and therefore a sufficient condition for the asymptotic form (37) (and thus the phase screen approximation) to be valid is

$$Ha/Ll_0 \ll 1. \quad (40)$$

It can be shown that treatment of an arbitrary light beam leads to a condition differing from (40) only in replacing the parameter a by a_θ, where $a_\theta \equiv \min \cdot \{1/k|\nabla\theta|, 1/k^{1/2}|\Delta\theta|^{1/2}\}$.

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Diffusion during multiple passage through a nonlinear resonance

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A simple model for the multiple passage through a nonlinear resonance is examined. Analytical expressions are derived for the boundary of the stochastic motion and for the diffusion rate in the stochastic region, and for both fast and slow passage through the resonance. The results of numerical simulations are reported. These results confirm the analytic results and are used to derive a refined semiempirical expression for the diffusion rate which may be used to analyze encounter effects in storage rings.

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The luminosity and thus the efficiency of colliding-beam devices are determined to a large extent by "encounter effects," i.e., by the electromagnetic interaction of the colliding beams (cf. Ref. 1). In the case of a weak interaction, the simplest approximation, the dynamics of a single particle in the given field of the oppositely directed bunch can be analyzed; this is the approach which has been taken in all previous analytic and numerical work on encounter effects. When the field of the bunch is taken into account, the betatron oscillations of a particle in the storage ring become nonlinear, and their stability is accordingly determined by nonlinear resonances. The interaction of these resonances can lead to a dangerous stochastic instability under certain conditions. A low-frequency modulation associated with particle synchrotron oscillations significantly lowers the threshold for this instability (see Ref. 2, for example). In this paper we will take the approach of Ref. 3 to study the effect of such a modulation as a result of a multiple passage through a nonlinear resonance of the betatron oscillations of a particle. This approach yields effective estimates of the growth rate of the stochastic instability under various conditions.

The passage through a resonance involves a change in the frequency of the betatron oscillations of the particle and/or a change in the perturbation frequency, so that at some instant these frequencies (or their harmonics) become equal, and the effect of the perturbation on the particle is sharply intensified. A single passage through resonance has been studied in many places (see Refs. ... for example). For nonlinear oscillations, the case of a... called slow passage through the resonance (see Ref. 6 ... Section 2 of the present paper) is the case of most int... In this case the rate of change of the frequencies due to the (given) external modulation is much smaller than th... rate of change which results from the phase oscillatio... caused at resonance by the perturbation.

If the perturbation is small, the effect of a single passage through a resonance is also small. According... a more important problem is that of the multiple pass... through a resonance, in which effects may accumulate. These accumulating effects may be either regular[5] or stochastic.[3,7,8]

It is a relatively simple matter to calculate the dif-fusion rate in the case of a fast passage through the res... nance, and these calculations have been carried out in several places.[3,7,8] The problem becomes much more complicated in the case of slow passage; in particular, ... widely used quasilinear approximation and the random phase approximation are completely inapplicable in thi... case.[7,8] Some rough estimates for this case were derived in Ref. 3.

Our major goal in the present work was to numeric... simulate diffusion during a slow multiple passage thro... resonance, in order to obtain some effective estimates of the rate of this diffusion. For this simulation we ad... what seems to be the simplest model, which is given by

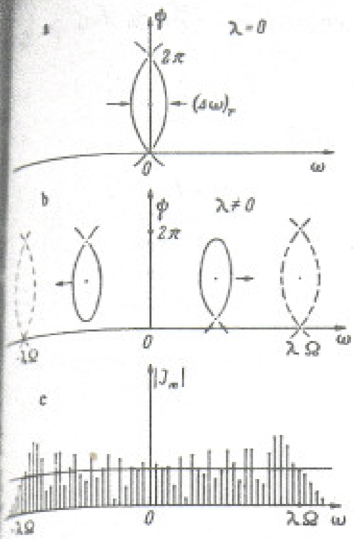

Fig. 1. a) The separatrix of a single resonance of width $(\Delta\omega)_r$; b) instantaneous configurations of the separatrix during passage through a resonance as a result of a frequency modulation of the perturbation (3) [the solid curves show the moving separatrix (the arrow shows the direction of the motion), while the dashed curves correspond to the two extreme positions of the separatrix, where the instantaneous velocity is zero]; c) spectrum of the frequency-modulated perturbation in (3) for $\lambda = 25$. The horizontal line shows the average value $\langle J_m^2 \rangle = 1/\pi\lambda$.

certain mapping (Section 1). The results of the numerical simulations (Section 3) confirm the results of Ref. 3, which may therefore be used along with the empirical coefficient found in the present paper to analyze encounter effects in storage rings and in other problems [expression (16)].

The results of both Ref. 3 and the present study refer to the case of a multiple passage through a single resonance. Multiple passage through several different resonances will be discussed briefly in Section 2.

1. MODEL

Let us consider a canonical mapping which simulates the effect on a particle of a single "jolt" from an oppositely directed bunch:

$$\bar{I} = I + k\sin(\theta + \lambda\sin\Omega t);\quad \bar{\theta} = \theta + \bar{I}, \tag{1}$$

where I, θ are the action-phase variables which describe the betatron oscillations of a particle with a single degree of freedom. As has been explained in detail in Ref. 10, for example, a mapping of this type gives an approximate representation of the local (in phase space) structure of the resonances of the nonlinear oscillations. The action I is normalized here in such a manner that it gives the phase shift (θ) of the betatron oscillations over the period (T) of the perturbation which corresponds to a single application of mapping (1). If we set T = 1, then the action becomes equal to the frequency of the betatron oscillations (ω = 1) and automatically incorporates the nonlinearity of these oscillations ($d\omega/dI \neq 0$). The perturbation parameter k is renormalized correspondingly. The low-frequency modulation associated with the synchrotron oscillations of the particle is described by the parameters λ and Ω. Tennyson[8] has discussed one method for transforming the equations of motion of a particle in the field of an oppositely directed bunch into a mapping like that in (1).

For system (1) we can write the exact Hamiltonian

$$H = I^2/2 + k\cos(\theta + \lambda\sin\Omega t)\,\delta_1(t) \tag{2}$$

with a periodic δ-function, $\delta_1(t) = 1 + 2\sum_{n=1}^{\infty}\cos 2\pi nt$. (the period is T = 1).

Let us assume $k \ll 1$. If, furthermore, we have $\lambda\Omega \ll 1$, then we can take the average of the δ-function in the Hamiltonian (2): $\delta_1(t) = 1$. The result is the average Hamiltonian

$$\bar{H} = I^2/2 + k\cos(\theta + \lambda\sin\Omega t), \tag{3}$$

which gives an approximate description of the dynamics of the original system, (1).

Let us now assume $\lambda = 0$. In this case we have the single resonance $\omega = 0$, and the motion in our model is the same as that of the pendulum (3). On the phase diagram for this type of motion there is a certain special trajectory: a separatrix (Fig. 1a), which bounds the region of the nonlinear resonance.[10]

The nonlinear resonance is characterized primarily by two parameters, which we will need below. These are the frequency of the (small) phase oscillations,

$$\Omega_{ph} = \sqrt{k} \tag{4}$$

(the frequency of small oscillations of a pendulum), and the width of the resonance (the total width along the frequency scale of the separatrix for the pendulum; see Fig. 1a),

$$(\Delta\omega)_r = 4\sqrt{k} = 4\Omega_{ph}. \tag{5}$$

If there is a phase (frequency) modulation of the perturbation ($\lambda \neq 0$), the resonance condition $\psi = 0$, where $\psi = \theta + \lambda\sin\Omega t$ is the resonant phase, may be written

$$\omega = \omega_M = -\lambda\Omega\cos\Omega t. \tag{6}$$

Here ω_M is the perturbation frequency, an explicit function of the time. If $|\omega| < \lambda\Omega$, then the two frequencies (ω; ω_M) become equal at certain times; i.e., a resonance is passed. Figure 1b is a rough sketch of the phase diagram for this motion. A modulation of the perturbation causes oscillations of the resonance along the ω axis between two extreme positions (the dashed curves in Fig. 1). The modulation also gives rise to a distortion of the fixed separatrix of the resonance; this distortion depends on the velocity and direction of the modulation. The phase diagram of the motion during passage of the resonance is conveniently analyzed by the simple method of Ref. 6.

There is another way to study the effect of a periodic modulation of the perturbation. In this approach, the frequency-modulated perturbation in (3) is expanded in a Fourier series:

$$\cos(\theta + \lambda\sin\Omega t) = \sum_{m=-\infty}^{\infty} J_m(\lambda)\cos(\theta + m\Omega t), \tag{7}$$

where $J_m(\lambda)$ is a Bessel function. Substitution of this expression into (3) yields a system of stationary resonances $\omega = \omega_m = m\Omega$ or a multiplet. Formally, the number of resonances in the multiplet is infinite, but the amplitudes of the Fourier harmonics and thus the widths of the resonances fall off rapidly (exponentially) at $|m| > \lambda$. The result is that only about 2λ resonances are "effectively working" in the frequency band $|\omega| \lessdot \lambda\Omega$.

Figure 1b shows an example of the frequency dependence of the harmonic amplitudes $[J_m(\lambda)]$, for $\lambda = 25$. On the (ω, ψ_m) phase plane $[\psi_m = \theta + m\Omega t$; see (7)] there is a separatrix corresponding to each of these resonances, similar to that shown in Fig. 1a, but with a width

$$(\Delta\omega)_m = 4\sqrt{k|J_m(\lambda)|} \approx 4\Omega_{ph}/(\pi\lambda)^{1/4}. \tag{8}$$

In this latter expression we have used the rms value $\langle J_m^2(\lambda)\rangle \approx 1/\pi\lambda$ $(\lambda \gg 1;\ |m| < 1)$; this value is shown by the horizontal line in Fig. 1b. The stationary resonances $\omega = m\Omega$ in our model correspond to synchrotron–betatron resonances of a particle in the storage ring, while the moving resonance in Fig. 1b is a betatron-oscillation resonance.

The approach based on the system of stationary resonances is particularly convenient for finding the nature of the motion, which is determined by the parameter describing the overlap of the nonlinear resonances:

$$s = (\Delta\omega)_m/\Omega \approx 4\Omega_{ph}/(\pi\lambda)^{1/4}\Omega. \tag{9}$$

Here Ω is the distance between adjacent resonances of the multiplet, which is equal to the modulation frequency (Fig. 1c), and we have used (8) for the width of the resonance. Stochastic motion arises if[10] $s > 0.63$.

In stochastic motion, the system undergoes diffusion in the action I (or the frequency $\omega = I$) within the interval of separatrix oscillations shown in Fig. 1b (only here is the resonance passed) or (equivalently) within the width of the multiplet in Fig. 1c, i.e., in the interval $|I| = |\omega| \lessdot \lambda\Omega$. This is apparently the type of diffusion which was observed in Ref. 7. More dangerous, however, is the case in which several adjacent betatron resonances are passed, since in this case the diffusion extends over a considerably larger region. In terms of synchrotron–betatron resonances, this situation corresponds to an overlap of adjacent multiplets. This case does not occur in the averaged system (3), which simulates only a single betatron resonance, but it may occur in the original model (1). To see this, we write the Hamiltonian (2) in the form

$$H = \frac{I^2}{2} + k \sum_{m,\, n=-\infty}^{\infty} J_m(\lambda)\cos(\theta + m\Omega t + 2\pi n t). \tag{10}$$

We see that the complete system of resonances $(w + m\Omega + 2\pi n = 0)$ of model (1) is a sequence of multiplets separated by a distance $\delta\omega = 2\pi$ (Fig. 2). The condition for the overlap of adjacent multiplets is

$$s_M = 2\lambda\Omega/\delta\omega \gg 1. \tag{11}$$

In terms of the passage of the system through the resonance (Fig. 1b), the case $s_M \lessdot 1$ mans that only a single resonance is passed, while with $s_M \gg 1$ the system passes through many different resonances, corresponding to different multiplets.

It becomes considerably more complicated to study the dynamics of the system in this latter case, but for the diffusion to extend to the region of several multiplets it is sufficient that s_M be only slightly greater than the critical value $s_M^{(cr)} \approx 1$. On the other hand, it is clear that in this case the diffusion rate is determined quite accurately by only one of the multiplets. Certain aspects of the diffusion in the case $s_M \gg 1$ will be discussed briefly at the end of the following section.

2. CALCULATION OF THE DIFFUSION RATE

The diffusion rate for the case of multiple passage through a single resonance under stochastic conditions was calculated in Ref. 3. The results of that paper depend strongly on the rate at which the resonance is passed, the rate being conveniently represented by the dimensionless parameter

$$v = |\dot{\omega}_M|/\Omega_{ph}^2, \tag{12}$$

where ω_M is the perturbation frequency, which depends explicitly on the time by virtue of the frequency modulation in (6), and Ω_{ph} is the frequency of the phase oscillations in (4) of the moving betatron resonance (Fig. 1b). The parameter v is the rate of displacement of the resonant value of the frequency due to the modulation of the perturbation, divided by the rate of change of the frequency due to the phase oscillations (at a high amplitude of these oscillations).

If the resonance is passed rapidly $(v \gg 1)$, the frequency ω, which is changed by the perturbation because of the nonlinearity of the oscillations, cannot keep up with the perturbation frequency ω_M. The result of the passage is thus (in a first approximation) the same as that for a linear oscillator[6]:

$$\Delta I \approx \sqrt{2\pi}\frac{k}{\sqrt{|\dot{\omega}_M|}}\sin\left(\psi_r \pm \frac{\pi}{4}\right), \tag{13}$$

where ψ_r is the value of the resonant phase ψ at the time of the exact resonance $(\psi = 0)$, and the sign is determined by the direction in which the resonance is passed.

In this case of fast passage, the value of ψ_r is arbitrary,[6] and under the condition for stochastic motion, $s > 0.63$ (see the discussion above), the sequence ψ_r is approximately random and is distributed uniformly over the interval $(0, 2\pi)$ (Refs. 3 and 10). The diffusion rate

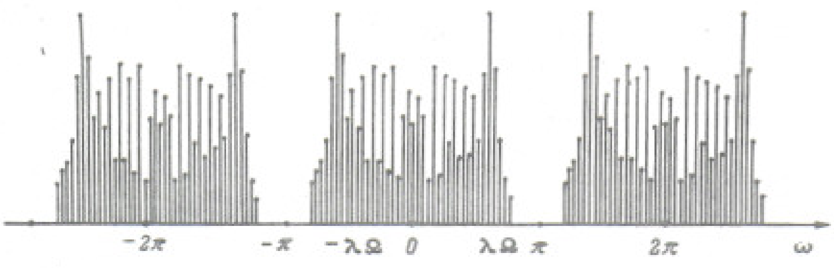

FIG. 2. Perturbation spectrum for model (1) [see (10)].

is thus

$$D_f = \frac{\langle(\Delta I)^2\rangle}{T/2} \approx \frac{\Omega k^2}{|\dot\omega_M|} = \frac{\Omega k^2}{v\Omega_{ph}^2}, \tag{14}$$

where $T = 2\pi/\Omega$ is the modulation period, during which the resonance is passed twice.

This approximation is valid if the rate at which the resonance is passed, v, changes only slightly during the passage, i.e., over the time during which the separatrix in Fig. 1b intersects some point in the phase plane which represents the system. For this condition to hold, the width of the separatrix must be at least smaller than its oscillation amplitude: $(\Delta\omega_r) \ll \lambda\Omega$.

Even under this condition, however, the rate v will change in the course of the diffusion (for different passages), from a maximum value at the center of the multiplet to zero at its edges. It can be shown that this effect has a negligible influence on the average diffusion rate over a broad range of parameters.[11] Below, we will understand v as representing its maximum value $\lambda\Omega^2/\Omega_{ph}^2$ [see (6) and (12)].

For a slow passage through the resonance[3,6] $(v \ll 1)$ we can write

$$\Delta I \approx \pm\frac{8}{\pi}\Omega_{ph}\sqrt{1+2\pi v-\frac{1}{4}\psi_r^2} \pm 2v\Omega_{ph}\ln[(v+\psi_r)(4\pi v-\psi_r^2)], \tag{15}$$

where the phase ψ_r can now take on values only in the narrow interval $-v < \psi_r < (4\pi v)^{1/2}$, and the sign in (15) is determined by the direction in which the resonance is passed.

In the limit $v \to 0$ the change $\Delta I = \pm 8/\pi\Omega_{ph}$ does not depend on the passage rate and is completely reversible (if Ω = const). This result[1)] agrees with a conclusion reached by Symon and Sessler.[5]

On the other hand, the condition for stochastic motion, which may be written as $s^2 \sim \Omega_{ph}^2/\Omega^2\sqrt{\lambda} = \sqrt{\lambda}/v \gg 1$ in the present case, shows that the motion is always stochastic if the resonance is passed slowly.

The explanation for this apparent contradiction is[3] that for any finite $v \neq 0$ the passage through the resonance is not completely reversible, because of the change in the phase ψ_r. It is clear that the diffusion rates will depend on precisely this irreversible increment (δI). The latter is determined primarily by the second term in (15), which contains a large logarithm. From (15) we find the rough estimate $\delta I \sim v\Omega_{ph}\ln v$ and thus the diffusion rate[3]

$$D_s \approx C\frac{k^2\Omega}{\Omega_{ph}^2}v^3(\ln v)^2, \tag{16}$$

where C = 2 is a numerical factor whose value is found from the numerical simulations (Section 3).

A method commonly used to calculate the diffusion rate for a dynamic system is the so-called quasilinear approximation (see Ref. 9, for example). In the problem at hand, this approximation is equivalent to replacing the discrete spectrum of the perturbation, kJ_m (Fig. 1c), by a continuous spectrum [because the phase θ becomes stochastic in Hamiltonian (3)] with the same spectral density $P = k^2J_m^2/\Omega$. This change leads to the diffusion rate

$$D_{ql} = \pi P = \pi k^2 J_m^2/\Omega \approx k^2/\lambda\Omega = D_f, \tag{17}$$

which agrees with the direct calculation in (14) for fast passage of the resonance.

For slow passage of the resonance, however, the diffusion rate drops off sharply (at a given spectral density of the perturbation):

$$D_s/D_{ql} \approx Cv^3(\ln v)^2. \tag{18}$$

This result shows that the quasilinear approximation is completely inapplicable in this region because of the strong correlations of the phase θ. Furthermore, for a slow passage of the resonance there is a wide range of initial conditions which lie within the separatrix and which move along with it (Fig. 1b) for which diffusion does not occur at all. This is the well-known region of phase stability or locking, in which the motion is quasiperiodic (Section 3).

To conclude this section we will briefly discuss the extent to which the results above may be changed in the case of a pronounced overlap of multiplets $[s_M = 2\lambda\Omega \cdot (\delta\omega)^{-1} \gg 1]$.

In the diagram of stationary resonances (Figs. 1c and 2) the average density of resonances along the frequency scale increases by a factor of about s_M. The parameter representing the overlap of adjacent resonances s in (13), increases by the same factor:

$$s \approx 8(\lambda^3/\pi)^{1/4}(\Omega_{ph}/\delta\omega); \quad s_M \gg 1. \tag{19}$$

This increase significantly lowers the stability boundary.

In the quasilinear approximation, the diffusion rate would increase by the same factor. For fast passage, for example, we find, instead of (17),

$$D_f \approx 2k^2/\delta\omega; \quad s_M \gg 1. \tag{20}$$

In terms of the passage of resonances, this result corresponds to an independent effect of the value of s_M for the different resonances through which the system passes under these conditions. If this independence is retained in the case of slow passage also, then the diffusion rate also increases to a value s_M times that in (16):

$$D_s \approx C(2k^2/\delta\omega)v^3(\ln v)^2; \quad s_M \gg 1. \tag{21}$$

The conditions under which the different resonances act independently, however, require further study. In particular, it is not clear at this point whether the quasilinear approximation remains valid even in the case of fast passage of the resonances if $s_M \gg 1$.

3. NUMERICAL SIMULATIONS

The basic purpose of the numerical simulations was to determine the rate of the diffusion in I as a function of the parameters of the model. The diffusion rate was calculated from

$$D = \langle(I_k - I_l)^2/(t_k - t_l)\rangle, \tag{22}$$

where I_k and I_l are the values of I averaged over certain time intervals centered at t_k and t_l, and the angle brackets denote the average of D over the different intervals. This double averaging significantly suppresses the effect of bounded oscillations (the "background"), which are always present, and it permits reliable measurements of an extremely small diffusion rate (see Fig. 3 and the discussion

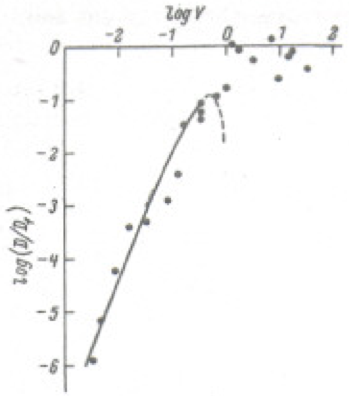

FIG. 3. Dependence of the normalized diffusion rate D/D_f on the resonance passage rate ν (log is the common logarithm). Points) Results of the numerical calculation; curve) estimate for $\nu \ll 1$ in (18) with the empirical value C = 2.0.

below). This technique has been described in detail elsewhere.[10,12]

In the numerical calculation, the initial parameters of the model were varied over the intervals $k = 10^{-3}$–10^{-1}, $\lambda = 3.3 \cdot 10^{1}$–$3.3 \cdot 10^{4}$, and $\Omega = 10^{-4}$–10^{-1}. The resonance overlap parameter (9) lay in the range s = 2.2–705; the maximum resonance passage rate was $v = 3.3 \cdot 10^{-3}$–$3.3 \cdot 10^{1}$; and the ratio $\lambda\Omega/(\Delta\omega)_r = 2.6$–26. The product $\lambda\Omega = 3.3$ remained the same in all cases (Fig. 3). The choice of the value of $\lambda\Omega$ was discussed above in Section 1.

A typical time required for calculation of a trajectory was $t = 10^6$ iterations of mapping (1). For small values of the resonance passage rate, the calculation time was increased to $3 \cdot 10^6$ iterations because of the very low diffusion rate with respect to the background level. For the minimum value $v = 3.3 \cdot 10^{-3}$ and k = 0.1, for example, the diffusion rate for $t = 10^6$ was $D \approx 9 \cdot 10^{-8}$, while for $t = 3 \cdot 10^6$ the rate dropped off by a factor of more than 20, to the value $D \approx 3.8 \cdot 10^{-9}$. Clearly, the previous high "diffusion" rate is actually determined exclusively by the background. On the other hand, as the calculation time increases further observed diffusion rate changes only very slightly. At $t = 6 \cdot 10^6$, for example, we find $D \approx 4.3 \cdot 10^{-9}$. The difference which does remain between the two latter values is apparently due to statistical fluctuations. The absence of dependence of the measured diffusion rate on the time of the motion, which was also checked in other cases, confirms that the process under study is a diffusion. At the rate $v = 8.3 \cdot 10^{-3}$ and above, 10^6 iterations prove sufficient for a reliable determination of the diffusion rate. For example, we find $D \approx 2.9 \cdot 10^{-7}$ ($t = 10^6$) and $D \approx 1.8 \cdot 10^{-7}$ ($t = 3 \cdot 10^6$) for $v \approx 8.3 \cdot 10^{-3}$. The background level depends not only on v but also on other parameters, primarily k. Consequently, the very small diffusion rate $D \approx 1.7 \cdot 10^{-8}$ at v = 0.33, for example, is much higher than the background because of the small value $k = 10^{-3}$.

As an additional control we carried out two test simulations. In the first, the multiplet overlap parameter was $s_M = 0.80 < 1$, so that the diffusion region was limited to the width of a single multiplet. Under these conditions the observed diffusion rate should decrease as the time of the motion increases. This effect is in fact observed. At v = 0.17 and $t = 10^4$, for example, we find the ratio $D/D_f \approx 1.5 \cdot 10^{-2}$, which is not very different from that the case $s_M = 1.05$ ($D/D_f \approx 3.3 \cdot 10^{-2}$). The reason for this result is that over this short time the diffusion does not manage to fill the multiplet. As the diffusion time is increased to $t = 10^6$, however, the ratio $D/D_f \approx 1.9 \cdot 10^{-4}$ falls off by nearly an order of magnitude.

In the second control simulation, the initial conditions were chosen to correspond to a point near the center of the oscillating resonance separatrix (Fig. 1b), where there is a stable region of regular motion at sufficiently low rates v. In this region there is no diffusion, of course, and the observed value is determined exclusively by the background. With v = 0.17, for example, the rate of the "diffusion" (the background) at the center of the resonance turns out to be $D \approx 10^{-9}$, while in the stochastic component we have $D \approx 10^{-4}$.

Comparison of the numerical results (Fig. 3) with analytic results shows that the latter give a satisfactory description of the dependence of the diffusion rate on the parameters of the model. For a fast passage of the resonance ($v \geq 1.3$) the average value is $\langle D/D_f \rangle = 0.74 \pm 0.11$. In the case of a slow passage ($v \leq 0.66$), the average value of the numerical factor in (18) is $\langle C \rangle = 2.0 \pm 0.45$. The region near v = 1 is not described by either (16) or (17). In this region ($0.66 \leq v \leq 1.3$) the diffusion rate drops off sharply with v, by approximately an order of magnitude.

There is significant scatter in the points in Fig. 3, for reasons which are not completely clear. The scatter is not, at least, a result of statistical fluctuations alone. For $v = 8.3 \cdot 10^{-2}$, for example, the significant decrease in the diffusion rate with respect to the analytic value $D/D_s = 1/6$ (C = 2) is seen for various trajectories. This question requires further study.

We wish to thank F. M. Izrailev and J. Tennyson for useful discussions.

1)If $\Omega^{+}_{ph} \neq \Omega^{-}_{ph}$, there is a systematic change in I at an average rate $\dot{I} = (8/\pi)(\Omega^{+}_{ph} - \Omega^{-}_{ph})$. Symon and Sessler[5] studied the extreme case $\Omega^{-}_{ph} = 0$.

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