Working copy

1979 г.

A UNIVERSAL INSTABILITY OF MANY-DIMENSIONAL OSCILLATOR SYSTEMS

Boris V. CHIRIKOV

Institute of Nuclear Physics, 630090 Novosibirsk, U.S.S.R.

NORTH-HOLLAND PUBLISHING COMPANY - AMSTERDAM

Straightforward integration of this equation for $H < U_0$ (oscillations) gives (see, e.g., ref. [58]):

$$\dot{\varphi} = 2\omega_0 \sin(\varphi_0/2)\, \mathrm{cn}(\omega_0 t) \tag{2.3}$$

where $\mathrm{cn}(u)$ is the Jacobian elliptic cosine; φ_0 stands for the amplitude of pendulum displacement, and the time $t = 0$ corresponds to crossing the point $\varphi = 0$ with a positive velocity. Term by term integrating the Fourier series for $\mathrm{cn}(u)$ gives:

$$\varphi(t) = 4\omega \sum_{n=1}^{\infty} \frac{\sin(\omega_n t)}{\omega_n \cosh(K'\omega_n/\omega_0)} . \tag{2.4}$$

Here $K' = K(k')$ is the complete elliptic integral of the first kind; $k' = \sqrt{1-k^2}$; $k = \sin(\varphi_0/2)$. The period of oscillations is $T = 2\pi/\omega$ where

$= \sqrt{(H+U_0)/2U_0}$

$$\omega(H) = \pi\omega_0/2K \tag{2.5}$$

and the oscillation spectrum is: $\omega_n = (2n-1)\omega$.

In the case of rotation ($H > U_0$), the solution has the form:

$$\pm\varphi(t) = 2\,\mathrm{am}(\omega_r t) = 2\omega t + 4\omega \sum_{n=1}^{\infty} \frac{\sin(\omega_r t)}{\omega_n \cosh(K'\omega_n/\omega_r)} . \tag{2.6}$$

Here $\mathrm{am}(u)$ is the Jacobian elliptic amplitude;

$$k = \sqrt{2U_0/(H+U_0)} = \omega_0/\omega_r; \qquad \omega_r = \sqrt{\tfrac{1}{2}(H+U_0)},$$

and we have introduced half the frequency of the mean rotation: $\omega(H) = \pi\omega_r/2K$ to make the expressions (2.4) and (2.6) as similar to each other as possible; we shall use this in section 2.4. The rotation frequency spectrum is: $\omega_n = 2n\omega$.

Relation (2.5) shows that the pendulum frequency does depend on the oscillation amplitude, or the energy so that the pendulum is generally a non-isochronous oscillator. The latter property is of paramount importance for the problem of motion stability as we shall see below.

Let us consider the case of small pendulum oscillations and relate the argument $k = \sin(\varphi_0/2) \ll 1$ of the elliptic integral to the pendulum energy E reckoned from the minimum of the potential energy: $-U_0$ (2.1). We have: $E = 2U_0k^2$. Expanding the expression $[K(k)]^{-1}$ as a power series in k^2 we find:

$$\omega \approx 1 - \tfrac{1}{8}E - \tfrac{5}{256}E^2 - \tfrac{11}{2048}E^3 \tag{2.7}$$

where we have put $U_0 = 1$ ($\omega_0 = 1$). We shall need such a high accuracy in the next section to compare eq. (2.7) with the result of the perturbation theory.

Let us introduce the dimensionless parameter of the nonlinearity:

$$\alpha = \frac{I}{\omega} \cdot \frac{d\omega}{dI} . \tag{2.8}$$

We define α via the action I since we shall use below, as a rule, the action-angle variables (I, θ). To the accuracy $\omega \approx 1 - E/8$ the action $I \approx E/\omega_0$ as for a harmonic oscillator. Hence: $\alpha \approx -I/8$. For $E \ll 1$ the nonlinearity of small oscillations ($E, I \ll 1$) is small ($\alpha \ll 1$).

Another characteristic property of nonlinear oscillations is the *anharmonicity*, i.e. the presence of higher harmonics of the basic frequency ω (2.5). For small oscillations the amplitudes of higher harmonics are small as one can see from eq. (2.4) with $K' \approx \ln(4/k) = \frac{1}{2}\ln(32/E)$. Hence the small oscillations are nearly harmonic.

Let us consider the pendulum motion near the separatrix. We will describe the distance from the separatrix by the relative energy:

$$w = \frac{H - U_0}{U_0} \approx \frac{p^2}{2U_0} + \frac{(\pi - \varphi)^2}{2} \ll 1. \tag{2.31}$$

Using the expressions in section 2.1 we find for both oscillation ($\omega < 0$) and rotation ($\omega > 0$): $k \approx \sqrt{|w|/2}$; $K(k) \approx \frac{1}{2}\ln(32/|w|)$; $K' \approx \pi/2$, and $\omega_r \approx \omega_0$. Hence for both kinds of motion, i.e. at both sides of the separatrix, the solution has the form (see eqs. (2.4), (2.6)):

$$\varphi(t) \approx 4 \sum_n \frac{\sin(n\omega t)}{n \cosh(\pi n \omega / 2\omega_0)} \tag{2.32}$$

for rotation only even and for oscillation only odd harmonics being present. The frequency:

$$\omega(w) \approx \frac{\pi\omega_0}{\ln(32/|w|)} \tag{2.33}$$

is decreasing indefinitely on approaching separatrix ($|w| \to 0$). From the spectrum (2.32) one can conlude that the motion *near* the separatrix is approximately the same as *on* the separatrix except that the motion is of a finite period. It is clear also that the nonlinearity is growing indefinitely when approaching the separatrix (2.33).

3. Nonlinear resonance

Free oscillations in a conservative system with one degree of freedom are always stable (if the motion is finite, of course) and therefore are of less interest for us. What happens if one switches on an external perturbation? The Hamiltonian may be written in this case as:

$$H(I, \theta, t) = H_0(I) + \epsilon V(I, \theta, t). \tag{3.1}$$

The external perturbation is described here via an explicit dependence of the Hamiltonian on time. We will assume below that the perturbation is periodic in time with a period T, and the basic frequency $\Omega = 2\pi/T$. The frequencies of the perturbation spectrum are $n\Omega$ in this case. An almost periodic perturbation with an arbitrary discrete spectrum (Ω_n) does not lead to any qualitatively new effects. A perturbation with a continuous spectrum but restricted in time (a perturbation pulse) is of less interest since it causes only a small ($\epsilon \ll 1$) change in the oscillation energy. Finally, a stationary perturbation with a continuous spectrum, for example, an irregular sequence of pulses, causes a diffusion-like process in the system. The theory of such processes, which are very important for applications, comprises now a vast section of the theory of both linear and nonlinear oscillations. The latter problem is, however, beyond the framework of the present paper. So, we assume the perturbation to be periodically dependent on the phase

$$\tau = \Omega t + \tau_0 \tag{3.2}$$

where τ_0 is the initial phase. We expand the perturbation in a double Fourier series:

$$H(I, \theta, \tau) = H_0(I) + \epsilon \sum_{m,n} V_{mn}(I)\, e^{i(m\theta + n\tau)}. \tag{3.3}$$

where the new perturbation

$$V_1 = \frac{\Phi_\theta^2}{2} = \frac{1}{2}\sum_{m,n} \frac{\cos(\theta - mt)\cos(\theta - nt)}{(m - J_1)(n - J_1)} \tag{5.14}$$

and $\theta(J_1, \theta_1, t)$ is determined by the second relation of eq. (5.10).

The perturbation (5.14) has terms resulting in half-integer resonances: $J_1 = J_{1r} = (2p + 1)/2$ with any integer p. Corresponding resonance phases are $2\theta_1 - (m + n)t$, $m + n = 2p + 1$ where we have set approximately $\theta \approx \theta_1$ since the difference $|\theta - \theta_1| \sim k$ is small (see eq. (5.10) and below). Note that the new Hamiltonian is inapplicable near integer resonances ($J_{1r} = 2p$) owing to small denominators in eq. (5.14). Characteristics of a half-integer resonance $J_{1r} = p + \frac{1}{2}$ are determined by the sum:

$$S_2 = \sum_{m+n=2p+1} \frac{1}{(m - J_{1r})(n - J_{1r})} = -\sum_n \frac{1}{(n - p - \frac{1}{2})^2} = -\pi^2. \tag{5.15}$$

Since this sum is independent of p all half-integer resonances are alike except a shift in J. This is obvious also just from the periodicity of the phase space structure in I discussed at the beginning of this section: there is only one half-integer resonance per unit square of the reduced mapping (5.2). Substituting $J_1 = J_{1r} = \frac{1}{2}$ into eq. (5.14) and neglecting non-resonant terms we deduce from eq. (5.13) the Hamiltonian for a half-integer resonance as

$$H_1^{(2)} \approx \tfrac{1}{2}J_1^2 - (\tfrac{1}{2}\pi k)^2 \cos(2\theta - t). \tag{5.16}$$

Applying the technique of section 3.2 we find the separatrix to be described by:

$$J_s^{(2)} = \tfrac{1}{2} \pm \pi k \cos(\theta - \tfrac{1}{2}). \tag{5.17}$$

Substituting now $\theta(\theta_1)$ according to eq. (5.10) into the perturbation (5.14) we may represent H_1 as a series in the small parameter k. However, it makes sense to retain only terms $\sim k^3$ in addition since following a new canonical transformation according to Kolmogorov the perturbation will be $\sim k^4$ (section 2.2). To get terms $\sim k^3$ it suffices to set in eq. (5.10)

$$\theta \approx \theta_1 - k\,\Phi_{J_1}(\theta_1) = \theta_1 - k\sum_n \frac{\sin(\theta_1 - nt)}{(n - J_1)^2}.$$

After substitution of this expression into eq. (5.14) we get terms $\sim k^3$ as

$$k^2 V_1^{(3)} = \frac{k^3}{2}\sum_{m,n,l} \frac{\sin(2\theta_1 - (m+n)t)\sin(\theta_1 - lt)}{(m - J_1)(n - J_1)(l - J_1)^2}. \tag{5.18}$$

Some terms here result in third harmonic resonances with phases $3\theta_1 - (m + n + l)t$; $m + n + l = p$ and $J_1 = J_{1r} = p/3$ ($p \neq 3q$ for any integer q to avoid integer resonances). All the resonances are alike again except a shift in J_1. Indeed, the denominator in eq. (5.18) may be written as: $(3m - p)(3n - p) \times (3m + 3n - 2p)^2$. It suffices to consider only $p = 1$; 2 within the interval of periodicity $\Delta J_1 = 1$ (see below). But quantities $(3m - 1)$ and $(3m - 2)$ are exchanged under $m \to 1 - m$, the sum (5.18) being unchanged. The same is true also for n. Thus, each of third harmonic resonances is determined by the sum:

$$S_3 = \sum_{m+n+l=1} \frac{81}{(3m-1)(3n-1)(3l-1)^2} = \sum_{m,n} \frac{81}{(3m-1)(3n-1)(3m+2n-2)^2} \approx -86.4. \tag{5.19}$$

The Hamiltonian describing a third harmonic resonance has the form:

$$H_1^{(3)} \approx \tfrac{1}{2}J_1^2 - \tfrac{1}{4}S_3k^3\cos(3\theta - t) \tag{5.20}$$

and the resonance separatrix is

$$\begin{aligned} J_s^{(3)} &= \tfrac{1}{3} \pm (\Delta J)_3 \sin(\tfrac{3}{2}\theta - \tfrac{1}{2}t) \\ (\Delta J)_3 &= |S_3|^{1/2}k^{3/2} \approx 9.30k^{3/2}. \end{aligned} \tag{5.21}$$

Resonances of the first three harmonics are outlined in fig. 5.4 at $t = 0$. One can see that the maxima of all the resonances coincide in θ (cf. fig. 5.2). Therefore, the overlap condition corresponds to the maximal width of resonance separatrices:

$$(\Delta J)_1 = 2\sqrt{k}; \qquad (\Delta J)_2 = \pi k; \qquad (\Delta J)_3 = |S_3|^{1/2}k^{3/2}. \tag{5.22}$$

Now we can improve the theoretical estimate for critical perturbation. First, we take account of half-integer resonances only. Then the perturbation is critical if a half-integer resonance separatrix just touches separatrices of two adjacent integer resonances, overlapping the gap between them: $(\Delta J)_1 + (\Delta J)_2 = \frac{1}{2}$, whence:

$$\sqrt{k_{12}} = \frac{\sqrt{1+\pi/2}-1}{\pi} \approx 0.192; \qquad K_{12} \approx 1.46 \tag{5.23}$$

where the subscript indicates the resonance pair determining critical perturbation. The value (5.23) is considerably lower than the old one due to integer resonances: $K_{11} \approx 2.5$, yet the former still much exceeds the experimental $K_s \approx 1$.

Taking account of third harmonic resonances we need to consider the two pairs of adjacent resonances: $(0, \frac{1}{3})$ and $(\frac{1}{3}, \frac{1}{2})$. In the first case touching condition is: $(\Delta J)_1 + (\Delta J)_3 = \frac{1}{3}$, whence:

$$\sqrt{k_{13}} \approx 0.151; \qquad K_{13} \approx 0.90. \tag{5.24}$$

For the second pair: $(\Delta J)_3 + (\Delta J)_2 = \frac{1}{6}$ which gives

$$\sqrt{k_{23}} \approx 0.185; \qquad K_{23} \approx 1.35. \tag{5.25}$$

The latter value of K is decisive, but it improves only slightly the estimate (5.23).*

Further progress in theoretical estimates may follow two lines:

1. Taking account of still higher harmonic resonances: $m > 3$. I wonder if one of the readers would like to try this.

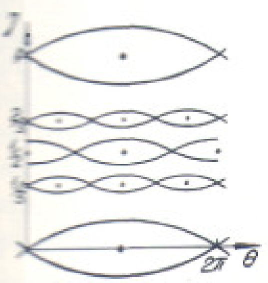

Fig. 5.4. Diagram of separatrices for resonances: $J_r = 0$; 1/3; 1/2; 2/3; 1.

*As was shown by Cary [140] the estimate (5.25) can be improved by taking account of the frequency shift due to $\langle V_1(\theta)\rangle \neq 0$ (see eq. (5.14)); he arrives at $\sqrt{k_{23}} = 0.180$; $K_{23} = 1.28$.

in the previous method, see table 6.1) and for 6×10^6 iterations (cf. the values of N in table 6.1). We have obtained that the quantity s_1 lies within the interval (1.004–1.013) that is compatible with the value of $s_{\max} = 1.006$.

We may use this result to check the critical value of $K = K_1$ for the standard mapping found in section 5.1: $K_1 = 0.989$. Applying the relation $K_1 \approx 1/|s_r|$ (6.11), where s_r are the two nearest to s resonant values we have got from the computation data:

$$0.98 \leqslant K_1 \leqslant 1. \tag{6.15}$$

Let us consider still another method to determine the width of a stochastic layer. The method is based upon the measurement of the mean rotation period (or a mean oscillation half-period) T in a stochastic layer. For the free oscillations of a pendulum near the separatrix (see section 2.4):

$$T(s) \approx \frac{1}{\omega_0} \ln\left(\frac{32}{|w|}\right) \to \frac{1}{\sqrt{K}} \ln\left(\frac{32}{|s|\, w_s}\right). \tag{6.16}$$

The last expression is again related to the case in which the pendulum represents an integer nonlinear resonance of the standard mapping (5.1), period T being measured in the number of iterations. From the numerical experiments one can easily find the mean value $T_E = t_N/N$ where t_N is the time interval of motion corresponding to exactly N periods, or to the $N + 1$ crossings of the surface $\theta = \pi$.

To deduce the theoretical mean (T_a) one needs to average eq. (6.16) over time, which $s(t)$ depends on, or, due to the ergodicity, over the stochastic component in the layer. Neglecting stability domains inside the layer, which are of importance near the layer edges only, we can average eq. (6.16) over the whole layer. Recall now that the original mapping, whose properties we are studying, is the standard one (5.1). Since the transformation of variables $(I, \theta) \to (s, \tau)$ by transition from the standard to whisker mapping is not canonical we need the Jacobian of this transformation. The latter may be represented as a sequence of successive transformations: (I, θ) (5.1)$\to (J, \theta)$ (4.18)$\to (I_s, \varphi_s) \to (w, \tau)$ (6.1)$\to (s, \tau)$ (6.9) where I_s, φ_s are the action-angle variables of the unperturbed system (4.18) with the Hamiltonian:

$$H_0 = \tfrac{1}{2}J^2 + k \cos\theta = k(1 + w) \tag{6.17}$$

describing the oscillations of frequency $\omega(w)$ (see eq. (2.33)) near the separatrix. Jacobians of all the transformations but one $(I_s, \varphi_s \to w, \tau)$ in the above chain are constants independent of the dynamical variables. So we need to evaluate only the Jacobian $\partial(w, \tau)/\partial(I_s, \varphi_s) = (\partial w/\partial I_s)\ (\partial\tau/\partial\varphi_s)$. The last expression is due to the fact that w depends only on I_s but not on φ_s. The deviative $\partial w/\partial I_s$ may be found from the relation $\partial H_0/\partial I_s = \omega(w)$, whence: $\partial w/\partial I_s = \omega/k$ (see eq. (6.17)). To evaluate $\partial\tau/\partial\varphi_s$ we write: $\tau = \Omega t^0$ and $\omega t^0 = \varphi_s = \text{const}$ – the value of φ_s at the surface $\theta = \pi$; t^0 is the time at which the system crosses this surface. Eliminating t^0 we get: $\tau = -\Omega\varphi_s/\omega$, and $\partial\tau/\partial\varphi_s = -\Omega/\omega$, whence the Jacobian we are interested in $\partial(w, \tau)/\partial(I_s, \varphi_s) = -\Omega/k$ does not depend on the dynamical variables, and so does the full Jacobian $\partial(s, \tau)/\partial(I, \theta)$. Hence we can evaluate any average simply over the phase plane (τ, s) of the whisker mapping. Since $T(s)$ does not depend on τ we get:

$$T_a \approx \int_0^1 \mathrm{d}s\, T(s) = \frac{1}{\sqrt{K}} \ln\left(\frac{32\mathrm{e}}{w_s}\right) \approx \frac{\pi^2}{K} - \frac{1}{\sqrt{K}} \ln\left(\frac{2r\pi^4}{\mathrm{e}K^{3/2}}\right)$$

$$w_s = 32 \exp(+1 - \sqrt{K}\, T_a). \tag{6.18}$$

Table 6.2
The KS-entropy in the stochastic layer

K	h_1 linear map	$h_s = h_1 T_a$	h_s/h_W (6.21)
0.15	0.0231	1.050	1.58
	(0.00643)	(0.292)	(0.44)
0.2	0.0295	0.964	1.45
	(0.0200)	(0.654)	(0.99)
0.3	0.0355	0.725	1.09
0.5	0.0686	0.765	1.15
0.7	0.0920	0.684	1.03
1	0.132	0.638	0.96
1.3	0.227	0.799	1.21
2	0.425	0.896	1.35
3	0.672	0.896	1.35
4	0.833	0.824	1.24

21)). For very small K the value of h_s grows, apparently, due to an insufficient motion time (see ɔve). The values of h_s for $t = 10^6$ are given in braçkets. In the case of $K = 0.2$ h_s value has escended" down to the theoretical one but for $K = 0.15$ it did so still much lower. This is caused, rhaps, by a "sticking" of the system in a peripheral part of the layer. For $K > 1$ the ratio h_s/h_W ɔws up appreciably, probably, due to the overlapping of different stochastic layers. Summarizing, can conclude that the idea of a constant KS-entropy in the stochastic layer (per motion period T_a) rmits us to describe satisfactorily and, what is both important and pleasant, very simply the stability rate inside the stochastic layer up to $K \sim 1$ and even, strange though it may seem, for fairly ge K, with less accuracy though (see table 5.1).

4. Again about the border of stability

Now we can turn back to the evaluation of the border of gross instability for the standard mapping ection 5.1). The best estimate deduced from the overlap condition for the resonances of the first ree harmonics gives: $K_T \approx 1.35$ (5.25). Taking account of the stochastic layer around separatrix rmits us to improve this estimate.

Below we will confine ourselves to a simplified scheme for the overlap taking account of the integer ıd half-integer resonances only and neglecting the stochastic layers of the latter. The relation etween the dimensionless energy w and displacement δI from the unperturbed separatrix can be ound from the Hamiltonian (6.17). Since $w = \delta H_0/k = J(\delta J)/k = I(\delta I)/K$ we get: $(\delta I)/I = Kw/I^2 \to$ /4, the latter expression corresponding to the maximal width of separatrix at $\theta = \pi$ $(I \to I_m = 2\sqrt{K})$ hich determines the overlap condition. The edge of the stochastic layer is related to $w = s_T w_s$, s_T eing given by eq. (6.13). It is convenient to describe the influence of stochastic layer on the esonance overlap by a factor giving an effective width of a resonance:

$$l(K) = 1 + \frac{\delta I}{I_m} = 1 + \frac{s_T w_s}{4} = 1 + r s_T \frac{16\pi^4}{K^{3/2}} e^{-\pi^2/\sqrt{K}} \qquad (6.23)$$

$\sim K^{5/2}$ since in (6.7) $K \neq 1$ ✓

here $r = 2.15$ is the empirical correction (6.5) due to higher approximations.

but this doesn't change much numerical results