MARGINAL LOCAL INSTABILITY OF QUASI-PERIODIC MOTION ☆

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In this paper, we analytically prove a long suspected link between integrable hamiltonian systems and average linear growth with time of separation distance between initially close phase space states. Specifically, it is shown that almost all solutions to the linearized variational equations derived from bounded, integrable hamiltonian systems exhibit an average linear growth with time, becoming unbounded at $t \to \infty$. The orbits of bounded, integrable hamiltonian systems are thus always locally marginally unstable, forever lying on that sharp border which divides completely stable from completely unstable motion.

For classical hamiltonian systems, the time-averaged rate at which initially close states separate in phase space is a hallmark frequently used [1–7] to establish the generic stability type of the systems. Unstable systems [8] for example, whose motion displays extremely stochastic behavior [9], are actually defined [10] in terms of everywhere exponentially separating states. On the other hand, for KAM stable systems [8] (integrable or near-integrable) which exhibit only a very limited and/or weak type of stochasticity, numerical experiments [7, 11] have so frequently revealed an average linear separation of initially close states that linear separation is now regarded as an extremely strong indicator of integrability or, at worse, near-integrability even though no formal proof of any such connection has been given. In this paper, we analytically establish one-half of this previously missing proof; namely, we show that bounded, integrable hamiltonian systems do indeed yield an average local separation of states which grows linearly with time. Stated this way, however, it might appear that we are here only adding a mathematical footnote to the literature of an already well-established fact. However, if we slightly restate our results, they become somewhat startling, even unbelievable one might say. Certainly we ourselves were incredulous at first, as were several of our colleagues in pure mathematics [‡1]. We now turn to a development of this alternative statement of our results.

Perhaps the simplest way to study the local stability properties of time evolving states for a hamiltonian system is to consider the behavior of solutions to the associated linearized variational equations

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta q_i) = \sum_j \left(\frac{\partial^2 H}{\partial p_i\, \partial q_j}\, \delta q_j + \frac{\partial^2 H}{\partial p_i\, \partial p_j}\, \delta p_j \right), \tag{1a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta p_i) = -\sum_j \left(\frac{\partial^2 H}{\partial q_i\, \partial q_j}\, \delta q_j + \frac{\partial^2 H}{\partial q_i\, \partial p_j}\, \delta p_j \right), \tag{1b}$$

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‡1 Of the several mathematicians with whom we discussed our results, only V.I. Arnol'd anticipated our conclusions.

where (q_i, p_i) denote the positions and momenta of the system having the hamiltonian H, and where $(\delta p_i, \delta p_i)$ denote the infinitesimal variations of the coordinates and momenta. Eqs. (1a, b) are linear, coupled, time-dependent differential equations; the explicit time dependence appears in the coefficients, such as $\partial^2 H/\partial q_i\, \partial p_j$, on the right hand side of eqs. (1a, b) because these coefficients are to be evaluated on a specified reference orbit of the system. But now linear differential equations with periodic coefficients – such as Mathieu's or Hills equation – have been extensively studied [12] (see also the first paper of ref. [3]) by mathematicians; and although eqs. (1a,b), in general, have quasi-periodic coefficients, the integrable systems under consideration do nonetheless posses an everywhere dense set of phase space surfaces entirely composed of periodic orbits. Thus when any one of these periodic orbits is used as a reference orbit in eqs. (1a, b), this well-known mathematical theory asserts that the reference orbit is locally stable when the solutions of eqs. (1a, b) execute bounded oscillations, is locally unstable when they grow exponentially with time, and is locally marginally unstable for the exceptional, borderline case in which the solutions grow linearly with time.

Armed with this well-known information, we numerically integrated eqs. (1a, b) for a reference orbit of the integrable, equal-mass Toda lattice [13]. Since the Toda lattice has only bounded orbits and since any two sufficiently close states must evolve on two everywhere adjacent, nested surfaces in phase space, we anticipated that the associated solutions to eqs. (1a, b) would oscillate indicating local stability of the reference orbit. We were thus quite shocked when the computer "overflowed" indicating that the solutions to eqs. (1a, b) had become numerically too large to be contained in the computer. In short, we found that the reference orbit was locally marginally unstable and that the variational solutions were linearly (in time) headed for infinity when the computer ceased to function. We then immediately showed analytically that this type behavior is not unique to the Toda lattice. Specifically, all orbits of every bounded, integrable hamiltonian system are locally marginally unstable, neglecting orbit sets of small or zero measure; moreover, the solutions to the associated variational equations become unbounded as $t \to \infty$, exhibiting an average linear growth with time. Let us now turn to the proof.

By definition [8], for a bounded, integrable hamiltonian $H = H(q_i, p_i)$ there exists an analytic canonical transformation to new variables (J_k, θ_k),

$$q_i = q_i(J_k, \theta_k), \quad p_i = p_i(J_k, \theta_k)\,, \tag{2a,b}$$

bringing H to the θ_k-independent form $H = H(J_k)$ alone. In terms of the (J_k, θ_k) variables, the solutions to the equations of motion take the especially simple form

$$J_k = J_{k0}\,, \tag{3a}$$

$$\theta_k = [\omega_k(J_{10}, ..., J_{l0}, ...)]\, t + \theta_{k0}\,, \tag{3b}$$

where J_{k0} and θ_{k0} are initial conditions and where $\omega_k = \partial H/\partial J_k$. We may now immediately write down the solution to eqs. (1a, b) by differentiating eqs. (2a, b) and by inserting eqs. (3a, b) to find

$$\delta q_i = \sum_l \left[\frac{\partial q_i}{\partial J_l}\, \delta J_{l0} + \frac{\partial q_i}{\partial \theta_l}(t\delta\omega_l + \delta\theta_{l0})\right]\,, \tag{4a}$$

$$\delta p_i = \sum_l \left[\frac{\partial P_i}{\partial J_l}\, \delta J_{l0} + \frac{\partial p_i}{\partial \theta_l}(t\delta\omega_l + \delta\theta_{l0})\right]\,, \tag{4b}$$

where δJ_{l0} and $\delta\theta_{l0}$ are the initial variations of the (J_l, θ_l) variables and where $\delta\omega_l$ $(= \Sigma_k(\partial\omega_l/\partial J_k)\delta J_{k0})$ is the variation of ω_l due to the variation of the J_{l0}. In eqs. (4a, b), δJ_{l0} and $\delta\theta_{l0}$ are all constants, all coefficients $\partial q_i/\partial J_l$, etc., are at worst quasi-periodic functions, and $\delta\omega_l$ is not zero in general; thus, eqs. (4a, b) make it explicitly transparent that δq_i and δp_i exhibit an average linear growth with time, becoming unbounded as $t \to \infty$. Fig. 1 shows a typical plot of separation distance versus time for two initially close, integrable system states. One here clearly sees the quasi-periodic oscillations superimposed on the average linear growth.

In order to emphasize that this linear average growth is not an artifact due to quasi-periodic reference orbits and in order to illustrate the physical source of the average linear growth with time of variational equation solutions, let us consider any bounded, periodic reference orbit of the integrable, hamiltonian simple pendulum. The variational equation has the form

$$\mathrm{d}^2(\delta q)/\mathrm{d}t^2 + F(t)\,(\delta q) = 0\,, \tag{5}$$

where $F(t + T) = F(t)$ with

$$F(t) = (\omega^2 - \beta^2) + \beta^2\, \mathrm{cn}^2(\omega t)\,, \tag{6}$$

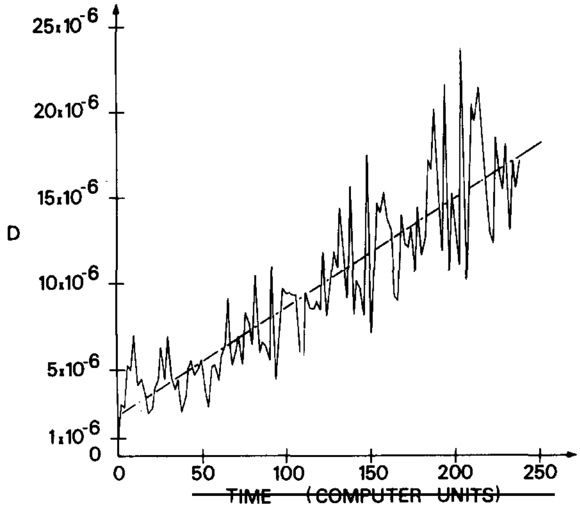

Fig. 1. A graph of phase space separation distance D versus time for two initially close states of an integrable system. As predicted by eqs. (4a, b), small quasi-periodic oscillations appear superimposed on the long term, average linear growth.

$$T = \frac{4}{\omega} \int_0^{\pi/2} [1 - (\beta^2/2\omega^2)\sin^2\alpha]^{1/2}\, d\alpha\,, \tag{7}$$

provided $0 < \beta^2 < 2\omega^2$. Eq. (5) is an example of Hill's equation and, as such, might be expected to yield solutions which oscillate indicating stability, solutions which grow exponentially indicating instability, or solutions which grow linearly with time indicating marginal instability. In the general theory of Hill's equation, marginal instability is the rare and exceptional case and is, therefore, largely ignored in the theory. However, when (as here) the variational eq. (5) is derived from a bounded hamiltonian, then eqs. (4a, b) insure that the solutions to eq. (5) fall precisely in this exceptional case. The surprise of this result can be somewhat removed by noting that Hill's equation is normally used to determine the stability character of isolated equilibrium points or of isolated periodic orbits to which the present theory does not apply as we discuss later; whereas in eq. (5) we are examining the stability properties of a one parameter family of non-isolated periodic reference orbits.

Turning now to the physical or intuitive explanation of the linear growth of the solutions to eq. (5), let us consider fig. 2 which shows a sketch of two adjacent phase space orbits surrounding the stable equilibrium point of the pendulum. The sequential arrows

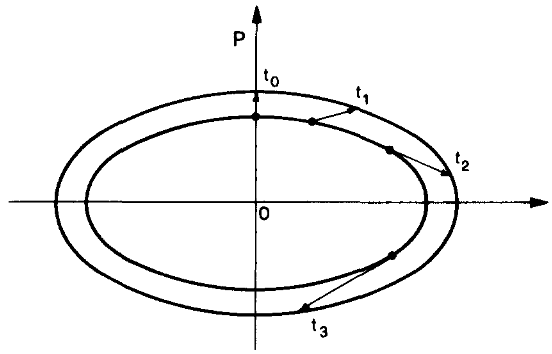

Fig. 2. Two close phase space orbits for the pendulum yielding the variational eq. (5). The sequentially growing arrows indicate the relative positions of the two, initially close, time evolving states.

in this figure depict the average linear growth with time of the separation distance between two initially close states. Here each time evolving state moves periodically along its respective oval, but the rate of rotation is different on each oval, yielding the linear average growth of arrow length. Quite clearly in fig. 2, the evolving arrow will eventually reach a maximum length following which the arrow size will oscillate; in short, the average linear growth will saturate and thereafter oscillate. But as the spacing between the two orbits shown in fig. 2 decreases, the time interval of average linear growth in arrow size becomes increasingly long, until, for the infinitesimally close orbits implied by eq. (5), the average linear growth continues without limit. Thus, despite its paradoxical features, the average linear growth and unbounded behavior of the solutions to integrable system variational equations is intuitively quite obvious.

As a final application of the results presented here, let us discuss a simple, analytical scheme for predicting stable and unstable behavior in hamiltonian systems which has been proposed by Toda [14] and used [15] and amplified [16] by others. It is perhaps easiest to discuss this matter using the integrable hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + q_1^2 q_2 + \tfrac{1}{3} q_2^3\,, \tag{8}$$

which yields variational equations that we choose to write in the matrix form

$$\frac{d^2}{dt^2}\begin{pmatrix}\delta q_1\\ \delta q_2\end{pmatrix} = \begin{pmatrix} -(1+2q_2) & -2q_1 \\ -2q_1 & -(1+2q_2)\end{pmatrix}\begin{pmatrix}\delta q_1\\ \delta q_2\end{pmatrix}. \tag{9}$$

The Toda scheme now computes the instantaneous eigenvalues of the matrix on the right hand side of eq. (9) for a specified configuration space point (q_1, q_2). If the resulting eigenvalues are negative, then δq_1 and δq_2 locally oscillate and the reference orbit passing through the point (q_1, q_2) is claimed to be locally stable; if the eigenvalues are positive, then δq_1 and δq_2 locally exponentiate and the reference orbit is claimed to be locally unstable. For the case at hand in eq. (9), the instantaneous eigenvalues are

$$\lambda = -1 \pm 2q_1 - 2q_2. \tag{10}$$

Recalling hamiltonian (8), we may show that these eigenvalues are strictly negative for system energies $E < 1/24$; there are configuration space regions of positive eigenvalues for $1/24 < E < 1/12$; above $E = 1/12$ the system motion can become unbounded. As a consequence of the theory developed here, we immediately note that the Toda predictions are totally incorrect for this integrable hamiltonian, since its bounded orbits are locally neither stable nor unstable; they are, in fact, all locally marginally unstable. However, our intent here is not merely to present another counter-example to the Toda criterion, since this has been done before by others [17,18]. We wish to make it abundantly clear that the Toda criterion as well as its later modifications [15,16] is neither right nor wrong for any example; it is simply irrelevant. For as eqs. (4a, b) and/or fig. 1 show, it is not the instantaneous behavior of solutions to eqs. (1) that determines the stability character of the reference orbitals; it is rather the average behavior of these solutions, computed over an interval long enough to "wash out" the short term oscillations, that establishes stability type. In short, no matter how alluringly simple the Toda criterion is and no matter how many example systems are discovered for which this method "works", this criterion is, at best, inadequate.

Let us now conclude this paper with a brief discussion of certain exceptional cases for which eqs. (4a, b) do not apply or do not yield linear time growth. We here adopt the attitude of physicists, listing some interesting exceptions without trying to be mathematically exhaustive. First, let us note that the $\delta\omega_l$ in eqs. (4a, b) is zero for integrable harmonic systems or for two initially close states lying on the same orbit; for these cases the reference orbit is stable and the solutions to eqs. (4a, b) are either constant or oscillate. Perhaps the most interesting exceptional reference orbits are isolated periodic orbits or equilibrium points (such as the origin in fig. 2). All such isolated orbits correspond to singularities in the transformation eqs. (2a, b) and, for them, eqs. (4a, b) are not valid. For example, the unstable periodic orbits of integrable systems are "linked" by separatrix surfaces across which eqs. (2a, b) must change their analytical form; or for a stable equilibrium point such as that of fig. 2, the transformation eqs. (2a, b) are singular because the angle variable becomes undefined at the equilibrium point. In short, without entering into all the mathematical details of these singularities, our formalism says nothing about the stability properties of isolated periodic or equilibrium orbits which form a set of measure zero in the totality of all orbits. Our results show that marginal local instability prevails for all the remaining complementary set containing almost all orbits. In a sense, it is our neglect of this commonly considered and significant zero-measure set that gives our conclusions an element of surprise.

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