using the perturbation theory proposed by Kolmogorov and worked out by Arnol'd and Moser (the KAM theory), Arnol'd proved rigorously that if the magnetic field was strong enough the charged particle would be trapped in it for an indefinitely long time. Braun[7] reported an analogous proof of this permanent stability of the motion of a particle in a magnetic mirror confinement system. Arnol'd's proof is of fundamental importance to the problem, but for practical applications it is important to have an accurate estimate of the stability boundary. This estimate cannot be found using the rigorous KAM theory because of technical difficulties. In the present paper we review the recent work in which a different approach is taken to the problem, via a semiempirical condition for the stability of the motion in terms of the "overlap of nonlinear resonances."[8-10]

This approach grew out of attempts to solve the problem of the conservation of magnetic moment, formulated by Budker. For many years the present author benefited from Budker's invaluable counsel on these and many other questions of modern physics.

## 1. RESONANT PROCESSES IN MAGNETIC CONFINEMENT SYSTEMS

The role played by resonances in the changes in the adiabatic invariant was apparently first pointed out by Andronov et al.[11] They examined the simple case of a linear oscillator with a sinusoidally modulated frequency. In this system we know that for any integer

$$n = 2\bar{\omega}/\Omega_m, \tag{1.1}$$

where $\bar{\omega}$ is the average oscillator frequency and $\Omega_m$ is the modulation frequency, there is a parametric resonance, and the action of the oscillator changes in an unbounded manner. Since $\Omega \to 0$ in the limit $n \to \infty$, this simple example shows that the fact that a perturbation is slow does not in itself guarantee conservation of the adiabatic invariant. The resonances are the crucial considerations. The role played by resonant processes becomes particularly apparent in quantum mechanics, according to which a change in the adiabatic invariant implies transitions between unperturbed levels of the system.

The resonant processes in a magnetic mirror confinement system were studied in Ref. 8. Below we begin with the case of an axisymmetric magnetic field, and for simplicity we assume that the magnetic field varies quadratically along a line of force:

$$B(s) = B_0\left(1 + \frac{s^2}{l^2}\right), \tag{1.2}$$

where $l$ is a constant – a measure of the characteristic dimension of the magnetic field inhomogeneity (the field doubling length). In the approximation $\mu$ = const, the longitudinal motion of the particle is described by the Hamiltonian

$$H(p_\parallel, s) = \frac{p_\parallel^2}{2} + \mu B(s); \quad p_\parallel = v_\parallel = \dot{s}. \tag{1.3}$$

Using (1.2) we see that the longitudinal oscillations are sinusoidal with a frequency[1)]

$$\Omega = \frac{\sqrt{2\mu B_0}}{l} = \frac{v}{l}\sin\beta_0. \tag{1.4}$$

The changes in $\mu$ result from resonances of these oscillations with the Larmor rotation. The resonance condition is

$$n = \frac{\bar{\omega}}{2\Omega} = \frac{l}{4\rho_0}\cdot\frac{1+\sin^2\beta_0}{\sin^3\beta_0}, \tag{1.5}$$

where n is any integer greater than zero, $\rho = v/\omega$ is the "total Larmor radius," and the average value of the Larmor frequency over a period of the longitudinal oscillations is

$$\bar{\omega} = \frac{\omega_0}{2}\left(1 + \frac{1}{\sin^2\beta_0}\right). \tag{1.6}$$

Since the magnetic field is symmetric with respect to the median plane, the frequency of the perturbation due to the longitudinal oscillations is $2\Omega$ in (1.5). The straight lines in Fig. 1 show the positions of the resonances in an axisymmetric magnetic confinement system.

Although both the longitudinal and radial oscillations (the Larmor revolution) are sinusoidal for a magnetic field of the type in (1.2), their frequencies ($\Omega$, $\bar{\omega}$) depend on the magnetic moment [see (1.4) and (1.6)]. In this situation we speak in terms of nonisochronous or nonlinear oscillations. Significantly, the ratio of the frequencies ($\bar{\omega}/2\Omega$) also changes with a change in $\mu$ or in the slope of the velocity, $\beta_0$, in (1.5). Accordingly, an individual nonlinear resonance simply leads to bounded oscillations of $\mu$ and correspondingly of the frequencies $\Omega$ and $\bar{\omega}$. These oscillations are shown schematically by the arrows in Fig. 1; they are usually called "phase" oscillations.

Another qualitative explanation can be offered for the mechanism of the phase oscillations: As a particle moves in a constant (over time) axisymmetric magnetic field, there are two exact integrals of motion: the energy, or the velocity modulus

$$v = \text{const}, \tag{1.7}$$

and the angular momentum, which can be written approximately as

$$M \approx \frac{\omega r_c^2}{2} - \mu = \frac{\omega}{2}(r_c^2 - \rho_\perp^2) \approx \text{const}, \tag{1.8}$$

where $r_c$ is the distance from the Larmor center to the symmetry axis of the field, and $\rho_\perp = v_\perp/\omega$ is the Larmor radius. These integrals define a four-dimensional hypersurface in the six-dimensional phase space of the system. The particle moves along this hypersurface. Since the frequencies $\Omega$ and $\bar{\omega}$ are independent of the (three) coordinates of the particle, the projection of this integral hypersurface onto the frequency plane is a curve, shown





$B(s) = B_0\left[1 - a\cos\left(\frac{2\pi s}{L}\right)\right]$

...tically by the dashed curve in Fig. 1. In the case ... we conclude from (1.8) that $r_c \approx$ const and thus ... const. Then from (1.4) and (1.8) we find, eliminating ... the following approximate equation for the integral curve in Fig. 1:

$$\bar{\omega} \approx \frac{\omega_0(r_c)}{2}\left(1+\frac{v^2}{l^2\omega^2}\right) \quad (1.9)$$

This expression depends on two parameters of the particle motion: v and $r_c$.

In general, and for this problem in particular, the integral curve intersects the resonance lines. Accordingly, each individual resonance can lead to only bounded phase oscillations. If the resonance perturbation is large, however, the regions of phase oscillations of neighboring resonances will overlap (Fig. 1), so there is the possibility that the particle may "wander" somewhat among the resonances. In particular, for a displacement to the left along the invariant curve in Fig. 1 the magnetic moment decreases, and the particle ultimately escapes through the magnetic mirrors. The boundary of the instability which occurs in this manner is determined for the overlap of the nonlinear resonances. This condition was formulated in Ref. 8 for the present problem, and it has subsequently been generalized to several other cases of nonlinear oscillations.[9,10]

## 2. CHANGE IN THE MAGNETIC MOMENT OF THE PARTICLE IN THE CONFINEMENT SYSTEM

The typical behavior of $\mu$ as a function of time during one half-period of the longitudinal oscillations of the particle in the confinement system (from one reflection to the next) is shown schematically in Fig. 2. It is clear that there are two qualitatively different changes in $\mu$: oscillations ($\Delta\tilde{\mu}$) at the Larmor revolution frequency and a more or less rapid change ("jump"), $\Delta\mu_1$, as the median plane is crossed (t = 0 in Fig. 2). This behavior of $\mu$ was found in the very first numerical simulation of particle motion in magnetic confinement systems[12] (see also Ref. 13).

The rate of change of the magnetic moment is[13]

$$\dot{\mu} = \frac{v_\perp}{BR_c}\left(v^2-\frac{v_\perp^2}{2}\right)\sin\Phi - \frac{v_\parallel v_\perp^2}{2B^2}\frac{\partial B}{\partial s}\sin 2\Phi, \quad (2.1)$$

where $R_c$ is the radius of curvature of a magnetic line of force, and $\Phi$ is the perturbation phase, which is related to the Larmor phase $\theta$ by the approximate relation[10]

$$r\sin\Phi \approx r_c\sin\theta. \quad (2.2)$$

Equation (2.1) is exact; it holds, in particular, for trajectories which enclose the symmetry axis of the field ($r_c < \rho_\perp$).

Since the perturbation field $\Phi$ is fast (it varies at the Larmor frequency $\omega$), the oscillations of $\mu$ are given in order of magnitude by

$$\frac{\Delta\tilde{\mu}}{\mu} \sim \frac{\rho}{l} \sim \frac{\Omega}{\omega} \sim \varepsilon \ll 1, \quad (2.3)$$

where we are assuming $R_c \sim l$ for simplicity. The small

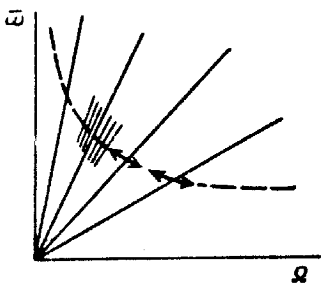

FIG. 1. Overlap of nonlinear resonances in the $\bar{\omega}$, $\Omega$ frequency plane. Solid lines) resonance lines, $\bar{\omega} = 2\Omega n$; dashed curve) projection of the surface of the integrals of motion (v = const; M = const) onto the frequency plane; arrows) phase oscillations at the nonlinear resonances. This splitting results from the azimuthal inhomogeneity of the magnetic field (Sec. 7).

dimensionless quantity $\varepsilon$ is usually called the "adiabatic parameter." This parameter is defined more accurately below.

Significantly, the oscillations of $\mu$ in (2.3), which are generally not small, do not grow in time. An accurate proof of this assertion runs into certain difficulties. The proof was first given by Krushkal,[14] who managed to analyze all orders of the asymptotic perturbation theory in the small parameter $\varepsilon$ in (2.3).

It is possible, however, to simply bypass the oscillatory part of the variation in $\mu$, by transforming from the differential equations of motion of the particle to the so-called Poincaré mapping or difference equations which describe the states of the particle at certain finite time intervals (Sec. 4). For the present problem, the characteristic time interval is conveniently chosen as the half-period of the longitudinal particle oscillations ($\pi/\Omega$). This approach implies, in particular, a transformation from the continuous equation in (2.1) for $\dot{\mu}$ to the finite variation $\Delta\mu_1$ over a time $\pi/\Omega$.

To find $\Delta\mu_1$ we must simply integrate (2.1) over a half-period of the longitudinal oscillations. The integrand [the right side of (2.1)] corresponds to high-frequency oscillations ($\omega$) with a low-frequency ($\Omega$) amplitude and phase modulation. If the right side of (2.1) is an analytic function of t, and it of course always is for real fields, the integral $\Delta\mu_1$ will be an exponential function of the adiabatic parameter:

$$\frac{\Delta\mu_1}{\mu} \sim e^{-1/\varepsilon_a}. \quad (2.4)$$

The quantity $\varepsilon_a$, which determines the argument of the exponential function, is naturally adopted as the adiabatic parameter.

The functional relationship in (2.4) is easily derived on the basis of the following considerations: We consider the right side of (2.1) as a function of t and continue it into the plane of the complex variable $\tau = t + it_1$; that is, we simply make the replacement $t \to \tau$. If the function of a complex variable found in this manner is analytic in some band along the real axis, the integration path ($t_1 = 0$) can be displaced along the imaginary axis. Converting to the Larmor phase, $\theta \sim \omega t \to \omega\tau$, in (2.1), we see that this displacement leads to the appearance of an exponential





→ $r\sin\Phi \approx r_c \sin\phi$

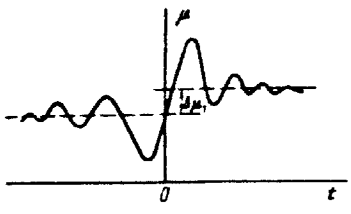


FIG. 2. Schematic behavior of the magnetic moment of the particle as a function of time. The median plane of the confinement system is crossed at t = 0.

factor:

$$\sin\theta \to \mathrm{Im}\,(e^{i\theta}) \sim e^{-\omega t_1}. \tag{2.5}$$

The magnitude of the displacement, $t_1$, is governed by that singularity of the integrand which is nearest the real axis. For example, for the "quadratic" field in (1.2) this singularity is the zero of the magnetic field at some point $\tau_p$: $B(\tau_p) = 0$ [see (2.1)]. Since the "jump" $\Delta\mu_1$ actually occurs near the median plane (at the minimum value of B; Fig. 2), we can set $s \approx v_\parallel t \to v_\parallel \tau = \tau v \cos\beta_0$ and $\theta \approx \theta_0 + \omega_0 t \to \theta_0 + \omega_0 \tau$. Hence $\tau_p \approx il/(v\cos\beta_0)$ and $\Delta\mu_1 \sim \mathrm{Im}\,(e^{i\theta_p}) \sim e^{-1/\varepsilon_a}\sin\theta_0$, where the adiabatic parameter satisfies

$$\frac{1}{\varepsilon_a} \sim \frac{l\omega_0}{v\cos\beta_0} = \frac{l}{\rho_0\cos\beta_0}. \tag{2.6}$$

Finally, we can write

$$\frac{\Delta\mu}{\mu} = \xi\sin\theta_0;\quad \xi = Ae^{-1/\varepsilon_a}, \tag{2.7}$$

where A is some coefficient which depends on the parameters of the system (see below). The quantity $\xi$ is the actual small parameter of this problem of the accuracy with which the adiabatic invariant is conserved.

In the estimate for $\varepsilon_a$ we used only the first term in (2.1), which is related to the curvature of a magnetic line of force. The second, higher-frequency term ($2\omega$) leads only to an exponentially small correction ($\sim e^{-2/\varepsilon_a}$), and we will ignore it below. For the same reason, the variation of the right side of (2.1) with t can be found in only the zeroth approximation, that is, for $\mu$ = const.

The behavior in (2.7) was found empirically (in numerical simulations) many years ago.[12] The first accurate calculation was apparently that of Ref. 15. Curiously, the equation for $\Delta\mu_1$ in Ref. 15 was found there through a solution of the quantum-mechanical problem of the motion of a magnetically confined electron in the semiclassical approximation.

A simple approximate equation for $\Delta\mu_1$ was derived in Ref. 13 by directly integrating (2.1):

$$\frac{1}{\varepsilon_a} = \frac{2}{3}\,\frac{l}{\rho_0\cos\beta_0}. \tag{2.8}$$

The equation for the coefficient of the exponential function found there, however, was unnecessarily complicated. Furthermore, as was shown later,[16] the results of Ref. 13 hold only for small values of $\beta_0$ and only for magnetic fields which are similar to the "quadratic" field in (1.2). In the latter case, the result of Ref. 13 was generalized to arbitrary angles $\beta_0$ by Krushkal'.[16] According to Ref. 16, the adiabatic parameter in this case is

$$\frac{1}{\varepsilon_a} = \frac{l}{2\rho_0}\psi(\beta_0);\quad \psi = \frac{1}{\sin^2\beta_0}\left[\frac{1+\sin^2\beta_0}{2\sin\beta_0}\ln\frac{1+\sin\beta_0}{1-\sin\beta_0} - 1\right]. \tag{2.9}$$

The quantity $\Delta\mu_1$ has been calculated more recently for a broader class of magnetic fields.[7] It should be noted that the ratio of (2.9) and (2.8),

$$\varkappa(\beta_0) = \frac{3}{4}\psi(\beta_0)\cos\beta_0, \tag{2.10}$$

is approximately equal to unity at angles up to $\beta_0 \approx 60°$ [$\varkappa(50°) = 0.9$; $\varkappa(60°) = 0.83$].

In the limit $\beta_0 \to \pi/2$ we have the approximate relations

$$\frac{1}{\varepsilon_a} \approx \frac{l}{\rho_0}\ln\frac{2}{\delta_0};\quad \xi \approx A\left(\frac{\delta_0}{2}\right)^{1/\varepsilon_a}, \tag{2.11}$$

where $\delta_0 = \pi/2$.

For the field in (1.2), or for sinusoidal longitudinal oscillations, the final equation for $\Delta\mu_1$ is, according to Ref. 16 (see also Ref. 10),

$$\xi = -\frac{3\pi}{4}\,\frac{r_0}{\rho_0\sin\beta_0}e^{-1/\varepsilon_a}. \tag{2.12}$$

Although a slightly different expression was derived for the coefficient of the exponential function in Ref. 17, the simpler version in (2.12) also agrees with the results of the numerical calculations of Ref. 17.

## 3. ANHARMONIC LONGITUDINAL OSCILLATIONS

Corrections to the adiabatic parameter for the anharmonic nature of the longitudinal oscillations, due to a deviation of the function B(s) from the quadratic function in (1.2), were also calculated in Ref. 16. In this case the equation for $\varepsilon_a$ becomes extremely complicated, but in cases of practical interest the anharmonic correction turns out to be small. The correction is largest at $\beta_0 = 0$, since the oscillations are most anharmonic in this case. For a field of the type ($\eta \ll 1$)

$$B(s) = B_0\left(1 + \frac{s^2}{l^2} + \eta\frac{s^4}{l^4}\right) \tag{3.1}$$

this maximum correction is ($\beta_0 = 0$), according to Ref. 16,

$$\frac{1}{\varepsilon_a} = \frac{2}{3}\,\frac{l}{\rho_0}(1+0.3\eta). \tag{3.2}$$

For example, for the field configuration

$$B(s) = B_0\frac{1-a\cos(2\pi s/L)}{1-a}$$

with the "mirror ratio" $k = (1+a)/(1-a)$, the correction to $1/\varepsilon_a$ is $0.3\eta = -1/10(k-1)$.

As another example we consider the field of a magnetic dipole. In this case, in the approximation of a "quadratic" field, we have

$$l = \sqrt{\left(\frac{2B}{B''}\right)_0} = \frac{\sqrt{2}}{3}R_0 = \sqrt{2}\,R_0 \tag{3.3}$$





Pitch angle is small ($\beta \simeq 4$)

$$\frac{1}{\varepsilon_a}=\frac{R_0}{3\sqrt{2}\,\rho_0}\psi(\beta_0), \tag{3.4}$$

where $R_0$ is the distance from the center of the dipole to the Larmor center in the median (equatorial) plane of the dipole. In particular, for $\beta_0 = 0$ we have the ratio $\rho_0 \cdot (R_0\varepsilon_a)^{-1} = 0.314$. When the correction in (3.2) is incorporated, this ratio increases to 0.352, that is, by about 12%. The exact value is[16] 0.325, or only 3.5% away from the approximation in (3.4).

According to Ref. 16, the change in $\mu$ in the dipole field can be written in the following form[2] (after a correction of the arithmetic error in Ref. 16):

$$\xi=\frac{8\pi}{3^{11/24}\Gamma(1/4)}\left(\frac{R_0}{\rho_0}\right)^{1/4}\frac{e^{-1/\varepsilon_a}}{\sin\beta_0}\approx 4.18\left(\frac{R_0}{\rho_0}\right)^{1/4}\frac{e^{-1/\varepsilon_a}}{\sin\beta_0}, \tag{3.5}$$

where $\varepsilon_a$ can be taken in the "quadratic" approximation, (3.4), as shown above.

The anharmonicity of the longitudinal oscillations also leads to changes in the frequencies $\Omega$ and $\bar{\omega}$. For the magnetic field configuration in (3.1) we have the approximate results (see, for example, Ref. 18)

$$\Omega\approx\frac{v\sin\beta_0}{l}\left(1+\frac{3\eta}{8}\operatorname{ctg}^2\beta_0\right),$$
$$\bar{\omega}\approx\frac{\omega_0}{2}\left(1+\frac{1}{\sin^2\beta_0}+\frac{3\eta}{4}\operatorname{ctg}^4\beta_0\right). \tag{3.6}$$

These expressions hold under the condition $\eta\cot^2\beta_0 \ll 1$; for a dipole field, $\eta = 32/81 \approx 1/3$ and $\beta_0 \gtrsim 35°$.

## 4. MAPPING

According to (2.7), the change in the magnetic moment over a half-period of the longitudinal oscillations, $\Delta\mu_1$, depends on the Larmor phase $\theta_0$ at the time at which the median plane is intersected. For a complete description of the motion we must also find the change in the phase between one intersection of the median plane and the next. This change can be written approximately

$$\Delta\theta_0=d(\mu)\approx\frac{\pi\bar{\omega}}{\Omega}=\frac{\pi}{2}l\omega_0\frac{1+v^2/2\mu\omega_0}{\sqrt{2\mu\omega_0}}, \tag{4.1}$$

where the last expression holds for the field in (1.2). We thus find the mapping $\mu$, $\theta_0 \to \bar{\mu}$, $\bar{\theta}_0$, where

$$\bar{\mu}=\mu+\xi\mu\sin\theta_0,$$
$$\bar{\theta}_0=\theta_0+d(\bar{\mu}). \tag{4.2}$$

The "phase advance" $d(\bar{\mu})$ is defined here as a function of the new value of the magnetic moment $(\bar{\mu})$ after crossing the median plane with phase $\theta_0$.

This is the Poincaré mapping, which describes the change in the state of the particle (in terms of the variables $\mu$, $\theta$) after a half-period of the longitudinal oscillations.

The resonances $\bar{\omega} = 2n\Omega$ [see (1.5) and Fig. 1] correspond to certain resonant values $\mu = \mu_n$, which can be found from the condition [see (4.1)]

$$d(\mu_n)=2\pi n. \tag{4.3}$$

We expand the function $d(\mu)$ near the resonant value and introduce a new variable (the impulse) I:

$$d(\mu)\approx d(\mu_n)+d_\mu'(\mu_n)(\mu-\mu_n)=d(\mu_n)+I. \tag{4.4}$$

This expansion is valid for $|\mu-\mu_n|/\mu < |\mu_{n+1}-\mu_n|/\mu \sim 1/n \ll 1$, that is, for resonances of a high harmonic of the longitudinal oscillations. We will see that these harmonics are present even in particle motion in the "quadratic" field in (1.2), because of the modulation of $\omega$ at the frequency of the longitudinal oscillations.

Using (4.4), we can replace the mapping in (4.2) by the following mapping, which has been linearized with respect to $\mu$:

$$\bar{I}=I+K\sin\theta,$$
$$\bar{\theta}=\theta+\bar{I}, \tag{4.5}$$

where we are omitting the subscript "0" from $\theta$. The sole parameter of this mapping is

$$K=\mu_n\xi(\mu_n)d_\mu'(\mu_n). \tag{4.6}$$

For the "quadratic" field in (1.2) we find from (2.12) and (4.1)

$$K=\frac{3\pi^2}{8}\frac{lr_c}{\rho_0^2}\frac{3+\sin^2\beta_0}{\sin^4\beta_0}e^{-1/\varepsilon_a}, \tag{4.7}$$

where $\varepsilon_a$ is given by (2.9). Analogously, for the field of a magnetic dipole we find, using (3.5) and the correction to the frequencies in (3.6),

$$K=2.23\left(\frac{R_0}{\rho_0}\right)^{5/4}\frac{1+1/2\sin^2\beta_0+\sin^4\beta_0}{\sin^4\beta_0}e^{-1/\varepsilon_a}. \tag{4.8}$$

## 5. STABILITY BOUNDARY

The properties of the mapping in (4.5) were studied in detail in Ref. 10, where this mapping was labelled the "standard" mapping since it arises in many problems of the theory of nonlinear oscillations.[10] Numerical simulations with (4.5) show[10] that the stability boundary corresponds to the following value of the mapping parameter:

$$K=K_1\approx 1. \tag{5.1}$$

The condition for the overlap of the first-approximation resonances leads to $K_1 \approx \pi^2/4 \approx 2.5$ (Ref. 10). When the second-approximation resonances and the stochastic layer of first-approximation resonances are taken into account, the agreement with the empirical value in (5.1) improves substantially: $K_1 \approx 1.05$ (Ref. 10). Below we will adopt the value in (5.1). In principle, this value can depend somewhat on N, the number of iterations of the mapping in (4.5). In the numerical simulations of Ref. 10 the value of N reached $10^7$; the variation in $K_1$ in this region is very slight. A rough estimate of the accuracy of the critical value in (5.1) is of the order of a few percent.[10]

The structure of the phase plane of the standard mapping in (4.5) is periodic not only in $\theta$ (with a period $2\pi$) but also in I (with the same period). It is therefore sufficient to analyze the motion of the system (4.5) on a $2\pi\times 2\pi$

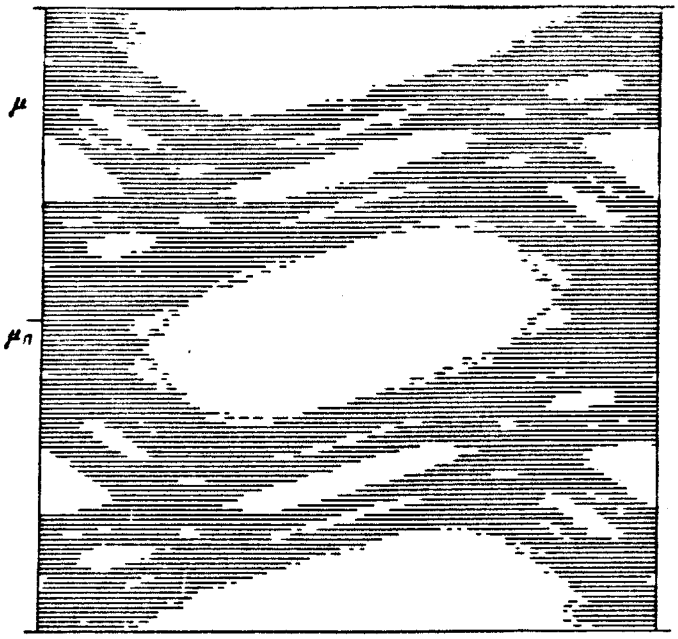


FIG. 3. Phase plane of the mapping in (4.5) for one trajectory after $10^5$ iterations. K = 1.16. One of the resonant values $\mu = \mu_n$ is indicated. The hatched region corresponds to an unstable (stochastic) variation in the magnetic moment of the particle.

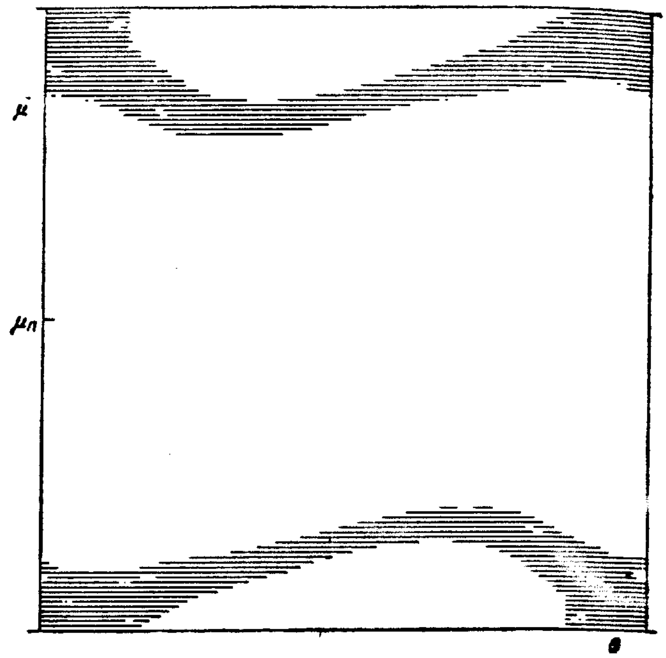


FIG. 4. The same as in Fig. 3, but for K = 0.96 ($10^6$ iterations). The stochastic motion in this case is confined to a narrow region along the separatrix of the nonlinear resonance. The variations of $\mu$ are strictly bounded.

phase square. The period over I corresponds to the distance between the adjacent resonances $\mu_n$ in (4.4). Figures 3 and 4 show examples of the phase diagram of the motion of system (4.5). The horizontal axis corresponds to one period of the Larmor phase (in the median plane of the confinement system), while the vertical axis shows two periods of the impulse I, which is proportional to $(\mu - \mu_n)$ [see (4.4)]. One of the resonant values $\mu = \mu_n$ is marked. The neighboring resonance coincides with the upper and lower edges of the figure, by virtue of the periodicity in I.

Figure 3 shows the region filled by one trajectory (hatched). This trajectory passes from one resonance to a neighboring resonance and, because of the periodicity in I, to any other resonance. The motion along I is thus unbounded; that is, this is an example of unstable motion (K = 1.16). Numerical simulations[10] show that the motion along I is irregular. The particle moves as if it were subjected to random forces, although the motion is actually described by the purely dynamic equations in (4.5). This motion has been labelled "stochastic" motion.[9,10]

In this case the motion can be described by a diffusion equation with the empirical diffusion coefficient[10]

$$D_I = \overline{(\Delta I)_1^2} \approx {}^1/_2 (K-1)^2, \tag{5.2}$$

which refers to a single iteration of the mapping in (4.5). Alternatively, for the magnetic moment [see (4.2)], we have

$$D_\mu = \frac{\overline{(\Delta\mu)^2}}{t} \approx \frac{\Omega\xi^2\mu^3}{2\pi}\left(1 - \frac{1}{K}\right)^2. \tag{5.3}$$

In the case $K \gg 1$ the diffusion coefficient corresponds to random and independent phases $\theta$. There is a "mixing" in terms of the phase $\theta$ (Ref. 9).

If the parameter K is only slightly larger than the critical value $K_1 = 1$, the unstable motion fills only a part of the phase plane, with an extremely unusual shape (Fig. 3). The rest of the phase plane (not hatched) corresponds to bounded oscillations of $\mu$.

Figure 4 shows an example of stable motion, for which the oscillations of $\mu$ are bounded for any initial conditions. In this case the trajectory does not move (at least in $10^6$ iterations) to an adjacent resonance (at the center of the figure; cf. Fig. 3). Nevertheless, the trajectory fills a region of finite width which is a more or less narrow layer running along the separatrix of the nonlinear resonance. Numerical and analytic study of the motion in this region shows[9,10] that it is stochastic; hence this layer is called the "stochastic layer." The fundamental distinction between this case and Fig. 3, however, is that the stochastic motion in the layer is bounded and causes only slight although irregular changes in $\mu$. Nevertheless, in an axially asymmetric magnetic confinement system the stochastic layers govern the stability of the motion (Sec. 8).

## 6. NUMERICAL SIMULATIONS

Let us compare these estimates of the stability boundary for the motion of a charged particle in a magnetic mirror confinement system with the results of numerical calculations of the trajectories. Figure 5 shows the results of such a calculation for the field of a magnetic dipole, taken from the data of Dragt.[20] The open circles show the parameters of "stable" trajectories, while the triangles show "unstable" trajectories. The calculations actually spanned only ~20 intersections of the median (equatorial) field plane, and the stability condition was the scatter of points on a phase plane like that in Figs. 3 and 4. Because of the complicated structure of the phase plane, this "local" stability condition does not guarantee the correct separation of trajectories into stable and unstable trajectories. Nevertheless, the stability boundary be distinguished quite clearly from Fig. 5. Its empirical position is shown by the dashed line

$$\frac{\rho_0}{R_0}=0.11\sin\beta_0. \tag{6.1}$$

The solid curve shows the estimate of the stability boundary based on the resonance-overlap condition (K = 1). The stability parameter K was calculated from Eq. (4.8) with $\varepsilon_a$ from (2.9). The agreement with the results of the numerical calculation can be judged satisfactory, particularly in view of the roughness of the empirical stability condition, as mentioned above. For the five stable points which stand out (Fig. 5) the average value is

$$\frac{\rho_0}{\rho_{cr}}=1.16, \tag{6.2}$$

where $\rho_{cr}$ corresponds to the theoretical boundary K = 1.

A more accurate comparison can be made by using the numerical data of Ref. 21 on the motion of particles combined by magnetic mirrors with the "quadratic" field in (1.2). Table I, taken from Ref. 10, shows the parameters of six unstable trajectories lying near the stability boundary. These trajectories were traced until they left the system (after N intersections of the median plane). The values of K listed in Table I were calculated from Eq. (4.7). In all cases we have K > 1, and it can be shown that (4.7) overestimates the stability parameter K. Actually, however, the boundary K = 1 corresponds to a very long particle lifetime in the system, while for the trajectories considered here the typical value is N ~ 300. According to Ref. 10, the value of K increases approximately in accordance with the empirical law

$$K_N\approx 1+(100/N)^{0.4}. \tag{6.3}$$

The values of $K_N$ are also listed in Table I. We now see that Eq. (4.7) underestimates K by an average of 30%, and the rms scatter in the values is ±10%. The last column of Table I shows the error in the calculation of the critical Larmor radius $\rho_0$ from Eq. (4.7) with correction (6.3):

$$\frac{\delta\rho_0}{\rho_0}=\frac{(K_N-K)/K_N}{1/\varepsilon_a-2}. \tag{6.4}$$

The error is apparently due mostly to the error of (2.12) for $\Delta\mu_1$ when the Larmor radius $\rho_\perp$ becomes comparable to the characteristic dimensions of the field, $l\approx 130$ cm in the present case.[22]

We wish to point out that the first three trajectories listed in Table I encircle the axis of the system ($r_c < \rho_\perp$), so that the theory derived above is also applicable in this case. Such particles can of course escape from the confinement system only under the auxiliary condition $r_c > R_s$ (Table I). The so-called Störmer radius $R_s$ can be written approximately as[21]

$$R_s^2\approx\rho_0^2\left(\sin^2\beta_0-\frac{1}{k}\right) \tag{6.5}$$

where k is the mirror ratio. In the opposite case ($r_c < R_s$), the particles are confined forever, regardless of the overlap of nonlinear resonances. This confinement is guaranteed by the exact integrals of the energy, (1.7), and of the angular momentum, (1.8). The mechanism responsible for the fact that the stochastic instability which arises upon resonance overlap does not remove the particles from the magnetic confinement system is as follows: The diffusion coefficient in (5.3) is proportional to the quantity $\xi^2\sim r_c^2$ in (2.12), but $r_c$ does not remain constant as $\mu$ changes because of the conservation of angular momentum. From (1.8) we have

$$r_c^2\approx\frac{2}{\omega_0}(\mu+M). \tag{6.6}$$

For trajectories which circumvent the axis of the system we have M < 0 [see (1.8)], so as $\mu$ decreases there is also a decrease in $r_c$, and this radius vanishes at some finite value $\mu=\mu_1=-M$ (the orbit becomes "centered," and the diffusion along $\mu$ or, more precisely, in the direction of a further decrease in $\mu$ is halted). If the minimum angle $\beta_1$ corresponding to $\mu_1$ does not lie on the adiabatic cone, i.e., if $\sin^2\beta_1 = 2\omega_0\mu_1/v^2 > 1/k$, the particle cannot escape from the system, despite the stochastic instability. This latter condition is precisely the same as the condition for absolute confinement, $r_c < R_s$, as is easily shown.

## 7. STRONG AZIMUTHAL INHOMOGENEITY: RESONANCE OVERLAP

Turning to the motion of a charged particle in a nonaxisymmetric magnetic configuration, which is both a more interesting problem and a problem more important for applications, we first consider the case of a pronounced azimuthal inhomogeneity of the magnetic field. The corresponding condition is given below in (7.3).

Because of the azimuthal field inhomogeneity, all characteristics of the particle motion, including the frequencies $\bar{\omega}$ and $\Omega$, are modulated at the azimuthal drift frequency $\Omega_d \ll \Omega$. As a result, each of the resonances $\bar{\omega}=2n\Omega$ (Sec. 1) splits into a multiplet, as shown schematically in Fig. 1. The distance between the lines in the multiplet is $\Omega_d$, while the total number of lines is governed by the modulation depth of the frequencies $\bar{\omega}$ and $\Omega$.

Alternatively, it can be shown that the system of main

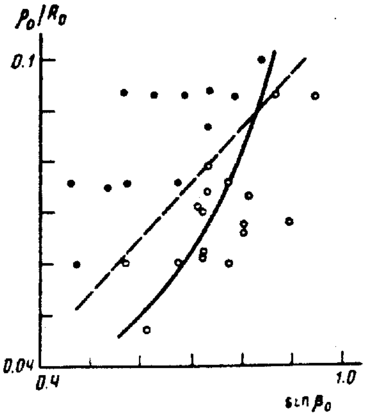


FIG. 5. Stability boundary for the motion of a particle in the field of a magnetic dipole according to the data of Dragt.[20] Open circles) Stable trajectories; filled circles) unstable trajectories; dashed line) empirical stability boundary ($\rho_0/R_0$ = 0.11 sin $\beta_0$); solid curve) stability boundary according to the resonance-overlap condition (K = 1).



$R_s^2 = \rho_0^2(\sin^2\psi_0 - \frac{1}{R})$, where $\rho_0 = \frac{V}{\Omega}$ is Larmor radius

TABLE I

| $\rho_\perp$, cm | $r_c$, cm | $R_s$, cm | $\beta_0$, deg | $\varepsilon_a$ | $N$ | $K_N$ | $K$ | $\frac{K}{K_N}$ | $\frac{\delta\rho_0}{\rho_0}$ |
|---|---|---|---|---|---|---|---|---|---|
| 40.7 | 19.3 | 16 | 63 | 0.314 | 370 | 1.60 | 1.20 | 0.75 | 0.21 |
| 36.6 | 23.4 | 15 | 64 | 0.271 | 340 | 1.62 | 1.05 | 0.65 | 0.21 |
| 32.5 | 27.5 | 14 | 64 | 0.239 | 310 | 1.64 | 1.01 | 0.62 | 0.18 |
| 28.1 | 31.9 | 11 | 62 | 0.219 | 270 | 1.68 | 1.14 | 0.68 | 0.13 |
| 24.1 | 35.9 | 7 | 59 | 0.208 | 220 | 1.73 | 1.40 | 0.81 | 0.07 |
| 20.2 | 39.8 | 4 | 56 | 0.192 | 180 | 1.79 | 1.50 | 0.84 | 0.05 |
| | | | | Average | | | 1.22 | 0.72 | |
| | | | | Scatter | | | ±0.18 | ±0.10 | |

resonances is now governed by three frequencies:

$$\bar{\bar{\omega}}-2n\bar{\Omega}+m\Omega_d=0, \qquad (7.1)$$

where $\bar{\Omega}$ is the frequency of the longitudinal oscillations, averaged over the drift motion, and $\bar{\bar{\omega}}$ is the Larmor frequency, averaged over both the longitudinal oscillations and the drift. The maximum index of the drift-frequency harmonics in the case of frequency modulation is (see, for example, Ref. 22)

$$|m|\approx\frac{|\Delta\bar{\omega}|+2n|\Delta\Omega|}{\Omega_d} \approx\left(\frac{|\Delta\bar{\omega}|}{\bar{\bar{\omega}}}+\frac{|\Delta\Omega|}{\bar{\Omega}}\right)\frac{\bar{\bar{\omega}}}{\Omega_d}=\eta\frac{\bar{\bar{\omega}}}{\Omega_d}, \qquad (7.2)$$

where $|\Delta\bar{\omega}|$ and $|\Delta\Omega|$ are the modulation amplitudes of the frequencies $\bar{\omega}$ and $\Omega$ on the drift surface.

Provided

$$2\bar{\Omega}\lesssim|m|\Omega_d\approx\eta\bar{\bar{\omega}} \qquad (7.3)$$

the adjacent multiplets overlap (Fig. 1). This is the case of a pronounced azimuthal inhomogeneity of the magnetic field of the system.

The overlap of adjacent (in terms of m) resonances in (7.1) is now aided considerably, since the distance between these resonances is much smaller: $2\Omega \to \Omega_d \ll \Omega$. Then the critical value K in (4.6) is now much smaller than $K_1 = 1$ in (5.1) for an axisymmetric configuration. The new critical value, $K = K_2$, can be estimated as follows. The value of K in (4.5) is proportional to the square of the ratio of the width of the nonlinear resonance to the distance between adjacent resonances (see, for example, Refs. 8 and 9). Because of the frequency modulation by the drift motion, the distance between the resonances decreases by a factor $2\Omega/\Omega_d$, and the square of the resonance width, which is proportional to the amplitude of the corresponding perturbation harmonic,[8,9] decreases by a factor of about $(2|m|)^{1/2}$ (Ref. 22). As a result, the stability boundary according to the overlap of nonlinear resonances, in (7.1), is now governed by the condition

$$K=K_2\approx\sqrt{2|m|}\left(\frac{\Omega_d}{2\bar{\Omega}}\right)^2 =\frac{1}{2\sqrt{2}}\sqrt{\eta\frac{\bar{\bar{\omega}}}{\Omega_d}}\left(\frac{\Omega_d}{\bar{\Omega}}\right)^2\sim\varepsilon_a\sqrt{\eta}. \qquad (7.4)$$

Here K is found from (4.7) or (4.8), and the last – extremely rough – estimate is found from the condition $\bar{\Omega}/\bar{\bar{\omega}}\sim\Omega_d/\Omega\sim\varepsilon_a$.

The rate of diffusion in $\mu$ in the limiting case of a strong resonance overlap ($K \gg K_2$) is again given by (5.3), since the sequential values of the phase $\theta$ can be assumed random and independent (Sec. 5). Near the new stability boundary, (7.4), we can write

$$D_\mu\approx\frac{\bar{\Omega}\xi^2\mu^2}{2\pi}\left(1-\frac{K_2}{K}\right)^2. \qquad (7.5)$$

## 8. SLIGHT AZIMUTHAL INHOMOGENEITY: ARNOL'D DIFFUSION

The most interesting case is that of a slight azimuthal inhomogeneity, defined by the condition [cf. (7.3)]

$$\eta<\frac{2\bar{\Omega}}{\bar{\bar{\omega}}}\sim\varepsilon_a. \qquad (8.1)$$

In this case the overlap of adjacent resonances does not make the critical value of K lower than that for an axisymmetric system; that is, $K_2 \sim 1$. However, any (arbitrarily small) azimuthal field inhomogeneity leads to the possibility of a qualitatively different behavior of the particle: motion along the resonance $\bar{\omega} \approx 2n\Omega$ (Fig. 1). In an azimuthally inhomogeneous field, only one of the two exact integrals of motion, (1.7) and (1.8), is retained: the energy integral (1.7). The position of the center of the Larmor circle ($r_c$) can now vary in an arbitrary manner; in particular, it can vary in a manner such that, along with a variation in $\mu$ or in the angle $\beta_0$, a constant ratio $\bar{\omega}/2\Omega$ = const is maintained. For the "quadratic" field in (1.2) we have $\omega_0(r) \approx \omega_{00}(1 - r^2/2l^2)$. Also, using (1.5), we can write an approximate condition for motion along the resonance:

$$\left(1-\frac{r_c^2}{2l^2}\right)\frac{1+\sin^2\beta_0}{\sin^2\beta_0}\approx\text{const.} \qquad (8.2)$$

This equation shows that a decrease in the angle $\beta_0$ is accompanied by an increase in $r_c$. The particle ultimately reaches a mirror or disappears at a side wall of the chamber.

The possibility of motion of this type makes the Arnol'd theorem[6] on the permanent confinement of a charged particle to the magnetic confinement system inapplicable as soon as the system becomes anything less than strictly axisymmetric. This circumstance was always emphasized by Arnol'd.[6] Furthermore, Arnol'd discovered a specific mechanism for motion along a resonance,[23] subsequently

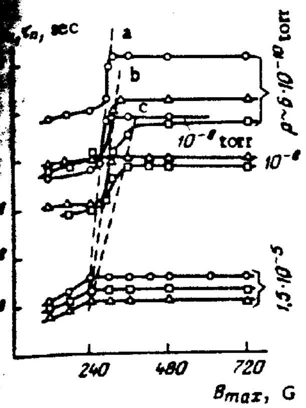


FIG. 6. Particle lifetime in the magnetic confinement system as a function of the magnetic field for various residual gas pressures.[27]

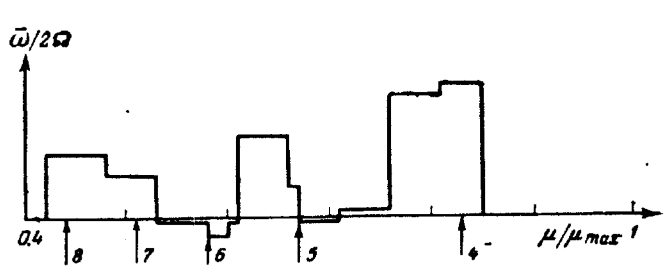


FIG. 7. Differential $\mu$ spectrum of the electrons in the confinement system according to Ref. 27. $\mu_{max} = v^2/2B_0$, p = 5 · 10 torr [sic], t = 3.4 sec after injection. The arrows show the resonant values $\mu = \mu_n$.

named "Arnol'd diffusion."[24] An upper limit on the rate of this diffusion was derived rigorously by Nekhoroshev.[25] A nonrigorous but (the author hopes) better estimate (Sec. 10) was found in Ref. 24 and, more accurately, in Ref. 10. This estimate is ultimately based on the condition for nonlinear-resonance overlap.

The mechanism for Arnol'd diffusion involves the existence of the so-called stochastic layer near the separatrix of a nonlinear resonance.[9,10] This layer can be seen clearly in, for example, Fig. 4. It turns out that this stochastic layer exists for any arbitrarily small perturbation; for example, for the mapping (4.5) it exists for any $K \to 0$. For this reason, Arnol'd diffusion can be called a universal instability of multidimensional nonlinear oscillations.[10] In this case the term "multidimensional" means that the number of degrees of freedom is greater than 2. For two degrees of freedom, e.g., in the motion of a particle in an axisymmetric confinement system, the motion is confined to within the stochastic layer because of the particular topology of the phase space, and no real instability occurs, as explained above (Sec. 5).

In the case of a multidimensional system, on the other hand, e.g., a particle in an axially asymmetric confinement system (three degrees of freedom), there is the possibility of motion along the stochastic layer, i.e., motion perpendicular to the plane of Fig. 4, or approximately along a resonance line in Fig. 1.

In the case $K \ll 1$ the width of the stochastic layer and the diffusion rate are exponentially small [see (10.10)]. For this reason, Arnol'd diffusion cannot be studied in a real system (e.g., a magnetic confinement system) through numerical integration of the trajectories at the present state of computer technology. A special, very simple, model system was used in Ref. 26; this system is compatible with numerical simulations of Arnol'd diffusion. The results of this work show that the relatively simple semiempirical theory of Arnol'd diffusion,[24,10] which is ultimately based on the resonance-overlap condition, gives an extremely satisfactory description of this unusual phenomenon.

## 9. EXPERIMENTS ON THE MOTION OF ELECTRONS IN A MAGNETIC CONFINEMENT SYSTEM

The first experiments[3] on prolonged confinement of electrons combined in magnetic mirrors were followed by studies of this topic in many places. In the absence of a theory, however, the results were of an empirical nature. It was suggested in Ref. 24 that the extremely unusual features of the nonadiabatic behavior of electrons in a magnetic confinement system could be attributed to Arnol'd diffusion. Below we will analyze this hypothesis in more detail, making use of the most complete experimental data obtained in Refs. 27 and 5.

Figure 6, from Ref. 27, shows the typical lifetime of electrons combined between magnetic mirrors, measured by various methods for various residual gas pressures, as a function of the magnetic field. A characteristic feature of this behavior is the presence of two plateaus. The "upper plateau" can be explained in a natural manner by arguing that in a sufficiently strong magnetic field the nonadiabatic effects are negligibly weak, and the lifetime is determined completely by scattering of electrons in the residual gas. At some critical magnetic field the electron lifetime falls rapidly by more than an order of magnitude (obviously because of nonadiabatic effects), and then remains nearly constant as the field is reduced further (the "lower plateau"). The lifetime at this lower plateau is roughly proportional to the pressure, like that at the upper plateau. Similar variations were also found in Ref. 5, in an experimental study of electron motion in the field of a magnetized sphere, simulating the geomagnetic field.

To explain the reason for the appearance of the lower plateau, we consider the electron spectra over $\mu$, measured in Ref. 27. A typical spectrum from Ref. 27 is shown in Fig. 7. This spectrum was recorded 3.4 sec after the injection of electrons into the system. The magnetic field was slightly below the critical value (Fig. 6). A characteristic feature of the spectrum is the presence of two "dips," i.e., intervals of $\mu$ in which there are essentially no electrons. On the other hand, because of the scattering in the gas there is a continuous flux of electrons into these intervals from the neighboring regions along the $\mu$ scale. These dips evidently correspond to the formation of some "holes" or "gaps" in phase space, through which the electrons "leak out" of the confinement system. It is seen from Fig. 7 that these gaps coincide approximately with resonances $\bar{\omega} = 2\Omega n$. As an argument in favor of this identification we point out that the histogram in Fig. 7 was constructed by differentiation of the directly measured integral spectrum,[27] and this procedure results in larger errors. In particular, the fact that the electron density becomes negative at certain places in Fig. 7 shows that there are errors. Nevertheless, the dips in the spectrum of electrons over $\mu$, at points corresponding to the resonance values $\mu_n$, definitely indicate leakage of electrons from the confinement system along the resonances. This behavior can be explained in a natural manner on the basis of Arnol'd diffusion (Sec. 8). We also find an explanation for the formation of the lower plateau in Fig. 6, i.e., for the halt to the rapid decrease in the electron lifetime as the magnetic field is reduced, which is an extremely surprising result at first glance. Specifically, Arnol'd diffusion spans only the relatively narrow stochastic layers, which contain a negligible fraction of the electrons. The bulk of the electrons arrive at one of the stochastic layers only as the result of scattering in the gas. In order to escape from the confinement system now, however, the electron need only reach the nearest stochastic layer; in other words, it is sufficient that the electron be scattered through an angle much smaller than that required for entrance into the loss cone ($\mu/\mu_{max} = \sin^2\beta_0 \approx 0.4$ in Fig. 7). For example, for electrons lying halfway between the fourth and fifth resonances in Fig. 7 ($\beta_0 \approx 60°$) scattering through an angle of about 5° is sufficient for reaching the newest stochastic layer, while the loss cone is about 20° away. As a result, the lifetime is reduced by a factor $\sim (20°/5°)^2 = 16$, in agreement with the data in Fig. 6.

The lifetime governed by Arnol'd diffusion is thus always proportional (for the overwhelming majority of the particles) to the characteristic time for scattering by the residual gas, but is much shorter than this scattering time. In contrast, the diffusion upon resonance overlap (Sec. 5), including the overlap of multiplets in an axially asymmetric confinement system (Sec. 7), is independent of scattering in the gas. The particle lifetime under the conditions remains finite (and relatively short) for an arbitrarily low gas pressure.

## 10. ESTIMATE OF THE RATE OF ARNOL'D DIFFUSION IN A MAGNETIC CONFINEMENT SYSTEM

The complete system of resonances for particle motion in an axially asymmetric confinement system is

$$\bar{\bar{\omega}}p - 2\bar{\Omega}n + \Omega_d m = 0, \quad (10.1)$$

where p, n, and m are arbitrary integers. In principle, Arnol'd diffusion can occur along any of these resonances, and that resonance along which the diffusion does occur is called the "leading" resonance. We will see below, however, that the maximum diffusion rate corresponds to the "strongest" or main resonances of the system [see (10.5)], which are the resonances $\bar{\bar{\omega}} = 2\bar{\Omega}n$ in the case under consideration here (Fig. 1). Diffusion along the leading resonance occurs under the influence of the perturbation terms with frequencies

$$\omega_m = \bar{\bar{\omega}}p - 2\bar{\Omega}n + \Omega_d m \quad (10.2)$$

because of the resonance of this perturbation with phase oscillations at the leading resonance. We note that in (1.2) we must necessarily have m ≠ 0 (p and n are arbitrary), since only such a perturbation (which depends on the azimuthal angle) will alter the angular momentum in (1.8), as is necessary for motion along the leading resonance, $\bar{\bar{\omega}} = 2\bar{\Omega}n$ (Sec. 8).

The frequency of the phase oscillations at the leading resonance is [see (4.5) and (7.8)]

$$\Omega_p \sim \Omega\sqrt{\bar{K}} \sim \omega e^{-1/\varepsilon_a}, \quad (10.3)$$

where we are assuming $\Omega/\omega \sim \varepsilon_a$. For the lower harmonics of the perturbation [in m and p; see (10.2)] and for a small value of $\varepsilon_a$ in (10.3) we have $\Omega_p \ll |\omega_m|$. The effect of the perturbation under these conditions is exponentially small (cf. Sec. 2):

$$\frac{\Delta\mu_A}{\mu} \sim \eta e^{-C\frac{|\omega_m|}{\Omega_p}}. \quad (10.4)$$

Here $\Delta\mu_A$ is a measure of the displacement of the particle along the leading resonance, $\eta \sim \Delta B/B$ is the azimuthal inhomogeneity of the magnetic field in (7.2), and $G \sim 1$ is a constant.

The coefficient of the diffusion along $\mu$ is proportional to $(\Delta\mu_A)^2$ and can be estimated very roughly from

$$D_\mu \sim (\Delta\mu_A)^2\omega \sim \omega\mu^2\eta^2 e^{-2C|\omega_m|/\Omega_p}. \quad (10.5)$$

The exact value of the coefficient of the exponential function is not important because $D_\mu$ is a very strong function of $\varepsilon_a$ [see (10.3) and (10.4)].

The diffusion rate along a resonance with m ≠ 0 in (10.2) is negligibly low, since the frequency of the phase oscillations at this resonance contains the additional small factor $\sim\sqrt{\eta} \ll 1$.

For high harmonics of the perturbation, the frequency $\omega_m$ falls off rapidly, roughly in accordance with (see, for example, Ref. 10)

$$\omega_m \approx \frac{\omega_r}{|m|^{N-1}}. \quad (10.6)$$

Here $\omega_r$ is a constant which depends on the main frequencies $\bar{\bar{\omega}}$, $\bar{\Omega}$, and $\Omega_d$ of the resonance system in (10.2), N = 3 is the number of main frequencies, and

$$|m|^2 = p^2 + n^2. \quad (10.7)$$

The amplitudes of the higher perturbation harmonics, however, fall off even more rapidly – exponentially (Sec. 2). Let us assume, for example, that these amplitudes are proportional to $\exp(-\sigma|m|)$, where $\sigma$ is some constant. Then in the argument of the exponential function which determines the rate of Arnol'd diffusion in (10.5) we have the additional term

$$-2C\frac{|\omega_m|}{\Omega_p} \to 2C\left(\frac{\omega_r}{\Omega_p}\right)\frac{1}{|m|^{N-1}} - 2\sigma|m| = E(|m|),$$

where we are using (10.6) for the perturbation frequencies $\omega_m$. The new argument $E(|m|)$ has a maximum at

$$|m| = m_0 = \left[(N-1)\frac{C}{\sigma}\frac{\omega_r}{\Omega_p}\right]^{1/N}, \quad (10.8)$$

which is equal to

$$E(m_0) = \frac{\sigma N}{N-1}\left[(N-1)\frac{C}{\sigma}\frac{\omega_r}{\Omega_p}\right]^{1/N}. \quad (10.9)$$

From (10.5) we find an estimate for the Arnol'd diffusion coefficient:

$$D_\mu \sim \omega\mu^2\eta^2 e^{-a(\omega_r/\Omega_p)^{1/N}}, \quad (10.10)$$

TABLE II

| $w$, keV | $B_0$, G | $p$, torr | $\tau$, sec | $\frac{l}{\rho_0}$ | $\beta_r$, deg | $\frac{1}{\varepsilon_a}$ | $b$ | $d$ |
|---|---|---|---|---|---|---|---|---|
| [illegible] | 118 | $10^{-8}$ | ~4 | 8.96 | 46 | 7.89 | 3.66 | 7.79 |
| [illegible] | 87 | $5\times10^{-9}$ | 4 | 6.61 | 55 | 6.78 | 4.38 | 8.86 |
| [illegible] | 87 | $10^{-8}$ | 4 | 6.61 | 66 | 8.25 | 3.37 | 7.29 |
| | | | | | Average | | 3,80 | 7.98 |

where

$$a = \sigma^{(N-1)/N}\frac{N}{N-1}[C(N-1)]^{1/N};\quad q(N) = \frac{1}{N}. \qquad (10.11)$$

This estimate of $D_\mu$ is of course highly simplified; this problem is analyzed in more detail in Ref. 10 (see also Ref. 26).

The rigorous upper limit derived for the Arnol'd diffusion rate by Nekhoroshev[25] has a structure similar to (10.10). The most important difference lies in the magnitude of q. According to Ref. 25, q(3) = 4/59 ≈ 1/15. The numerical simulations carried out in the second paper in Ref. 26 show that this value of q is probably much too low.[3)]

For the problem under consideration here, that of a charged particle confined by magnetic mirrors, we have N = 3, and the frequency of the phase oscillations, $\Omega_p$, at the leading resonance, $\bar{\omega} = 2\bar{\Omega}n$, is given by estimate (10.3). Also noting that $D_\mu \sim \mu^2/\tau$, where $\tau$ is the particle lifetime in the confinement system, we find from (10.10) the following estimate for the rate of Arnol'd diffusion in the system:

$$\tau\omega\eta^2 \exp(be^{1/\varepsilon_a}), \qquad (10.12)$$

where $b \sim a(\omega_r/\omega)^{1/3}$ varies weakly (as the cube root) with all the parameters other than $\sigma$ [$b \sim \sigma^{2/3}$; see (10.11)]. A characteristic feature of this estimate is the very strong dependence of the particle lifetime in the confinement system on $\varepsilon_a$ (a double exponential!). This result implies highly accurate conservation of the adiabatic invariant of the particle – the magnetic moment in the sense of the accumulated changes in it (Sec. 2).

We again emphasize that, in contrast with the more or less accurate theory for the stability of a particle confined in an axisymmetric system (Sec. 5), the estimate in (10.12) has been derived using a highly simplified picture of an extremely complicated process: Arnol'd diffusion (see Refs. 26 and 10 for more details). This estimate is thus crude, although the very weak dependence $\varepsilon_a(\tau)$ (a double logarithm) makes this crudeness somewhat less important in a determination of the critical parameters of the system.

Let us attempt a quantitative comparison of the analytic estimate in (10.12) with the experimental data from Refs. 27 and 5. The greatest uncertainty results from the absence of data on the azimuthal asymmetry of the magnetic field, which has not been measured specially. Nevertheless, judging the accuracy of control measurements of the magnetic field in Refs. 27 and 5, we can assume with some justification that we have $\eta \sim 10^{-2}$ for the field of an ironless solenoid[27] and $\eta \sim 10^{-1}$ near the surface of a magnetized sphere.[5] Here again, because of the very weak

TABLE III

| $p$, torr | $\tau$, sec | $\frac{l}{\rho_0}$ | $\beta_r$, deg | $\frac{1}{\varepsilon_a}$ | $b$ | $d$ |
|---|---|---|---|---|---|---|
| $5\times10^{-10}$ | ~4 | 17.6 | 35 | 13.6 | 1.75 | 3.36 |
| $5\times10^{-10}$ | 4 | 15.5 | 50 | 14.5 | 1.51 | 2.47 |
| $5\times10^{-10}$ | 4 | 14.6 | 70 | 20.5 | 0.55 | −3.59 |
| | | | | Average | 1.63 | 2.92 |

dependence $\varepsilon_a(\eta)$ in (10.12), the unavoidable errors in the values adopted for $\eta$ could not greatly affect the critical value of $\varepsilon_a$.

We will make the comparison with experimental data in the following manner: We adopt the parameters of the system for the critical magnetic field (Fig. 6) at which the proposed Arnol'd diffusion substantially reduces the particle lifetime in the confinement system. For $\tau$ we choose the value on the upper plateau, on the basis that the critical field corresponds to that case in which the rate of Arnol'd diffusion along the leading resonance is comparable to the diffusion rate due to scattering by the residual gas. As the leading resonance we choose that which is nearest the particle with the maximum angle $\beta_0$. For example, for the case in Fig. 7 we choose n = 5 (the angle corresponding to this resonance is $\beta_0 = \beta_r \approx 55°$). Finally, we calculate the adiabatic parameter $\varepsilon_a$ from Eq. (2.9). Then we find the only unknown, b, from (10.12).

The results of this comparison are shown in Table II (for the data of Ref. 27) and Table III (the data of Ref. 5). In the latter case the electron energy is ~100 eV, and the magnetic induction is ~30 g. The case in Table II corresponds to the spectrum in Fig. 7.

The values of b in Table II agree well. The last case in Table III does not fit the pattern, apparently because the angle spanned by the electrons is too large ($\beta_0 \approx 70°$). The adiabatic parameter $\varepsilon_a$ and the diffusion rate are very small. In this case the particle escapes from the system more rapidly if it is first scattered by the residual gas to smaller angles $\beta_0$, and it "misses" several resonances with a small value of $\varepsilon_a$. The average values of b are noticeably different apparently because of the behavior in (10.11): The spectrum of the particle motion in the field of the magnetic dipole is richer in higher harmonics, and there are correspondingly smaller values of $\sigma$ and thus b.

It should hardly be appropriate to use the complicated expression in (2.9) to evaluate $\varepsilon_a$ for use in the estimate in (10.12), because the latter estimate is so crude. We accordingly use the simpler expression in (2.8) and set $\beta_0 \approx 45°$. Then (10.12) can be written

$$\frac{1}{\varepsilon_0} \approx \frac{l}{\rho_0} \approx 6 \ln\ln(\tau\omega\eta^2) - d, \qquad (10.13)$$

where the quantities d = 6 ln b depend on the configuration of the magnetic field of the confinement system (Tables II and III). The estimates in (10.12) and (10.13) are of course meaningful only for small values of $\varepsilon_a$, far from a resonance overlap (Sec. 5). They are apparently suitable for a rough estimate of the effect of Arnol'd diffusion in a magnetic confinement system. It is nevertheless clear that further study is required, both theoretical (in particular, to evaluate the parameter d) and experimental, to test and refine the theory of Arnol'd diffusion.

[1]The subscript "0" is the value of the corresponding quantity in the median plane (s = 0), which is the symmetry plane of the magnetic confinement system.

[2]An analogous result was derived slightly earlier in Ref. 28, where the numerical coefficient of the exponential function was found to differ from that in (3.5) by a factor $3^{19/18}/2 = 0.994$.

[3]In a more recent paper, Nekhoroshev[29] gives a refined estimate, which leads to q(3) = 1/8 in the present case.


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