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DEGENERATION OF TURBULENCE IN SIMPLE SYSTEMS

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The results of numerical studies of a simple dissipative system are presented including the appearance of the stochastic attractor under sufficiently strong dissipation. The degeneration of turbulent motion into a periodical one under a weak dissipation is emphasized and studied both qualitatively and quantitatively including the dependence on the number of degrees of freedom.

Recently a large number of papers has been published in which the stochastic motion in dissipative systems was studied from various points of view. A broad interest to this problem was stimulated, in particular, by a hypothesis due to Ruelle and Takens /1/ who made an attempt to link the problem of turbulence and the modern ergodic theory. Apparently first the stochasticity in a simple dissipative model was observed in numerical experiments by Lorenz /2/ who discovered, in particular, that trajectories of this model were attracted to a set of very complicated structure. Such a set was called the 'strange attractor' in Ref./1/. This popular term seems to us stark unsatisfactory since from the standpoint of modern ergodic theory those structures are, on the contrary, perfectly natural. They are common, for example, in the well-known Anosov systems, or the C-systems /3/, which possess the full set of statistical properties including the strongest one - Bernoilli property.* Graphically speaking, that attractor seems strange only for a stranger. So, we are going to use below another term - the stochastic, or chaotic attractor.

The properties of stochastic attractor were studied in a number of works (see, e.g., review paper /4/). In one of our former studies /5/ we also did observe stochastic attractor which we called the foliation of the phase plane having borrowed the term from the theory of Anosov systems. At the same time we had discovered and studied in more detail later on /6/ the phenomenon of degeneration of stochastic motion into a periodical one. This phenomenon turned out to be fairly common in many simple models of turbulence including the original Lorenz model (see Ref./7/) as well as Henon's map which is similar to one studied in Refs./5,6/. The degeneration occurs due to the capture of a stochastic trajectory, under the influence of a weak dissipation, into one of the stable regions which are typical for Hamiltonian oscillatory systems. Apparently first such a capture was actually observed by Liberman and Lichtenberg /10/. Even though the degeneration of stochasticity is not a universal phenomenon as pointed out correctly in Ref./11/ - it is impossible, for example, in any Anosov system due to the structural stability - nevertheless that phenomenon proves to be quite common for simple dissipative models.

* Let us mention that the latter property was actually discovered and described already by Lorenz in his paper /2/.

In what follows we are going to consider this phenomenon using two models studied in Refs./5,6/.

1. 2-DIMENSIONAL MODEL

The model is described by a map

$$\bar{p} = \{p + k\cdot f(x) - \varepsilon\cdot(p-0.5)\}$$
$$\bar{x} = \{x + \bar{p} - 0.5\} \tag{1.1}$$

where braces denote the fractional part. The map models a nonlinear oscillator with angle coordinate x and momentum (action) p. The fractional part describes a periodical dependence (of period 1) in both coordinate and momentum, the latter being introduced to simplify the computation. As a result the oscillator phase plane is reduced to the unit square, or to be more precise, to a torus. The oscillator suffers a damping (parameter ε) and a periodic perturbation (of iteration period). The dependence of perturbation on coordinate is given by the function $f(x)$.

For a sort of perturbation which we use to call trivial (for more detail see Ref./12/), for instance

$$f(x) = x - 0.5 \tag{1.2}$$

model (1.1) is a C-system /3/, hence, a weak dissipation cannot destroy the stochasticity which takes place under condition

$$k > 0 \quad , \text{ or } \quad k < -4 \tag{1.3}$$

Instead, we are going to consider the perturbation

$$f(x) = x^2 - x + 1/6 \tag{1.4}$$

which is by no means a trivial one since without dissipation ($\varepsilon = 0$) and for any k there exist stable regions on the phase plane (x,p). The biggest of them are formed around the fixed point (a periodic trajectory of period $T = 1$): $\bar{p} = p = p^o$; $\bar{x} = x = x^o$. This point is stable under condition

$$-4 < k\cdot f'(x) < 0 \tag{1.5}$$

whence one can derive (see Ref./5/) the special k_o values for which the fixed point (x^o, p^o) is situated just in the middle of the stable region (1.5). According to Ref./5/ the stable area is estimated by

$$S(k) \sim \left[\frac{4}{k\cdot f''(x^o)}\right]^2 \tag{1.6}$$

For $k_o = 3.46$ and $\varepsilon = 0$ a phase plane picture is given in Fig.1. A region of stable trajectories as well as that of stochastic motion is clearly seen. It is important that the size of stable region decreases rapidly with increase in k even for special k (1.6). According to numerical data in Refs./13,5/ this size drops still more sharply for arbitrary $k \neq k_o$ (see Section 2 below).

The dissipative system (1.1) may be considered as the simplest model of the so-called auto- or self-oscillation. The latter differs from a 'passive' motion damping by the final state of the system which is a limit cycle with finite oscillation amplitude rather

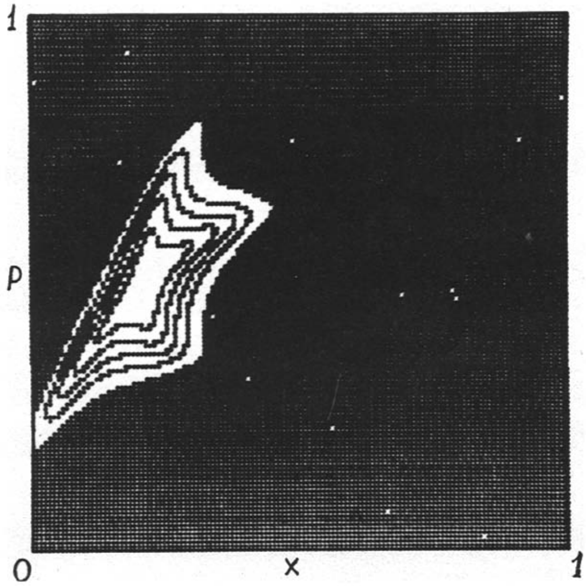

Fig.1 Phase plane of system (1.1,1.4) for $k_o = 3.46$; $\varepsilon = 0$. Four trajectories of different initial conditions (x_o, p_o) are shown inside a stable region. On the outside is the stochastic region filled up with a single trajectory which missed a few bins (white spots). The number of iterations for each trajectory $N = 10^5$.

than the stable equilibrium. For our model (1.1) both cases don't differ at all because of periodicity in p. In work /6/ the dependence of the capture time on various parameters was studied. It turned out that the following rather accurate empirical relation holds

$$N \cdot S \cdot \varepsilon \approx 1 \quad (1.7)$$

where N is the number of iterations of the map (1.1) until the capture (the stochasticity 'life' time); where S stands for the area of the stable regions (the phase square area is equal to 1) and where ε is the damping parameter. The data concerning relation (1.7) are summarized in Fig.2.

Relation (1.7) has been established via independent measurements of the mean capture time N and the area of the stable regions for values of $k_o = 3.46$; 5.56; 12.98. To measure the stable area the phase square (x, p) was subdivided into a number of bins the crossing of which by a stochastic trajectory having been registered. The number of missed bins did determine then the stable area. The finest subdivision of the phase plane amounted to 512 x 1024 = 524288 bins.

For $k = 7.66$ we failed to measure the stable region which turned out to be much stretched in one direction while shrinked in the other. Relation (1.7) leads to $S \approx 2.8 \times 10^{-3}$ in this case.

One could expect the capture time $N_1 \sim 1/S$. It is the period of time during which a moving at random system gets into a vicinity of stable area S. Watching the motion on display has shown, indeed, that the system used to encounter the stable area in N_1 iterations at average. However, the probability of capture proved to be very low ($\sim \varepsilon$). This is apparently related to the transition zone surrounding a stable region, the diffusion being very slow in the former. This transition zone as if defends the stable region against penetration there a stochastic trajectory. Fig.3 shows an example of motion in the transition zone followed an encounter of a stable region. The structure of the transition zone was studied in detail in Ref./14/. Zone relative area rapidly increases with decreasing the absolute size of stable region. For example, at $k_o = 120.1$ the transition zone area roughly equals that of the stable region itself ($S \approx 1.6 \times 10^{-4}$). We managed to measure the latter by means of on-line display. As soon as we saw on the display a trajectory penetrated into the transition zone and filled it up we stopped computation and recorded the distribution of trajectory

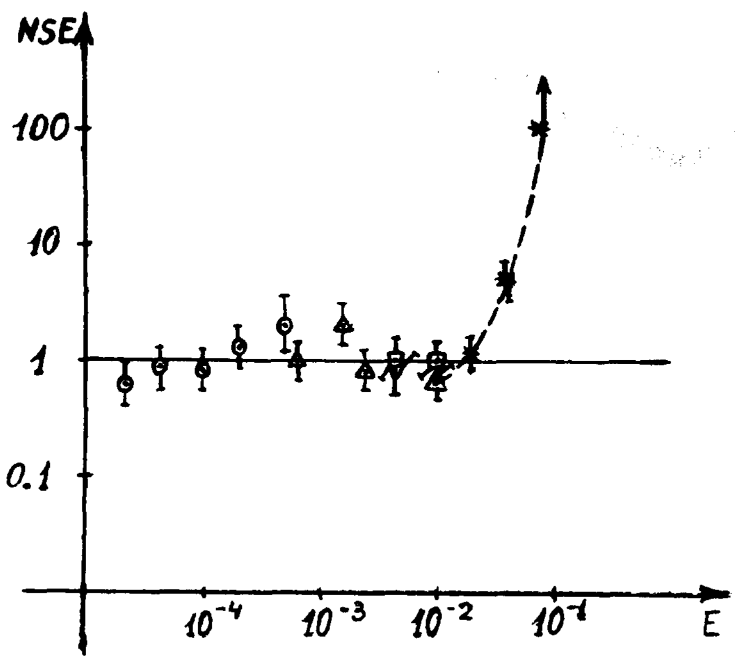

Fig.2 The stochasticity 'life time' vs. dissipation: ⊙ — k = 3.46; △ — k = 12.98; ⊡ — k = 5.56; ✱ — k = 7.66; straight line corresponds to relation (1.7); dotted line indicates transition to stochastic attractor. Arrow at the last point points out that the capture has not been actually observed, the point giving the lower border for the life time.

over the transition zone that allowed us just to estimate its area. In Fig.2 the dotted curve indicates a sharp increase in the capture time for a relatively strong damping ($E > 10^{-2}$). A detailed analysis of the phase plane structure has revealed (see Ref./6/) that a sort of stochastic attractor appeared in this case. As seen in Fig.4, there are some (white) regions which seem to be 'forbidden' for the motion. If initially the system happens to be within one of those regions it comes out immediately and never gets back. These regions have shape of strips with various width, and their central curves correspond to the iteration of the 'maximal damping' line $p = 0$. Successive iterations of this line are related to more and more narrow forbidden strips which form a fine structure of the foliation. This structure is clearly seen in Fig.4b.

The whole phase plane is foliated, thus, into two components one of which contains 'forbidden', or repulsion regions while the other, to which all the trajectories are attracting, is just the stochastic attractor of the Cantor structure and of zero measure. It is interesting to mention that for still bigger E the attraction structure looks like just a few lines. However, some Cantor structure appar-

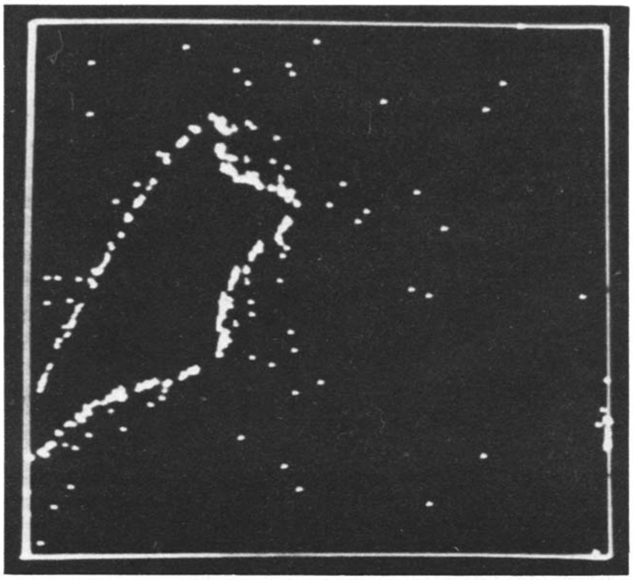

Fig.3 Motion of system (1.1,1.4) in transition zone, $k_0 = 3.46$; $\varepsilon = 0$. The picture was taken after the penetration of trajectory into transition zone. Number of points (iterations) is 400.

ently persists within these lines as well but it is more difficult to resolve the structure numerically. A similar observation has been made in Ref./8/.

The foliation occurs apparently for any small ε including those for which the degeneration happens later on. Hence the foliation picture observed on display does not prove, generally, that the stochastic attractor really exists. A sharp increase of the capture time for sufficiently strong dissipation (see Fig.2), and a corresponding coarse-grained foliation, is a more straightforward indication of the bifurcation to stochastic attractor. This increase in capture time is related apparently to the destruction, at least, of some stable regions. In Fig.2 the dotted curve has drawn assuming the independence of the stable region area from dissipation. If one would consider, on the contrary, that relation (1.7) persists the stable area would decrease down to $S_1 < 2.5 \times 10^{-4}$ (the last point in Fig.2).

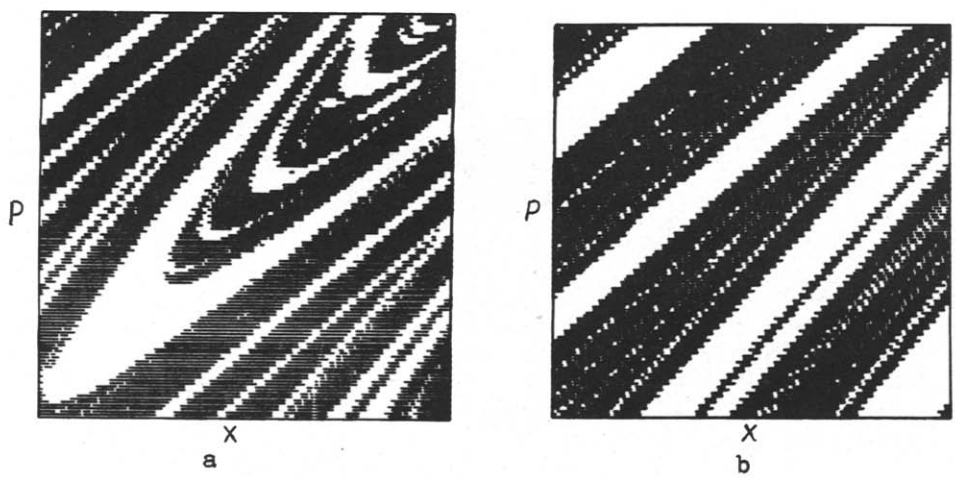

Fig.4 a - Phase plane foliation, or alleged stochastic attractor; $k \approx 9.76$; $\varepsilon = 0.2$; $N = 10^5$.

b - A part of the phase plane in Fig.4a, magnification x 16.

2. 4-DIMENSIONAL MODEL

The studies of 2-dimensional model (Section 1) have shown that the stochastic self-oscillations may take place within some interval of time (before the capture). It is true that the stochasticity life time grows rapidly with parameter k in Eq.(1.1) because of a sharp decrease in the area S of the stable regions.* According to Ref. /13/ the following estimate holds for $E = 0$:

$$S \sim \exp\left[-3\left(\ln\frac{k}{2}\right)\cdot\left(\sqrt{\frac{\pi k}{2}}-1\right)\right] \tag{2.1}$$

Nevertheless, for moderate $k \sim 4$ the stable regions are fairly big, and hence the stovhasticity life time is rather short (see Eq.(1.7)).

What might be the influence of the number of degrees of freedom? To answer this question we studied a 4-dimensional model described by the map:

$$\begin{aligned} \bar{p} &= \{p + k_1\cdot f(x) - E\cdot(p-0.5) + C\cdot(y-0.5)\} \\ \bar{q} &= \{q + k_2\cdot f(y) - E\cdot(q-0.5) + C\cdot(x-0.5)\} \\ \bar{x} &= \{x + \bar{p} - 0.5\} \\ \bar{y} &= \{y + \bar{q} - 0.5\} \end{aligned} \tag{2.2}$$

where C is coupling parameter. For $E = 0$ this map is canonical. Under $C \sim 1$ (strong coupling) the capture was not observed up to $N = 10^8$ iterations for any k used in computation including $k \approx 1$ when the corresponding 2-dimensional system ($C = 0$) has a big stable region. This suggests that the stable regions become very small. The stability conditions for a fixed point have now the form (comp. Eq.(1.5)):

$$-4 < \left(\xi_1 + \xi_2 \pm \sqrt{(\xi_1-\xi_2)^2 + 4C^2}\right) < 4 \tag{2.3}$$

where $\xi_1 = 2 + k_1 f'(x)$; $\xi_2 = 2 + k_2 f'(y)$. For stability both inequalities have to hold with both signs. Since conditions (2.3) are obviously more rigid as compared to Eq.(1.5) the size of stable regions drops considerably.

Numerical determination of the stable region volume is rather difficult even for $E = 0$. For example, with $k \approx 3$, $C = 0.9$ a trajectory of arbitrary initial conditions filled up all the bins of the phase hypercube subdivision 32 x 32 x 32 x 16 = 524288. Nevertheless, a stable fixed point does exist at $p_0 = q_0 = 0.5$; $x_0 = y_0 = 0.025$. The trajectory started near this point fills only about 3×10^{-5} fraction of the full phase volume. This seems to indicate that the stable regions may be much stretched and, thus, may occupy only a part of each bin.

For a weak coupling the capture does occur, yet it disappears sharply by a negligible increase in coupling parameter ($\Delta C \approx 10^{-3}$, $N = 10^5$ iterations). The critical coupling depends on dissipation as follows: $C_{cr} = 0.10$; 0.179; 0.30 for $E = 0.05$; 0.1; 0.2 relatively.

An example of 2-dimensional projection of a phase trajectory is given in Fig.5. Unlike the 2-dimensional model a distinctive feature of this motion is the presence of some clearly regular structure.

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* off the special intervals of k described in Section 1.

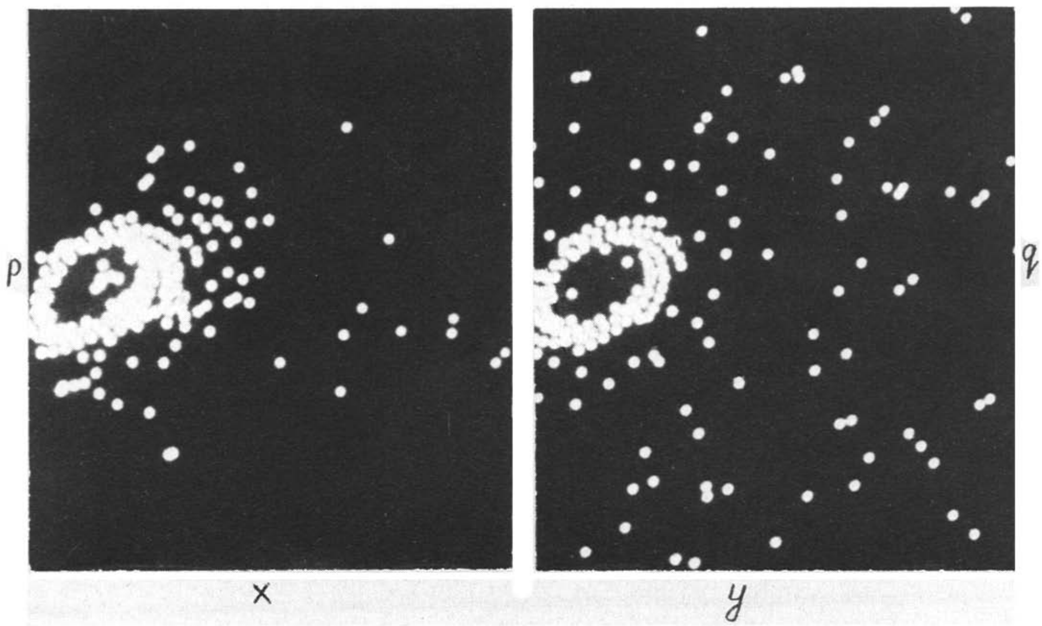

Fig.5 An example of 2-dimensional projections of a stochastic trajectory in 4-dimensional phase space of system (2.2): $k_1 \approx 1.278$, $k_2 \approx 1.25$; $C = 0.2$; $E = 0.1$. On the left - projection (x,p) ; on the right - projection (y,q) . Passing of a regular structure is recorded.

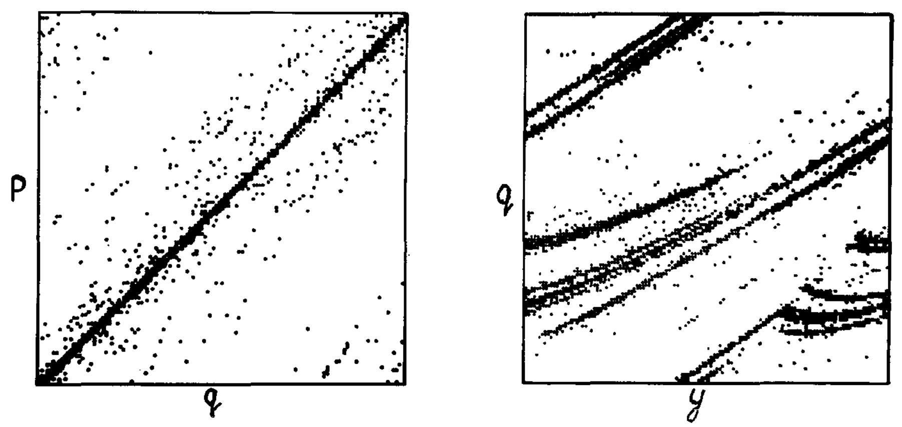

Fig.6 Stochastic attractor in system (2.2) under s strong dissipation: $k_1 \approx 1.278$; $k_2 \approx 1.25$; $E = 0.6$; $C = 0.8$. On the left, projection (q,p) ; on the right, projection (y,q) ; $N = 10^4$.

Watching the motion on display reveals that such structures always do appear in an irregular manner to exist a relatively short period of time ($\sim 10^4$ iterations). If the coupling is decreased down to the critical value that change of structures seems to become more regular, yet still not a periodical one.

For a strong dissipation ($\varepsilon \sim 1$) the stochastic attractor was observed in the model (2.2) as well that is a foliation of the 4-dimensional phase space (see Fig.6).

Thus, the reported numerical experiments show that the stochastic motion in a nonlinear oscillatory system may degenerate into a periodical one under a weak dissipation. The 'life time' of stochastic 'self-oscillations' depends on the area of remained stable regions which, no matter how small they are, do influence the global dynamics of the system. Increasing the number of degrees of freedom sharply enlarges the stochasticity life time.

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