Dynamics of some homogeneous models of classical Yang-Mills fields

B. V. Chirikov and D. L. Shepelyanskiĭ

Institute of Nuclear Physics, Siberian Division, USSR Academy of Sciences
(Submitted 8 July 1982)
Yad. Fiz. 36, 1563–1576 (December 1982)

The role of spatially homogeneous models in the theory of classical Yang-Mills fields is considered. The dynamics of some of these models is investigated. It is shown that chaotic motion is typical for their time evolution. In particular, the interaction with Higgs fields does not always lead to stabilization of the motion.

PACS numbers: 11.15.Kc

INTRODUCTION

The Yang-Mills (YM) gauge fields were introduced[1] for an isotopic-invariant description of strong interactions. Currently they find wider application in elementary particle theory. Therefore investigation of their properties and in particular their dynamics remains an interesting and important problem. The study of classical YM fields, apart from general knowledge of their properties and applications in several problems of quantization,[2] is of special interest in connection with the fact that essentially nonlinear processes in these fields lie in the quasi-classical domain of usual (pseudo-Euclidean) or Euclidean (imaginary time) space-time. In the latter case classical solutions describe strong vacuum fluctuations of the YM fields[3-5]. The essential nonlinearity of these fields leads to very interesting and unusual dynamics but at the same time significantly hinders its study. In the circumstances it is natural to turn to simple models which can clarify those or other properties of YM fields.

An important class of models was introduced and studied by Matinyan and collaborators.[6-8] In these models the field depends only on time, and we will call them homogeneous models (HM). A considerable simplification of the problem is attained here owing to the finite dimensions of the model, the number of degrees of freedom of which $N=3(n^2-1)$ is determined by the symmetry group $SU(n)$. The number of different fields (colors) in a multiplet (n^2-1) determines the dimensions of the internal space of YM fields. HM provide in "pure form" the internal dynamics of YM fields. In the general case this dynamics turns out to be chaotic (stochastic) or random (for classical YM fields).[7-9] The present work is basically devoted to the study of it. Below we restrict ourselves to the simplest but nontrivial case of the group $SU(2)$ and will consider almost exclusively free (without sources) YM fields.

1. HOMOGENEOUS MODELS OF YM FIELDS

It is of course possible to consider a homogeneous model simply as a very special limiting case of YM fields. It is useful, however, in such situations to imagine that the HM approximately correspond to more realistic inhomogeneous fields. For the group $SU(2)$ the internal space of the field is three-dimensional and every external[1)] component of the field (potential) is represented by the internal vector $\mathbf{A}_\mu (\mu=0,1,2,3)$ or equivalently every internal component is described by an external (usual) 4-vector $A^a (a=1,2,3)$. Correspondingly every external component of the tensor field (intensity) is an internal vector $(\hbar=c=1)$

$$\mathbf{F}_{\mu\nu}=\partial_\mu \mathbf{A}_\nu-\partial_\nu \mathbf{A}_\mu+\mathbf{A}_\mu\times\mathbf{A}_\nu. \tag{1.1}$$

Let us write the YM Lagrangian field density with a source in the form[10]

$$g^2L=\mathbf{E}_k\cdot\dot{\mathbf{A}}_k-{}^1/_2(E_k^2+B_k^2)+\mathbf{A}_k\cdot\mathbf{j}_k-\mathbf{A}_0\cdot\mathbf{C}, \tag{1.2}$$

where the Latin indices run over the values 1, 2, 3. Here the "electric" fields

$$\mathbf{E}_k=\dot{\mathbf{A}}_k-\partial_k\mathbf{A}_0+\mathbf{A}_k\times\mathbf{A}_0 \tag{1.3}$$

play the role of canonical momenta conjugate to the coordinates $\mathbf{A}_k$; the "magnetic" field is $\mathbf{B}_k=\frac{1}{2}\varepsilon_{klm}\mathbf{F}_{lm}$, and the vector $\mathbf{C}$ is

$$\mathbf{C}=D_k\mathbf{E}_k+\mathbf{j}_0=\partial_k\mathbf{E}_k+\mathbf{A}_k\times\mathbf{E}_k+\mathbf{j}_0=0. \tag{1.4}$$

The last equality follows, for example, from the fact that the term $\mathbf{A}_0\cdot\mathbf{C}$ can be removed by a gauge transformation $(\mathbf{A}_0\to 0)$. The equation $\mathbf{C}=0$ expresses (generalized to YM fields) the Coulomg law, which does not have the character of equations of motion for the field. It is possible to consider the equation $\mathbf{C}=0$ as a constraint.[10] However, for our purposes it is convenient (and natural) to interpret equation (1.4) as a conservation law differing from the usual ones only in that the value of this integral of motion is fixed (a restriction on the initial conditions).[2)] Explicit elimination of an integral of the equations of motion is not obligatory and is even undesirable (in any case in so far as this concerns classical fields), since reduced phase space (the integral manifold), as well as the Hamiltonian on it, can prove to be exceedingly complicated.[11,12] In addition, the symmetry of the original system would be lost, which would hinder the qualitative analysis of its dynamics.

In the following it will be convenient for us to use a gauge where $\mathbf{A}_0=0$. Then $\mathbf{E}_k=\dot{\mathbf{A}}_k$ (1.3), and the Hamiltonian density $(\mathbf{j}_\mu=0)$ is

$$H(\mathbf{E}_k, \mathbf{A}_k) = {}^1/_2 E_k^2 + {}^1/_4 \sum_{k,l} |(\partial_k \mathbf{A}_l - \partial_l \mathbf{A}_k) + \mathbf{A}_k \times \mathbf{A}_l|^2, \qquad (1.5)$$

:re we have made the substitutions $g^2 H \to H$ and t/g

'or $\mathbf{A}_k \to 0$, the Hamiltonian (1.5) describes a linear ;tem of plane waves, weakly coupled by the nonlinear :turbation $\mathbf{A}_k \times \mathbf{A}_l$. In the opposite case ($\partial_k = 0$) we get HM, which describes, in this way, the internal non- ear dynamics locally with respect to external space. e approximation of HM corresponds to long waves (λ) d strong fields: $\lambda A \gg 1$ or in a more descriptive ·m

$$g^2 n_g \lambda^3 \gg 1, \qquad (1.6)$$

ere n_g is the density of the (massless) quanta of YM lds.

Γhe Hamiltonian of the HM has the form[6,7]

$$H_{YM} = {}^1/_2 E_k^2 + {}^1/_4 |\mathbf{A}_k \times \mathbf{A}_l|^2 = {}^1/_2 (E^a)^2 + {}^1/_4 |\mathbf{A}^a \times \mathbf{A}^b|^2. \qquad (1.7)$$

ı the basis of isotropy of both external and internal ace (the potential energy depends only on the angles tween the vectors $\mathbf{A}^a$ or $\mathbf{A}_k$) two "moments" are con- :rved:

$$\mathbf{M} = \mathbf{A}^a \times \mathbf{E}^a = \text{const}, \quad \mathbf{M}_g = \mathbf{A}_k \times \mathbf{E}_k = 0. \qquad (1.8)$$

ıe last expression is a special case of the general re- tion (1.4) for $\mathbf{j}_0 = 0$ and $\partial_k = 0$. We notice that for a YM ɜld with sources, $\mathbf{M}_g = -\mathbf{j}_0 \neq 0$. If, in addition, $\mathbf{j}_k = 0$, en the Hamiltonian (1.7) and consequently the equa- ons of motion are not changed. We note that HM do ɔt describe charged states of free YM fields for which $_g \neq 0$ [see (1.4)].

The Hamiltonian (1.7) is symmetric also with respect) the transposition of the matrices A_k^a, i.e., relative to ıternal and external subspaces of the field. In so far s the Hamiltonian is an even function of the quantities $_k^a$, there exist (particular) symmetric and antisymmet- ic solutions (fields). In the first case $A_l^k = A_k^l$, i.e., the ıatrix is symmetric and by a rotation of the coordinate xes it is possible to bring it to diagonal form. For a ree YM field $M = M_g = 0$ in this case, from which it fol- ɔws that the principal axes of the matrix $\mathbf{A}_k^a$ are fixed nd the Hamiltonian (1.7) takes on the simple form

$$H_s = {}^1/_2 [E_1^2 + E_2^2 + E_3^2 + (A_1 A_2)^2 + (A_1 A_3)^2 + (A_2 A_3)^2]. \qquad (1.9)$$

:his system has a total of three degrees of freedom.

In the antisymmetric case ($A_l^k = -A_k^l$), it is possible by rotation of coordinate axes to eliminate two of the hree independent elements of the matrix A_k^a. For ex- ımple, let $A_1^b = -A_2^a \equiv A$; then the Hamiltonian (1.7) takes he form

$$H_a = E^2 + {}^1/_2 A^4, \qquad (1.10)$$

.e., only one degree of freedom remains.

Another class of HM, also introduced and investigated ın Ref. 8, is obtained by calculation of the gauge inter- ɑction of the YM field with a Higgs field. In the last case there arise in the Hamiltonian (1.7) (mass) terms quadratic in A:

$$H = H_{YM} + {}^1/_2 \omega^2 (A^a)^2. \qquad (1.11)$$

It is essential that on the basis of the isotropy of internal space the additional potential energy is also spherically symmetric. Therefore, as before, $\mathbf{M}_g = 0$ and the transposition symmetry of the matrix A_k^a is conserved. Generally speaking, in addition, there arises an additional degree of freedom corresponding to the dynamics of the Higgs field itself. Below we restrict ourselves only to models (1.11) which describe the interaction of YM fields with a Higgs vacuum.

2. LOCAL INSTABILITY OF MOTION AND CHAOS

Among the "simple" HM mentioned in the previous section only the dynamics of the system (1.10) turns out actually to be simple, even simpler than could be expected in the case of a strongly nonlinear equation of motion, namely[13]

$$A(t)/A_{max} \approx 0.96 \cos \omega t + e^{-\pi} \cos 3\omega t + \dots, \qquad (2.1)$$

i.e., the oscillations are almost harmonic although their frequency $\omega \approx 0.85 A_{max}$ depends substantially on the amplitude. Such a solution of the motion for YM fields was considered in Refs. 6 and 7.

The dynamics of the remaining HM with more than one degree of freedom turn out, generally speaking, to be very complicated and in some sense extremely complicated. The mechanism of this complexity is connected with the strong local instability of the trajectories, characteristic of nonlinear oscillations in general and of classical YM fields in particular. Let us illustrate this using the example of model (1.9) in the particular case where $A_1 \equiv 0$. The equations of motion have the form

$$\ddot{A}_2 = -A_2 A_3^2, \quad \ddot{A}_3 = -A_3 A_2^2. \qquad (2.2)$$

Let $A_i^0(t)$ be some solution of this system. Let us consider the behavior of the nearby trajectories in the linear approximation. Assuming $A_i = A_i^0(t) + a_i$ we obtain the linearized equation for $a_i(t)$:

$$\ddot{a}_2 = -[A_3^0(t)]^2 a_2 - 2A_2^0(t) \cdot A_3^0(t) \cdot a_3,$$
$$\ddot{a}_3 = -[A_2^0(t)]^2 a_3 - 2A_3^0(t) \cdot A_2^0(t) \cdot a_2. \qquad (2.3)$$

We introduce the "distance" between nearby trajectories in phase space

$$\rho^2 = a_2^2 + a_3^2 + \dot{a}_2^2 + \dot{a}_3^2 \qquad (2.4)$$

and consider the quantity

$$\lambda = \lim_{t \to \infty} \frac{\ln[\rho(t)/\rho(0)]}{t}, \qquad (2.5)$$

which is called the Liapunov exponent and generalizes the corresponding concept for periodic trajectories $A_i^0(t)$. On the basis of the ergodic theorem of Birkhoff and Khinchine (see, for example, Ref. 14) this limit always exists but depends, in general, on initial conditions both in phase space ($A_i, \dot{A}_i$) and in the tangent space ($a_i, \dot{a}_i$). From the linearity of Eqs. (2.3) it follows that there are $2N$ Liapunov exponents (λ_i) and their corresponding characteristic directions in the tangent space. From conservation of phase volume $\sum \lambda_i = 0$, but from time reversal it follows that all the λ_i are divided into pairs $\lambda_{\pm}^{(k)}$, such that for each pair, $\lambda_+^{(k)} + \lambda_-^{(k)} = 0 (k = 1, \ldots, N)$. For a closed system with only the energy integral, the trajectory lies completely on a $(2N-1)$-dimensional energy surface. One of the characteristic directions on this surface corresponds to a displacement along a trajectory associated with $\lambda = 0$. In this way, the number of positive $\lambda_+ > 0$ does not exceed $N - 1$, and $N = 2$ is already sufficient for an exponential, local instability of the motion. Example (2.2) has exactly the minimum dimensions.

The quantity λ does not depend on the inital conditions in the basic phase space in the range of each of the ergodic components of motion into which all phase space is divided for any dynamical system with an integral invariant. An ergodic component can envelope both the whole energy shell (for a closed system)—this is ergodicity in the usual sense from which comes the term—and (in the other extreme case) only, for example, a periodic trajectory. In any case the trajectory (chaotic or regular) is uniformly distributed over the whole ergodic component, i.e., the average time of occupation of a system in any element of an ergodic component is proportional to the invariant (conserved during the motion) measure of that element (ergodic theorem of Birkhoff and Khinchine). For Hamiltonian systems the phase volume is an invariant measure (Liouville's theorem) and in the presence of integrals of motion (I_k) the induced (on the integral shell) measure is

$$d\mu = \prod_k \delta(I_k - I_k^0)\, d\Gamma, \tag{2.6}$$

where the I_k^0 are the values of the integrals fixing the integral shell, and $d\Gamma$ is the total phase-volume element. We note that μ is proportional to the volume of a narrow "tube" in phase space given by dI_k.

The Liapunov exponent determines one of the most important characteristics of motion, the metric entropy[14]: $h = \Sigma\lambda_+ \; (\lambda_+ > 0)$. In numerical modeling it is simplest when the maximum exponent occurs because the initial tangent vector $(a_i, \dot{a}_i)$ can be chosen arbitrarily without bothering to search among the characteristic directions. For $\lambda_{max} > 0$ in practice any initial vector will quickly approach the characteristic direction with $\lambda = \lambda_{max}$. On the other hand the knowledge of λ_{max} is sufficient because it is essentially not the exact value h, but the fact that $h \geq \lambda_{max} > 0$, i.e., the fact that the trajectories have an exponential, local, instability. For the example (2.2) considered, $h = \lambda_{max}$.

The significance of the quantity h lies, above all, in the fact that according to the modern theory of dynamical systems the condition $h > 0$ is necessary and sufficient to have a probability of almost all trajectories.[15] Obviously this is related to the fact that the concept of a trajectory loses meaning in the conditions of strong, local instability of motion; thus we speak of the absence of unstable equilibrium states. If the point of view is raised to a more formal level, then it is possible to understand more deeply the nature of this curious phenomena of dynamic probability. Namely, we will assume that the system moves strictly according to the determined trajectory completely given by the initial conditions. Then, with exponential, local instability, the trajectory will, in the course of time, essentially depend more and more on the fine details of the exactly given initial conditions, details of which for stable motion can be completely neglected. Consequently, the source of randomness is imbedded in the initial conditions of motion, while the role of a strictly dynamical system is only to guarantee local instability. In such a capacity a dynamical system can be very simple, a fact appearing paradoxical even to the present day. The argument here that almost all initial conditions correspond precisely to random motion is nontrivial. Clearly, it is possible to present this as follows: any regularities of a trajectory seldom shrink in size from the corresponding initial conditions. A popular exposition of these questions can be found in Ref. 16.

Besides the fundamental meaning for the dynamics of systems, local instability appears to be the most convenient criterion of dynamical chaos (or its absence) for numerical models. Namely, there was the discovery[17] of the so-called Toda lattice, a strongly nonlinear multidimensional system of coupled oscillators.

A practical calculations of λ_{max} is carried out by means of a combined numerical integration of the fundamental Eq. (2.2) and the linearized Eq. (2.3). In order to avoid too large values of ρ it is necessary periodically to shorten the length of the tangent vector without changing its direction. A different method consists of integrating directly two very nearby trajectories of the system (2.2) also with a periodic reduction of the distance between them.

One should bear in mind that for quasi-periodic motion (a completely integrable system) local instability still arises at the expense of a dependence of frequencies on the initial conditions although at the same time $\rho \propto t$ (on the average) (see, for example, Ref. 18). In the limit $t \to \infty$, $\lambda = h = 0$, but it is necessary to continue a numerical calculation sufficiently long in time to distinguish, with confidence, an exponential from a linear dependence. From this it also follows that the conclusion that chaos of motion is present is more reliable than that it is absent (h can turn out to be too small).

3. MASSLESS YM FIELDS

We shall begin with the model (1.9) with $N = 3$ (degrees of freedom), in which chaos turns out to be simpler than in the particular case[9] $A_1 \equiv 0$ ($N = 2$). On the basis of the homogeneity of the potential energy a change of the quantity H results in a change of the time scale, $t \propto H^{-1/4}$, in particular $\lambda \propto H^{1/4}$. In the given case the energy shell $H = \text{const}$ is five-dimensional [both momenta (1.8) are exactly equal to zero], so that h can exceed the maximum Liapunov exponent.

Numerical modeling shows[9] that the limit in (2.5) is attained sufficiently fast (after times ~10^3), and the spread of values of $\lambda H^{-1/4}$ for individual trajectories is relatively small: $\langle \lambda H^{-1/4} \rangle = 0.38 \pm 0.04$ (for 22 trajectories with randomly chosen initial conditions), so that this argues for the existence of a single chaotic component of motion on every energy shell.

Let us consider some special trajectories. Let, for example, $A_2 = A_3 = \dot{A}_2 = \dot{A}_3 = 0$. Such initial conditions correspond to uniform motion along the A_1 axis and constant E_1.[7] Let us consider the nearby trajectory $A_1^2 \gg A_2^2 + A_3^2 \equiv A_\perp^2$. The Hamiltonian (1.9) assumes the form

$$H \approx {}^1/_2(E_1^2 + E_\perp^2 + A_1^2 A_\perp^2) \approx {}^1/_2 E_1^2 + A_1 J_\perp. \tag{3.1}$$

The last expression is correct in the adiabatic approximation when the transverse action $J_\perp = H_\perp / A_1 \approx \text{const}$ (the longitudinal coordinate A_1 plays the role here of the variable frequency of transverse motion). From (3.1) it is clear that for any $J_\perp > 0$ a reflection of the trajectory occurs when $A_1 = A_m = H/J_\perp$. After a reflection $A_1 = A_m - J_\perp t^2/2$ and the adiabatic condition $|\dot{A}| \lesssim A_1^2$ is violated when $A_1 \lesssim (A_1 A_m)^{1/2} \sim H^{1/4}$. As a consequence of that, $J_\perp$ substantially changes, and the trajectory is found to be chaotic. In this way, uniform motion along any of the axes turns out to be unstable, although the development time of the instability ($T \sim \sqrt{A_m/J_\perp} \sim 1/A_\perp$) grows indefinitely as $A_\perp \to 0$.

Cophasal oscillations $A_1(t) = A_2(t) = A_3(t)$ form another particular trajectory.[6,7] As shown in Ref. 9, these oscillations are stable, the mechanism of stability turning out to be extremely peculiar. Small transverse oscillations in a neighborhood of a periodic solution are strongly nonlinear, and their frequency tends to zero with a decrease of their amplitude. The region of stability is very small and has, apparently, a complicated form.

Let us turn now to the case $N = 2$ ($A_1 \equiv 0$). Just this model was studied numerically in Ref. 7, where the first data were obtained indicating chaotic motion (see also Ref. 8). According to our data,[9] the motion here is also locally unstable; however, we failed to obtain a definite numerical value of the entropy $h = \lambda$ because of strong fluctuations and the poor convergence of the expression (2.5). This is explained by the fact that the invariant measure (2.6) diverges on the energy shell. In fact

$$\mu_H = 2\pi \int \delta(E^2 - 2H^0 + A_2^2 A_3^2) E \, dE \, dA_2 \, dA_3$$
$$= \pi \int_{(A_2 A_3)^2 < 2H^0} dA_2 \, dA_3 \to 8\pi \sqrt{2H^0} \ln A \to \infty. \tag{3.2}$$

The divergence of the measure shows that for an overwhelming fraction of time a chaotic trajectory lies in regions with $|A_k| \to \infty$. Any distribution function will "recede" in the course of time further and further along the A_2 and A_3 axes, and its density decreases indefinitely at any point on the energy shell.

On the other hand for a sufficiently large (or small) ratio $|A_2/A_3|$ there is, as in the case $N = 3$, an additional integral of motion, which is the action $J_2 = H_2/A_3$, where $2H_2 = E_2^2 + (A_2 A_3)^2$, and we have assumed for definiteness that $A_3 \gg |A_2|$. Therefore in these regions the motion is regular and, in particular, exponential, local instability is absent. The latter occurs only in the nonadiabatic region $|A_3| \lesssim H^{1/4}$, the measure of which $\mu_\lambda \sim H^{1/2}$ is finite. From this for $t \to \infty$

$$h = \lambda \to H^{1/4} \frac{\mu_\lambda}{\mu_H} \to \frac{H^{1/4}}{\ln A} \sim \frac{H^{1/4}}{\ln t} \to 0. \tag{3.3}$$

Here the connection between A and t is obtained from an estimate of the rate of diffusion with respect to A: $D_A = \langle(\Delta A)^2\rangle/t \sim A\sqrt{H}$, from which $A \sim \sqrt{H}t$.

Although with respect to A the motion considered is infinite, the field intensities $B_1 = A_2 A_3$, $E_2 = \dot{A}_2$, and $E_3 = \dot{A}_3$ remain finite from conservation of energy. Moreover, the successive magnitudes $B_1 \sim E_2 \sim E_3$ even if, say, $A_3 \to \infty$. However, at the same time, the frequency of oscillations of A_2 and B_1 grows indefinitely, and the frequency of A_3 decreases in such a way that the product of the frequencies remains unchanged.

For $N = 3$ the invariant measure of the energy shell is finite, although the shell is not closed, i.e., its "size" with respect to A_k is not bounded. To be convinced of this we shall calculate the measure on one of the six "tubes" running off to infinity ($A_1^2 \gg A_\perp^2$):

$$\mu_H(A_1 > A) \propto \int_A^\infty A_\perp^2 \, dA_1 \propto \int_A^\infty \frac{dA_1}{A_1^2} \to 0$$

for $A \to \infty$. Therefore in the given case there is the usual relaxation to the equilibrium distribution with a nonzero density.

There is still one curious feature of the model (1.9) for $N = 2$ related to the initial conditions of motion. For a complete determination of the trajectory it is necessary to give the four quantities A_2, $\dot{A}_2$, A_3, and $\dot{A}_3$. Although in the problem there are four components of the stress tensor, the sum of two of them is exactly equal to zero in view of the antisymmetry of the tensor: $F_{23} = -F_{32} = B_1$. Consequently, in contrast to linear fields, specifying the tensor $F_{\mu\nu}$, generally speaking, still does not determine completely either the state of the nonlinear field or its dynamics. The first example of a similar nonuniqueness was constructed in Ref. 19 (see also Ref. 20). In our case it is necessary in addition to give the ratio A_2/A_3. A change of this ratio results in different dynamics not equivalent to a gauge transformation. If, for example, $A_2 = A_3$ and $E_1 = E_2 = 0$, the motion will be periodic (although unstable), while for $A_2 \neq A_3$ with $B_1 = A_2 A_3$ the motion becomes chaotic. Instead of the ratio A_2/A_3 it is possible to give $\dot{E}_2$ or $\dot{E}_3$ or finally their ratio $\dot{E}_2/\dot{E}_3$. If B_1 is given then, generally speaking, either two values or no value of A_2/A_3 is obtained.

There is a curious feature of the initial conditions in the case $N = 3$ as well. Let us assume $B_1 = A_2 A_3$, $B_2 = A_1 A_3$, and $B_3 = A_1 A_2$ from which ($B_k \neq 0$) we have

$$A_1^2 = \frac{B_2 B_3}{B_1}, \quad A_2^2 = \frac{B_1 B_3}{B_2}, \quad A_3^2 = \frac{B_1 B_2}{B_3}. \tag{3.4}$$

For given B_k the field can be found in two states differing by a change of sign of all the A_k. If, in addition, $E_k = \dot{A}_k \neq 0$, then the motion of the two initial states will be different (one trajectory is obtained from the other by $t \to -t$ and $A_k \to -A_k$). On the other hand the components B_k cannot be given arbitrarily but must satisfy the condition $B_1 B_2 B_3 \geq 0$ from (3.4), whereupon in the case of equality at least two components B_k must be zero. It is possible to show that a similar restriction survives in the case of arbitrary vectors $\mathbf{B}_k$.

4. MASSIVE YM FIELDS

Let us consider an HM of a YM field with mass in the simplest nontrivial case $N=2$ degrees of freedom.[8] The Hamiltonian (1.9) with the addition (1.11) has the form ($\omega=1$)

$$H=\tfrac{1}{2}(E_1^2+E_2^2+A_1^2+A_2^2+A_1^2A_2^2). \tag{4.1}$$

It is clear that for $A_k \gg 1$ the quadratic terms are unimportant (at least in the first approximation; see below), and we return to the case of a massless field, i.e., to chaotic motion (Sec. 3). For $A_k \ll 1$ it would appear possible to neglect the weak nonlinear coupling and to expect stable, regular oscillations. These simple considerations were expressed and verified by a numerical modeling of the system (4.1) in Ref. 8. Below we shall consider the dynamics of this model in more detail. Let us begin with the case $H \ll 1$. Let us pass to an alternative action, the phase of the nonperturbed system (linear oscillations):

$$A_k=\sqrt{2I_k}\cos\theta_k,\quad E_k=-\sqrt{2I_k}\sin\theta_k,\quad k=1,\,2.$$

The Hamiltonian (4.1) assumes the form

$$H=I_1+I_2+\tfrac{1}{2}I_1I_2[1+\cos 2\theta_1+\cos 2\theta_2 +\tfrac{1}{2}\cos(2\theta_1+2\theta_2)+\tfrac{1}{2}\cos 2(\theta_1-\theta_2)]. \tag{4.2}$$

Perturbation terms depending on the phases are divided into two groups: high-frequency ($\omega=2$) or nonresonance (the first three terms) and low-frequency or resonance (the last term, the unperturbed frequency of which is zero). It is clear that just the resonance perturbation will mainly determine the dynamics of the system. Therefore in the first approximation we shall discard all nonresonance terms (the so-called averaging method[21]). In the averaged system there remains a single resonance term of the perturbation, or briefly, a single resonance. In this case the system is always completely integrable, and its motion is quasi-periodic (see, for example, Ref. 13). This is related to the original resonance integral because one resonance always depends on a specific combination of phases which results in a symmetry of the perturbation. In the given case, for example, the Hamiltonian does not depend on a displacement of the two phases, and as a result of that the unperturbed energy $H^0=I_1+I_2$ is conserved. Since the total Hamiltonian is also conserved, their difference is also conserved, i.e., the perturbation $V=H-H^0$.

We shall introduce a canonical transformation which leads to the variables $J_1=H^0$ and $\varphi_2=2(\theta_2-\theta_1)$. Then $\varphi_2=\theta_1$, $J_2=I_2/2$, and the averaged or resonance Hamiltonian is

$$H_r=J_1+J_2(J_1-2J_2)(1+\tfrac{1}{2}\cos\varphi_2). \tag{4.3}$$

Since $J_1=\text{const.}$ the phase-space curves are determined by the ratio

$$P(1-P)(1+\tfrac{1}{2}\cos\varphi)=\frac{2V}{(H^0)^2}\equiv v, \tag{4.4}$$

where $P=2J_2/H^0$ and $0\leqslant P\leqslant 1$. Typical phase-space curves are displayed in Fig. 1. In the original vari-

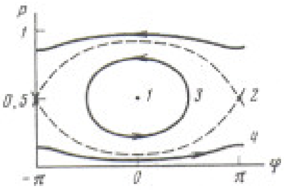

FIG. 1. Typical phase-space curves of model (4.3): 1—stable periodic trajectory, 2—unstable periodic trajectory, 3—quasi-periodic trajectories inside a resonance (oscillation of the phase φ), 4—rotation of the phase; the dashed line is the separatrix bounding the resonance region.

ables they turn into the curves in Fig. 1 of Ref. 8. Periodic trajectories of the system correspond to $\varphi=0,\ \pi$ (for this $\varphi_1=\theta_1=H^0t$). The first of them is stable (the frequency of small oscillations is $\omega_0=\sqrt{3/8}H^0\approx 0.61H^0$), but the second is unstable and determines the separatrix of the resonance on which the oscillations J_2 are maximum ($J_2^{\max}/J_2^{\min}=(1+\sqrt{2/3})/(1-\sqrt{2/3})\approx 10$). This last ratio shows that the effect of a weak nonlinear perturbation (exchange of energies between degrees of freedom) is always large and does not depend on $H\to 0$. This is related to the isochronism of unperturbed oscillations (for linear coupling the exchange of energy would be total).

The effect of the discarded nonresonance harmonics of the perturbation is twofold. On the one hand they cause high-frequency oscillations of the integrals of the averaged system J_1, V. The amplitude of these oscillations $|\Delta V/V|\sim H^0\to 0$; they do not change the character of the motion and could be eliminated in principle, since they do not depend on the time of change of variables (see, for example, Ref. 13).

On the other hand a "nonresonant" interaction can cause resonances with high harmonics of the frequency ω_0. The amplitude of the latter grows with approach to the separatrix (Fig. 1). As a result a stochastic layer forms around the separatrix, the relative width of which is $\sim\exp(-b/\omega_0)$, where $b\sim 1$ (for a detailed investigation of this phenomenon in a similar model see Ref. 13). With an increase of ω_0 the width of the stochastic layer grows fast and for $\omega_0 \gtrsim 1$ ($H\gtrsim 1$) the chaotic component of motion spreads to a large part of the energy shell. The critical value $H_{cr}\approx 6.7$ introduced in Ref. 8 corresponds to a resonance with a fourth harmonic of unperturbed oscillations [the term $\cos(2\theta_1+2\theta_2) =\cos(\varphi+4\theta_1)\approx \frac{1}{2}\varphi_0\cos(4\theta_1-\omega_0 t+\alpha)$]. In any case the quantity H_{cr} is a matter of convention, since the size of the chaotic component depends continuously on H.

For $H\gg 1$ the motion is chaotic, as it is for massless YM fields.[8] However, there are interesting perculiarities. First of all the energy shells are closed and have finite measure, and this means an entropy $h>0$ (see Sec. 3). An estimate of its dependence on energy can be obtained for $H\gg 1$ in the following way. From (4.1) the maximum value of the field is $A_m=\sqrt{2H}$. Substituting this value into (3.3), we find

$$h\sim H^{1/4}/\ln H. \tag{4.5}$$

Further, the motion along each of the axes will now be periodic: $A_1 = A_m \cos t$. The stability of this solution in the linear approximation is determined by the Mathieu equation ($A \equiv A_2$)

$$\ddot{A}+(1+H+H\cos 2t)A=0. \tag{4.6}$$

For $H \gg 1$ the stable and unstable intervals of H have approximately the same width (see, for example, Ref. 22). The centers of the intervals are given by the approximate relations

$$H_{\text{stab}} \approx {}^1/_8\pi^2(n+{}^1/_2)^2-1, \quad H_{\text{unstab}} \approx {}^1/_8\pi^2 n^2-1,$$

where $n > 1$ is an integer.

In this way, the mass terms in the Hamiltonian (1.11) actually stabilize the motion, so that for $H \to 0$ the chaotic component is preserved only in an exponentially narrow layer around the separatrix. However, the situation changes fundamentally with an increase in the number of degrees of freedom. Let us consider, for example, the model (1.9) with the mass addition (1.11) but for $N = 3$.

Passing to the action-angle variable as for the case $N = 2$, we arrive at the averaged Hamiltonian

$$\begin{gathered} H_r = I_1 + I_2 + I_3 + {}^1/_2 V, \\ V = I_1 I_2 [1 + {}^1/_2 \cos 2(\theta_1 - \theta_2)] + I_1 I_3 [1 + {}^1/_2 \cos 2(\theta_1 - \theta_3)] \\ + I_2 I_3 [1 + {}^1/_2 \cos 2(\theta_2 - \theta_3)]. \end{gathered} \tag{4.7}$$

The principal peculiarity of this model is the presence of, not one [as in (4.2)], but three resonances which are preserved for $H \to 0$. For complete integrability of the system two additional integrals are now needed. Nevertheless, (4.7) contains two linearly independent combinations of phases, so that there is only one cyclic combination of phases and correspondingly only one additional integral $H^0 = I_1 + I_2 + I_3$. In these conditions one can expect a sizeable chaotic component of motion for any $H \to 0$. Moreover, as for $N = 2$, the structure of the phase space generally does not depend on the quantity H, which determines only the time scale. Actually, thanks to the integral $H^0 = \text{const}$ the system can be reduced to two degrees of freedom. Then if we carry the scale transformation of time $H^0 t \to t$ and pass to the canonical variables

$$\begin{gathered} \varphi_1 = 2(\theta_1 - \theta_3), \quad \varphi_2 = 2(\theta_2 - \theta_3), \\ J_1 = I_1 / H^0, \quad J_2 = I_2 / H^0, \end{gathered} \tag{4.8}$$

the Hamiltonian of the reduced system assumes the form

$$\begin{gathered} H_R = J_1(1 - J_1 - J_2)(1 + {}^1/_2 \cos\varphi_1) \\ + J_2(1 - J_1 - J_2)(1 + {}^1/_2 \cos\varphi_2) + J_1 J_2 (1 + {}^1/_2 \cos(\varphi_1 - \varphi_2)) \end{gathered} \tag{4.9}$$

and does not depend on the energy of the initial system $H \approx H^0$. If the motion of this system is chaotic, then universal chaos in the initial system will be preserved for any weak nonlinear perturbation. This beautiful phenomenon was discovered and investigated in Ref. 23 in a similar model. We remark that the KAM (Kolomogorov-Arnold-Moser) theory is inapplicable in this case, since the unperturbed system (linear oscillator) is isochronous.[24]

The investigation of the dynamics of the system (4.9) was carried out by means of numerical modeling. The accuracy of the conservation of the integral H_R is in the interval from 10^{-3} to 10^{-8} and does not influence the characteristics of the system.

In the process of the numerical solution of the equations of motion (over the interval $t \sim 10^4$) by the method of two nearby trajectories (Sec. 2) we determined the entropy h_R, which is related to the entropy of the initial system by $h \approx h_R H$. Numerical experiments showed that in the system there is a chaoitc component, but the quantity h_R characterizing depends on the value of the integral H_R. The maximum value $h_R^{\max} \approx 0.15$ is attained for $H_R \approx 0.3$. It is possible for the quantity H_R to assume a value in the interval $0 \leqslant H \leqslant \frac{1}{2}$, which follows from (4.9) and the positivity of the action I_k.

For the approach of H_R to its extreme values, h_R decreases. For example, $h_R = 0.082$ for $H_R = 0.18$ and $h_R = 0.026$ for $H_R = 0.40$.

For a graphic representation of the picture of motion it is possible to draw a two-dimensional Poincaré cross section of the three dimensional energy shell of the system (4.9) if some additional condition is imposed on the dynamical variables (see, for example Ref. 23). We used the condition $\varphi_1 = \varphi_2$ for which the picture of motion will by symmetrical relative to J_1 and J_2. In these variables the surface of the cross section is represented by an equilateral triangle, since $H^0 = I_1 + I_2 + I_3 = \text{const}$ and $I_k > 0$. Let us change to the rectangular coordinates

$$X = {}^1/_2(1 + J_2 - J_1), \quad Y = {}^1/_2\sqrt{3}(1 - J_1 - J_2). \tag{4.10}$$

The energetically accessible region of motion is represented by the intersection of the region inside the circle

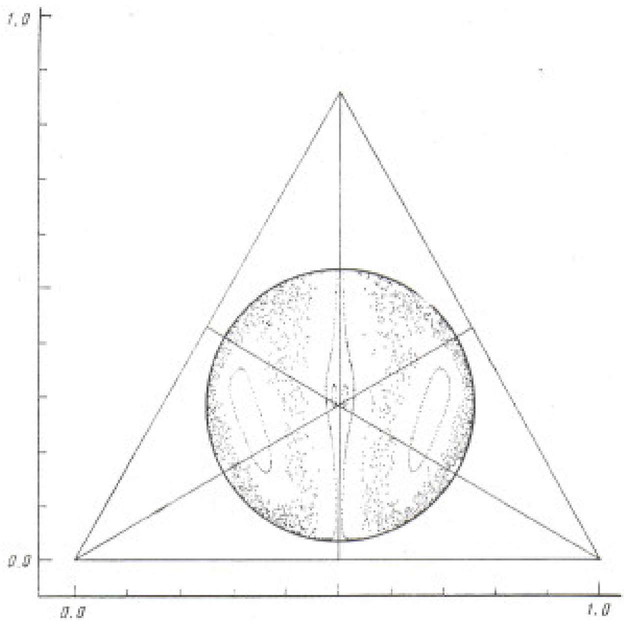

FIG. 2. Surface of the Poincare cross section for the system (4.9); $H_R = 0.404$. The picture of motion is symmetric relative to a vertical line. The center of the triangle coincides with the center of the circle, which bounds the energetically allowable region of motion. The irregularly distributed points belong to one chaotic trajectory; $h_R = 0.026$.

$$\left(X-\frac{1}{2}\right)^2+\left(Y-\frac{\sqrt{3}}{6}\right)^2=\frac{1-2H_R}{3} \tag{4.11}$$

and the region outside the ellipse

$$\left(X-\frac{1}{2}\right)^2+\frac{1}{9}\left(Y+\frac{\sqrt{3}}{2}\right)^2=\frac{1-2H_R}{3} \tag{4.12}$$

(see Fig. 3).

In Fig. 2 the example of the picture of motion for $H_R = 0.404$ is produced. The continuous circle corresponds to (4.11), but the points are from a numerical calculation of four different trajectories. Three of these for which the points lie on a smooth curve correspond to quasi-periodic motion, i.e., there exists yet one additional integral for them. The remaining irregularly distributed points belong to the chaotic trajectory which approximately envelopes the whole chaotic component of motion ($h_R = 0.026$).

Let us turn our attention to the considerable variation of the density of points of the chaotic trajectory. This shows the strong nonuniformity of the invariant measure on the surface of the cross section.

Another example of a picture of motion is given in Fig. 3. The relative area of the chaotic component reaches a maximum for $H_R \approx 0.3$ (as does h_R) and decreases for $H_R \to 1/2$ and $H_R \to 0$. In the latter case the possible region of motion is divided into three portions near the angles of the triangle which are not connected among themselves.

Some average statistical characteristics of the chaotic component of the initial model (1.9) and (1.11) for $H \to 0$ can be obtained in the following way. Let us take the group of trajectories, the initial conditions of which are distributed randomly and uniformly in a layer of phase space of thickness ΔH with an average value $H^0 \ll 1$. We calculate the entropy for each of these trajectories. Then let S be the ratio of the number of trajectories with $h > 0$ to the total number of trajectories, which is equal (approximately) to the relative size of the chaotic component on the energy shell ($H \to 0$).

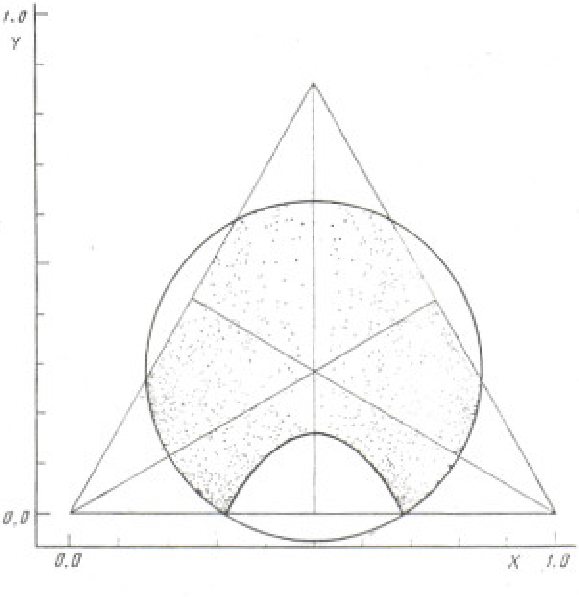

FIG. 3. Same as in Fig. 2; $H_R = 0.324$, $h_R = 0.14$.

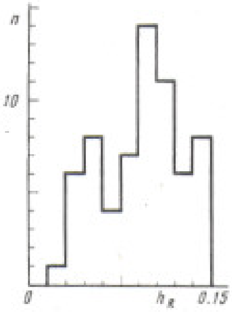

FIG. 4. Histogram of distribution of entropy $h_R \approx h/H$ for the model of (1.9) and (1.11) with $H \ll 1$; n is the number of trajectories with h_R in the corresponding interval.

Special measurements showed that for H_0 it is sufficient to take $H_0 \approx 0.25$ ($\Delta H \approx 0.1$). An inspection was carried out of the dependence on H of the dimensionless entropy $h/H \to h_R$ ($H \to 0$). Then, for an increase of H from 0.07 to 0.29 the ratio h/H remains constant to within several percent. From surveying 100 trajectories we obtain $S = 65\%$.

Besides this, the distribution of different values of the entropy $h_R = h/H$ was obtained, the histogram of which is produced in Fig. 4. The average value of the entropy on the energy shell is $\langle h_R \rangle \approx 0.093$. The sharp cutoff of the distribution at small values of h_R indicates a good separation of the stable and chaotic components of motion under the conditions of the numerical modeling. The near-maximum values of h_R for the original system (0.143) and the reduced system (0.151), in addition, confirm that the assumed value $H_0 \approx 0.25$ reflects rather well the limiting behavior of the original system for $H \to 0$. A significant decrease of H_0 is undesirable since that increases the calculation time.

The significant chaotic component discovered for $H \to 0$ is related to the degeneration of the nonperturbed system [all three frequencies of the linear oscillator (masses) are identical]. Only in this case are all resonances preserved for $H \to 0$. If for some reason the linear frequencies prove to be different there arises, as for $N = 2$, a critical energy $H_{cr} \sim |\Delta\omega|_{max}$, the maximum difference of linear frequencies ($|\Delta\omega| \ll \omega$). In particular, $H_{cr} > 0$ even if two frequencies coincide. In this case for $H \to 0$ only one resonance remains.

An analogous situation can arise for identical frequencies if the moment $M \neq 0$ in (1.8). Let us consider the following model. Let the YM field have the following components: $A_1^a \equiv A_1$, $A_2^a \equiv A_2$, and $A_3^a \equiv A_3$. Then $M_3 = A_1E_2 - A_2E_1 = \text{const}$ ($M_z \equiv 0$). The Hamiltonian has the form

$$H={}^1/_2[E_1^2+E_2^2+E_3^2+A_1^2+A_2^2+A_3^2+A_3^2(A_1^2+A_2^2)]. \tag{4.13}$$

In view of the axial symmetry, the symmetry can be reduced to two degrees of freedom. Let us set $A_\perp^2 = A_1^2 + A_2^2$. Then the Hamiltonian of the reduced system can be written in the form

$$H=\frac{1}{2}[E_3^2+E_\perp^2+M_3^2/A_\perp^2+A_3^2+A_\perp^2+A_3^2A_\perp^2]. \tag{4.14}$$

If $M_3=0$, the problem reduces to the case (4.1) considered above with one resonance and a critical energy. For $M_3\neq 0$, the energy has a minimum $H_m=|M_3|$ with $A_1=A_m=\sqrt{|M_3|}$ and $A_3=0$. Denoting $A_1-A_m=A$, we obtain a decomposition of the Hamiltonian (4.14) near the minimum:

$$H=|M_3|+{}^1/_2[E_3^2+(1+|M_3|)A_3^2+E^2+4A^2+2\sqrt{|M_3|}AA_3^2+A^2A_3^2]. \quad (4.15)$$

Frequencies of small oscillations are equal to $\omega_3=\sqrt{1+|M_3|}$ and $\omega=2$ (for A_3 and A respectively). The conditions of resonance are $\omega=2\omega_3$ (the perturbation term AA_3^2) and $\omega=\omega_3$ [the term $(AA_3)^2$]. Therefore, even one resonance is possible only for $|M_3|=0, 3$. In the opposite case, the motion of the system (4.15) will be even more stable than (4.1) in the sense that the oscillations of the unperturbed actions will be small: I_3, $I=$ const (compare Fig. 1). We note that there exists a symmetric solution $A_2\to A_1^b$, $M=0$, and $M_g^c\neq 0$) corresponding to a nonzero source density.

In conclusion we return once again to the previous model with $N=3$ and consider the case $H\gg 1$. As for $N=2$, the chaotic component of motion envelopes, in this case, almost the whole energy shell with the exception of small regions along the coordinate axes, i.e., when, for example, $A_1^2\gg A_2^2+A_3^2$. In this region it is possible to neglect the term $(A_2A_3)^2$ in the Hamiltonian [see (1.9)], so that the motions in A_2 and A_3 become independent, and each of them is described by the Mathieu equation (4.6) as in the case $N=2$. Correspondingly the motion will be stable or unstable (chaotic) depending on the value of H.

Numerical modeling confirms that, for example, for $H=51$ (stable interval, $n=6$) there actually exists a finite region of regular motion the size of which is

$$\sqrt{A_2^2+A_3^2}/A_1\sim 3\times 10^{-3}.$$

CONCLUSION

In this way, the dynamics of HM of YM fields is found in the general case to be chaotic. Chaos is intensified with an increase in the number of degrees of freedom of the model. Therefore for nonhomogeneous YM fields with an infinite number of (external) degrees of freedom one expects complete chaos of motion if and only if there are internal degrees of freedom. For one internal degree of freedom the question remains open, since in the approximation of HM the motion in this case is periodic [see (1.10) and (2.1)]. Such a system was investigated numerically in Ref. 25. However, the investigation failed to obtain a definite result about chaos of the motion. We note that it is possible for chaos in such a system to be local and not reduce to an equal distribution of energy over many modes[26] (the latter was the criterion of chaotic motion in Ref. 25).

We would like to express our sincere gratitude to V. N. Baier, A. I. Vaĭnshteĭn, and I. B. Khriplovich for many explanations of questions about the theory of YM fields and valuable comments.

Note added in proof (26 December 1982). Recently (Pis'ma Zh. Eksp. Teor. Fiz., 36, 176 (1982)) [JETP Lett. 36, 218 (1982)] the local nonintegrability of a YM model was established from a splitting of the separatrix, which, generally speaking, is unimportant for the dynamics of a system.

[1)] I.e., in the usual space-time with pseudo-Euclidean metric.

[2)] We recall a simple mechanical model with the following type of constraint (or conservation law): let one of the interacting particles have mass $m\to 0$; then for certain restrictions on the interaction the effective force on this particle $f\to 0$.

[1] C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).
[2] A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. 20, 430 (1974) [JETP Lett. 20, 194 (1974)]; Zh. Eksp. Teor. Fiz. 68, 1975 (1975) [Sov. Phys. JETP 41, 988 (1975)].
[3] A. M. Polyakov, Phys. Lett. 59B, 82 (1975); A. A. Belavin, A. M. Polyakov *et al.*, *ibid.*, p. 85.
[4] C. G. Callan, R. Dashen *et al.*, Phys. Rev. D17, 2717 (1978).
[5] A. I. Vaĭnsteĭn, V. I. Zakharov *et al.*, Usp. Fiz. Nauk 136, 553 (1982) [Sov. Phys. Usp. 25, 195 (1982)].
[6] G. Z. Baseyan, S. G. Matinyan, *et al.*, Pis'ma Zh. Eksp. Teor. Fiz. 29, 641 (1979) [JETP Lett. 29, 587 (1979)].
[7] S. G. Matinyan, G. K. Savvidi, *et al.*, Zh. Eksp. Teor. Fiz. 80, 830 (1981) [Sov. Phys. JETP 53, 421 (1981)].
[8] S. G. Matinyan, G. K. Savvidi, *et al.*, Pis'ma Zh. Eksp. Teor. Fiz. 34, 613 (1981) [JETP Lett. 34, 590 (1981)].
[9] B. V. Chirikov and D. L. Shepelyanskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. 34, 171 (1981) [JETP Lett. 34, 163 (1981).
[10] A. A. Slavnov and L. D. Faddeev, Vvedenie v kvantovuyu teoriyu kalibrovochnych poleĭ, Moscow, Nauk (1978) [Gauge fields, introduction to quantum theory, Reading, Mass., Benjamin/Cummings (1980)].
[11] A. G. Izergin, V. E. Korepin, *et al.*, Teor. Mat. Fiz. 38, 3 (1979) [Theor. Math. Phys. (USSR) 38, 1 (1979)].
[12] L. V. Prokhorov, Yad. Fiz. 35, 229 (1982) [Sov. J. Nucl. Phys. 35, 129 (1982)].
[13] B. V. Chirikov, Phys. Rep. 52, 263 (1979).
[14] I. P. Kornfel'd, Ya. G. Sinaĭ, *et al.*, Ergodicheskaya teoriya, Moscow, Nauka (1980) [I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, Ergodic theory, New York, Springer Verlag (1981)].
[15] V. M. Alekseeev and M. V. Yakobson, supplement in: Metody simvolicheskoĭ dinamiki [Methods of symbolic dynamics]; R. Bowen; A. A. Brudno, Candidate's dissertation, Moscow (1978).
[16] A. V. Gaponov-Grekhov and M. I. Rabinovich, Priroda, No. 2, 54 (1981); Ya. G. Sinaĭ, *ibid.* No. 3, 72; B. V. Chirikov, *ibid.* No. 7 (1982).
[17] J. Ford, S. D. Stoddard *et al.*, Prog. Theor. Phys. 50, 1547 (1973).
[18] G. Casati *et al.*, Phys. Lett. 77A, 91 (1980).
[19] T. T. Wu and C. N. Yang, Phys. Rev. D12, 3843 (1975).
[20] S. Coleman, Phys. Lett. 70B, 59 (1977).
[21] N. N. Bogolyubov and Yu. A. Mitropol'skiĭ, Asimptoticheskie methody v teorii nelineĭnykh kolebaniĭ, Moscow, Nauka (1974) [Asymptotic methods in the theory of nonlinear oscillations, Grodon (1961)].
[22] N. W. McLachlin, Theory and application of Mathieu functions, New York, Dover (1964).
[23] J. Ford and G. N. Lunsford, Phys. Rev. A1, 59 (1970)].
[24] V. I. Arnold, Usp. Mat. Nauk 18, 91 (1963) [Russian Math. Surveys 18, 85 (1963)].
[25] V. E. Zakharov, M. F. Ivanov *et al.*, Pis'ma Zh. Eksp. Teor. Fiz. 30, 39 (1979) [JETP Lett. 30, 34 (1979)].
[26] B. V. Chirikov, F. M. Izrailev *et al.*, Comp. Phys. Comm. 5, 11 (1973).

Translated by L. J. Swank