CHAOTIC DYNAMICS IN HAMILTONIAN SYSTEMS WITH DIVIDED PHASE SPACE

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1. INTRODUCTION

The subject of this lecture is related to a peculiar dynamical phenomenon in classical mechanics commonly termed among physicists as the chaotic, or stochastic, motion. Until recently the mathematicians used to speak just about ergodic properties of a dynamical system. However, nowadays the term "random motion" becomes also popular. I would like to emphasize from the beginning that the problem we are going to discuss is purely dynamical, without any random element either in the equations of motion or in the initial conditions. Hence, the term - dynamical, or intrinsic chaos. Below we restrict ourselves to only Hamiltonian dynamics for which the invariant measure (phase space volume) is known beforehand, unlike dissipative systems.

The interest in the dynamical chaos is twofold. First, it is a fundamental phenomenon in physics which, in particular, gives, at last, a long-awaited model for the true random process. Second, no matter how strange the random dynamics may appear it turns out to be fairly wide-spread in many fields of science and technology as, in particular, the present Conference demonstrates.

In a rare occasion, when chaos comprises all the phase space of a dynamical system or, at least, a whole invariant surface of the motion, a fairly simple statistical description is possible as contrasted to most complicated dynamical pictures of motion. In many cases, however, the situation is not that simple. A typical example is the so-called divided phase space,divided into regions of both chaotic and regular motions separated by highly intricate borders. It is the structure of that chaos border which considerably complicates statistical description of the motion. Even though the mathematical theory of dynamical systems admits divided phase space and, moreover, terms it by a special notion - ergodic component - not much is actually known thus far on the dynamical behavior therein. Below we are going to consider a number of selected questions related to this topic. I choose an old Poincaré problem, which is still not solved completely, to discuss some recent developments in this field. A general review of the modern mathematical theory can be found in [1-3], while related physical theory is surveyed, e.g., in [4,5].

2. POINCARE'S PROBLEM

We begin with a "simple" example considered by Poincaré [6] in his attempt to understand profound difficulties arising in the study of nonlinear dynamics, in general, and of the famous three body problem, in particular. Much later, this example has proven to typify a fairly general situation in Hamiltonian dynamics (see [4] and Section 3b below).

Consider the motion of the ordinary pendulum under a high frequency parametric perturbation as described by the Hamiltonian

$$H(p,\varphi,\theta)=\frac{p^2}{2}+\omega_0^2\cdot \cos\varphi\cdot(1+\varepsilon\cdot\cos\theta) \tag{2.1}$$

Here φ is the angular position of pendulum (φ = 0 corresponds to the unstable equilibrium); $p=\dot{\varphi}$ is the angular momentum and ω_0 is the frequency of small oscillation. The perturbation is characterized by two small parameters: that of strength ε, and of adiabaticity $1/\lambda=\omega_0/\Omega$, $\Omega=\dot{\theta}$. The motion of the unperturbed (ε = 0) pendulum, as is well known, is periodic for any initial conditions with one important exception corresponding to the value of $H=\omega_0^2$. The latter trajectory is called separatrix since it separates the pendulum oscillation ($H<\omega_0^2$) from its rotation ($H>\omega_0^2$). In what follows the separatrix is going to play a leading part in dynamical chaos. The motion period T is increasing indefinitely when approaching separatrix. In immediate vicinity of the latter

$$T\approx\frac{1}{\omega_0}\ln\frac{32}{|w|}\ ;\quad w=\frac{H}{\omega_0^2}-1 \tag{2.2}$$

The separatrix motion is, thus, aperiodic, and it has continuous Fourier spectrum which may be characterized by the integral [4]:

$$A_m(\lambda)=\omega_0\int_{-\infty}^{\infty}e^{i\left(\frac{m}{2}\varphi_s(t)-\Omega t\right)}dt\approx\frac{4\pi}{\Gamma(m)}(2\lambda)^{m-1}\cdot e^{-\pi\lambda/2} \tag{2.3}$$

The last expression holds for $\lambda\gg 1$; $\Gamma(m)$ is the gamma function with any positive real m, and [6]

$$\varphi_s(t)=4\cdot\arctan\left(e^{\omega_0 t}\right)-\pi \tag{2.4}$$

is the separatrix motion (in case of $m<0$ $A_m=A_{|m|}\cdot e^{-\pi\lambda}$).

What is the impact of perturbation on the pendulum motion? The first move would be to consider the perturbation as completely nonresonant because of the condition $\lambda \gg 1$. Then, in the first approximation of the asymptotic theory [7] the perturbation can be neglected, or averaged out. Yet, in the second approximation it changes the effective potential:

$$U(\varphi) = \omega_0^2 \cdot \cos\varphi \rightarrow \omega_0^2 \cdot \left(\cos\varphi - \frac{\varepsilon^2}{8\lambda^2}\cos 2\varphi\right) \qquad (2.5)$$

and shifts the frequencies at both stable and unstable equilibria.

Now, let us inspect the perturbation more carefully. Is it really completely nonresonant? And is the change (2.5) its only effect? Certainly, it is not on the separatrix , as is obvious from the boundless spectrum (2.3). Hence, in some vicinity around separatrix we also cannot neglect the perturbation even in the first approximation. That the motion here is very sensitive to perturbation, which makes it highly intricate, has been found out and well recognized already by Poincaré. He was very close to the discovery of chaotic dynamics although he did never use this sort of language, instead speaking just about homoclinic solutions, or trajectories. One of the problems he has left to future researchers was to find out the dimension, structure, and measure of the homoclinic region near separatrix.

3. SOLUTION OF THE POINCARÉ PROBLEM

a. Separatrix mapping

First, we construct a mapping describing the motion near separatrix in finite time steps. It is natural to choose the motion period T as the time step. Then, the change in energy w over this step is given by the integral of the type (2.3) while the change in perturbation phase θ is determined by the dependence (2.2). Thus, we arrive at the separatrix mapping $(w, \theta) \rightarrow (\bar{w}, \bar{\theta})$[4]:

$$\bar{w} = w + \xi \cdot \sin\theta; \qquad \bar{\theta} = \theta + \lambda \cdot \ln\frac{32}{|\bar{w}|} \qquad (3.1)$$

The new perturbation parameter ξ is given by the expression

$$\xi = -4\pi\varepsilon\lambda^2 e^{-\pi\lambda/2} \qquad (3.2)$$

While ξ is proportional to small parameter ε , it cannot be expand in powers of adiabaticity parameter $1/\lambda$. Hence, as is commonly believed, the expression (3.2) as well as the map (3.1) go beyond the asymptotic perturbation theory. However, one can argue in a different way: it is not so much a fault of asymptotic theory but, rather, our own failure to choose the proper, adequate perturbation parameter. In other words, the true small parameter of the adiabatic perturbation is not the usual one $1/\lambda$, which enters the original Hamiltonian, but another one which explicitly takes account of weak resonances present in spite of adiabatic conditions. An important point of this philosophy relates to the fact that there is no principal difficulty in evaluating this ξ . The evaluation actually follows the usual asymptotic procedure of successive approximations since the unperturbed separatrix motion (2.4) is used. The really crucial difference from earlier unsuccessful approaches to Poincaré's and similar problems lies in seeking out the resonances even if they do appear to be absent.

Parameter ξ immediately gives the so-called splitting of separatrix, i.e. a gap between the two branches of separatrix going up and down in time (the first corresponds to $w=0$, and the second does so to $\bar{w}=0$, the maximal gap being $2|\xi|$). This effect has also been discovered by Poincaré [6] (Section 401). In our time it was further studied by Melnikov [8], Shilnikov [9] and others.

Separatrix splitting is a very important dynamical phenomenon. Yet, it does not tell us anything about a long-term evolution of the system. Are variations of w restricted or unbounded?

Before we proceed further we transform (3.1) introducing a new variable $y=w/\xi$, that is we take half of separatrix splitting as the unit for w . Ignoring a constant phase shift in the second equation (3.1) we arrive at the reduced map

$$\bar{y} = y + \sin\theta; \qquad \bar{\theta} = \theta - \lambda\cdot \ln|\bar{y}| \tag{3.3}$$

b. <u>The standard map</u>

For treating the separatrix mapping (3.3) analytically we introduce another approximate model [4] by linearizing the second equation (3.3) in y around one of resonant values of $y=y_r$ where $\lambda\cdot\ln y_r = 2\pi r$, and r is any integer. We get the map

$$\bar{I} = I + K\cdot\sin\theta; \qquad \bar{\theta} = \theta + \bar{I} \tag{3.4}$$

which is called the standard map since it is the final reducing step for a number of particular problems in nonlinear dynamics [4]. The new momentum $I=(y_r-y)\lambda/y_r$, and the perturbation parameter:

$$K = -\frac{\lambda}{y_r} \tag{3.5}$$

The standard map provides a local (in y) description for the previous model (3.3) under the condition: $|y_r - y_{r-1}| \ll y_r$, or $\lambda \gg 2\pi$. Note an additional symmetry of this map: $I \to I + 2\pi r$, which is not present in (3.3). It is just this symmetry that considerably simplifies the motion analysis since it makes the motion structure periodic in momentum I.

We replace, further, a discrete system (3.4) by the completely equivalent continuous one with Hamiltonian [4]

$$H(I, \theta, t) = \frac{I^2}{2} + K \cdot \sum_{r=-\infty}^{\infty} \cos(\theta - 2\pi r t) \qquad (3.6)$$

which has an infinite series of (integer) resonances $I = I_r = 2\pi r$. If we single out one of them, say, $r = 0$, and ignore (average out) all the others, we just come back to the pendulum whose motion we intended to study in this way. It is easy to see that leaving two more terms in series (3.6) ($r = \pm 1$) we completely recover the original problem (2.1) with the parameters:

$$\omega_0^2 = K; \quad \Omega = 2\pi; \quad \lambda = \frac{2\pi}{\sqrt{K}}; \quad \varepsilon = 2 \qquad (3.7)$$

Yet, it is not a vicious circle but a spiral of cognition! In a more formal language it is called renormalization.

Now, let us mention, first of all, that the dynamics of a single nonlinear resonance can be described as a pendulum motion, or in the "pendulum approximation". As is shown in [4], this approximation is applicable under fairly broad conditions. Moreover, the original problem (2.1) relates to the dynamics of several (three) resonances and, hence, does include also the resonance interaction. Here, precisely, lies the importance of the Poincaré example and of the Poincaré problem.

Renormalized system (3.6) is not completely equivalent to the original one (2.1) in that the former has infinitely many resonances instead of three only for the latter. At the first glance, the problem becomes, thus, much more complicated, yet this is not the case. Just due to periodicity in I, the standard map, unlike the perturbed pendulum, has a sharp critical value of its parameter $|K| = K_{cr}$ which separates the bounded and unbounded variation of I. What is this critical value? First, we may just refer to the numerical simulation [4] which gives $K_{cr} = 0.989 \approx 1$ to the accuracy within a few percent. Using a completely different approach, based on a combination of analytical as well as numerical procedures, Greene [10] has found $K_{cr} = 0.971635$. This latter result has been confirmed also in [11]. The accuracy of this value is open to criticism [12], yet, at any event, it is fairly close to the above numerical result.

The critical K value can be also estimated, in order of magnitude, from a simple resonance overlap criterion [4]

$$s = \frac{(\Delta\omega)_r}{\delta\omega_r} \approx \frac{4\sqrt{K}}{2\pi} \approx \frac{4\Omega_\phi}{2\pi} \sim 1 \tag{3.8}$$

where Ω_ϕ is the frequency of small phase oscillation on a resonance, $(\Delta\omega)_r$ is the resonance width, and $\delta\omega_r$ is the spacing of resonances under consideration. Even though Eq.(3.8) gives the correct order it considerably overestimates $K_{cr} \sim 2.5$ because only integer resonances ($\omega_r = I_r = 2\pi r$) are taken into account. Meanwhile, in higher approximations of perturbation theory the full set of resonances $(\omega_{rq} = 2\pi r/q)$ does appear which obviously lowers K_{cr}. A partial consideration of those higher order resonances results in a much better estimate [4]: $K_{cr} \approx 1.1$.

Below the threshold, that is for $|K| < 1$ (we neglect the above discrepancies in K_{cr}), the I variation is strictly bounded by the resonance width: $|\Delta I| \lesssim 4\sqrt{K}$. Above the threshold the motion is generally (depending on initial conditions) unlimited in I and chaotic [4]. From Eq.(3.5) we immediately see that the motion near separatrix is chaotic within the layer $|y| \lesssim \lambda$, or:

$$|w| \lesssim w_s = \lambda|\xi| = 4\pi\varepsilon\lambda^3 e^{-\pi\lambda/2} \tag{3.9}$$

This relation resolves [4] the Poincaré problem as to the dimension of a homoclinic region. Thus, the whole homoclinic structure generated by the two branches of split separatrix is chaotic and occupies a layer whose width is about λ times the separatrix splitting. That layer is commonly termed as the stochastic layer.

c. Numerical evidence

The first numerical verification of estimate (3.9) was undertaken [4] using the standard map as a model. Indeed, we have seen above that the latter is essentially equivalent to the original system (2.1) with parameters (3.7). As to the other resonances in (3.6), their contribution is exponentially small according to (3.9). There is an additional complication with the standard map related to the fact that parameter $\varepsilon = 2$ is no longer small. On the other hand, numerical simulation is much simpler, of course, for a map than for a continuous system like (2.1). The first numerical experiments showed, however, that Eq.(3.2) is not exact for the map, and an additional factor has to be introduced:

$$\xi \to \xi R_e ; \qquad R_e \approx 2.15 \tag{3.10}$$

Even though this factor can be calculated analytically as an effect of higher approximations [4], its actual evaluation seems to be formidable and constitutes an unsolved problem. This shows also that the above assumed condition $\varepsilon \ll 1$ is generally essential for the validity of Eqs. (3.2) and (3.9). Taking into account the factor (3.10), we arrive at the expression

$$w_s = 64\pi^4 R_e \frac{e^{-\pi^2/\sqrt{K}}}{K^{3/2}} \tag{3.11}$$

to be compared with numerical data. In a completely different approach this estimate has been confirmed in [13] except for the correction (3.10). New and more accurate data, obtained by Vecheslavov, are presented in Fig.1 as the dependence of w_s on motion time (the number of map iterations) for both the outer (curve 1) and the inner (curve 2) parts of stochastic layer (K = 0.5). Note unusually big fluctuations which we are going to discuss below (Section 4c). The values of w_s were calculated from the mean motion period T_m for a single trajectory in the layer, using the relation [4]

$$w_s = 32 \cdot \exp\left(1 - \sqrt{K} \cdot T_m\right) \tag{3.12}$$

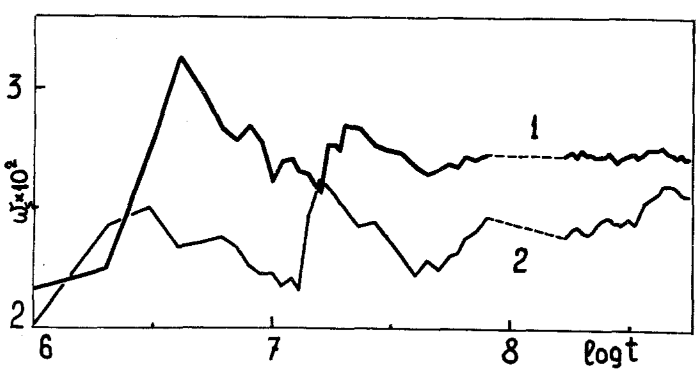

Fig. 1 Stochastic layer half-width vs. motion time

It is obtained via simple averaging of Eq.(2.2) assuming the uniform distribution of trajectory over the layer. Eq.(3.11) gives in this case w_s = .0329. Thus, the accuracy of simple estimate (3.11) is about 20 percent for this not a very big λ = 8.89.

The agreement can be improved by taking account of: i) frequency shift (2.5); ii) nonuniform equilibrium distribution in the layer (see Fig.2 as an example [14]); iii) the effect of marginal resonance inside the layer [4], the final accuracy achieved being about 4 percent. Note a slight asymmetry of the layer in Fig.1 which indicates the accuracy of asymptotic relation (2.2).

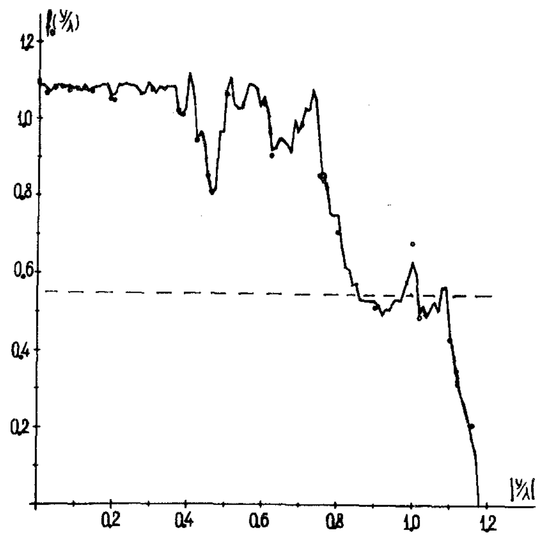

Fig. 2 Equilibrium distribution of a single trajectory in stochastic layer: $t = 10^7$ (broken line); $t = 4\times10^6$ (circles).

4. ON STRUCTURE OF THE CHAOS BORDER

Chaotic motion, particularly that in a stochastic layer, is, in principle, undistinguishable from a true random process according to the algorithmic theory of dynamical systems [3]. The random means here unpredictable, or uncomputable, which appears to be in conformity with our intuitive ideas of what the random is like. However, the randomness does not yet determine the statistical properties of motion. As is well known, the most fundamental of them is correlation.

a. Diffusion near the border

Consider, first, the standard map (3.4). The "force" correlation is defined by

$$C_F(\tau) = \langle \sin\theta(t+\tau)\cdot \sin\theta(t)\rangle \qquad (4.1)$$

where averaging is performed either along a trajectory (in motion time t) or over an ergodic component of the motion. For sufficiently large K , when regular component of the motion is negligible, this correlation is known to decay fairly fast [15], so that the sum

$$S_F = \sum_{\tau=1}^{\infty} C_F(\tau) \tag{4.2}$$

does certainly converge. Hence, a simple statistical description of the chaotic motion is possible by means of the diffusion equation:

$$\frac{\partial f(I,t)}{\partial t} = \mathcal{D} \cdot \frac{\partial^2 f(I,t)}{\partial I^2} \tag{4.3}$$

where the diffusion rate

$$\mathcal{D}(|K|) = \lim_{t \to \infty} \frac{\langle (\Delta I)^2 \rangle}{2t} = \frac{K^2}{4}\left[1 + 4 S_F(|K|)\right] \tag{4.4}$$

For large $|K|$ the correlation correction $S_F \sim |K|^{-1/2}$ (4.2) vanishes, and the diffusion rate approaches its limiting, uncorrelated value $\mathcal{D}^{\infty} = K^2/4$. In the opposite case $|K| \to 1$ the correlation dominates, and diffusion rate rapidly decreases [4]

$$\mathcal{D}(|K|) \approx a \cdot (|K| - 1)^{\alpha} \tag{4.5}$$

where $a \approx 1/5$, and $\alpha \approx 2.55$ according to numerical simulation.

For separatrix mapping (3.3) the diffusion becomes inhomogeneous since the dependence $D(|K|)$ turns into $D(|y|)$ according to Eq.(3.5). Generally, the diffusion equation includes an additional (drift) term. Indeed, the Fokker-Plank-Kolmogorov (FPK) equation can be written in the form (see, e.g. [16]):

$$\frac{\partial f}{\partial t} = -\frac{\partial Q}{\partial y}; \qquad Q = -\mathcal{D}(y)\frac{\partial f}{\partial y} + \mathcal{U}(y) f \tag{4.6}$$

Here $Q(y,t)$ is the flux, and $\mathcal{U}(y)$ is the drift velocity related to equilibrium distribution $f_0(y)$ by the expression

$$\mathcal{U}(y) = \frac{d}{dy}\mathcal{D}(y) + \mathcal{D}(y)\frac{d}{dy} \ln f_0(y) \tag{4.7}$$

Inspection of Fig. 2 shows that there are two regions within a stochastic layer where the drift can be neglected:

i) near the layer center where $f_0(y)$ = const exactly (variations of f_0 seen in Fig. 2 are due to fluctuations) and where $\mathcal{D}(y) \approx \mathcal{D}^{\infty}$ = 1/4;

ii) near the layer border where $f_0(y) \approx$ const approximately only (see below), and where

$$\mathcal{D}(y) \approx a \cdot \left(1 - \frac{|y|}{\lambda}\right)^{\alpha} \tag{4.8}$$

(see Eqs. (4.5) and (3.5)). Since the border line is of a complicated shape, the y variable above (as well as θ) is assumed to have been transformed in such a way as to "straighten out" this line ($|y_\ell| = \lambda$).

In a model like (2.1) the diffusion spreads across the layer, and is obviously restricted by a finite layer width. Neglecting so far the slow diffusion (4.8) at the layer edges, it takes $t_r \sim \lambda^2$ iterations for a trajectory to get across the layer, or for a distribution function to relax. Since, however, Eq.(4.4) still holds, a long time correlation does arise due to the boundary conditions. How simple the nature of that correlation may appear, it led to a paradox (or, rather, misunderstanding) [17-19] that the mixing precludes the diffusion instead of implying it. A formal reason for such a surprise conclusion is in that the mixing does provide existence of the limit in (4.4), while the paradox is a result of too literal understanding of this limit. It reminds us of an additional (besides the mixing) condition for the diffusion description of relaxation in a chaotic system to be applicable. Namely, there must exist two different time scales of the motion

$$t_c \ll t_r \tag{4.9}$$

that of correlation decay (t_c) on which the limit (4.4) is asymptotical, and the other one of relaxation (t_r) on which the same limit is local. For example, the motion in a stochastic layer has $t_c \sim 1$, and $t_r \sim \lambda^2$, so that the condition (4.9) requires $\lambda \gg 1$.

The long time correlation within stochastic layer is of a primary importance in many-dimensional systems where the diffusion along the layer (the so-called Arnold diffusion [4]) does generally occur. For the latter diffusion to be long-range, it has to be independent of the diffusion across the layer (due to different perturbation terms involved, for example) to get rid of that correlation.

Now, what would be the impact of the slow diffusion (4.8) on the motion in stochastic layer? It turns out to be crucial if the exponent $\alpha > 2$. Assume the following diffusion equation near the layer border

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x} x^{\alpha} \frac{\partial f}{\partial x} \tag{4.10}$$

where we have introduced a new variable $x = 1 - y/\lambda$ $(y>0)$, and rescaled t appropriately. First, let us try to find the eigenfunctions, that is to solve the equation

$$\frac{d}{dx} x^{\alpha} \frac{df_{\varkappa}}{dx} + \varkappa^2 f_{\varkappa} = 0 \tag{4.11}$$

It admits a solution via the cylindrical functions $Z_p(z)\ (\alpha \neq 2)$:

$$f_{\text{æ}}(x) = x^{\frac{1-\alpha}{2}} Z_p\left(\frac{2\text{æ}\, x^{1-\frac{\alpha}{2}}}{\alpha-2}\right); \quad p = \left(\frac{\alpha-1}{\alpha-2}\right)^2 \qquad (4.12)$$

If $\alpha < 1$ the solution is regular at $x = 0$, and the relaxation is exponential. However, for $\alpha \geqslant 1$ the solution is generally singular, and one would expect a nonexponential relaxation.

The general solution of this diffusion problem is not known. However, we may analyze a particular self-similar solution to Eq.(4.10) which, as is easily verified, reads:

$$\varphi_s' \equiv \frac{d\varphi(s)}{ds} = C\cdot s^{-\alpha} \cdot \exp\left(-(\alpha-2)^{-2} \cdot s^{2-\alpha}\right) \qquad (4.13)$$

Here C is an arbitrary constant, and $s = x \cdot t^{1/(\alpha-2)}$. At $x = 0$ the flux

$$Q(x,t) = -x^{\alpha}\frac{\partial\varphi}{\partial x} = -x^{\alpha-1} s\,\varphi_s' = -\frac{C}{t^{\frac{\alpha-1}{\alpha-2}}}\cdot \exp\left(-(\alpha-2)^{-2} \cdot s^{2-\alpha}\right) \qquad (4.14)$$

is always zero, while density $\varphi(s)$ may be non-zero (for $C < 0$). Due to the self-similar nature of this solution the second boundary condition cannot be imposed at any fixed x (e.g., at the layer center, $x = 1$). However, asymptotically as $t \to \infty$ it doesn't matter since the diffusion mainly proceeds in an ever narrowing region at the layer edge. The size x_D of this region ($s \sim 1$) scales with t as $x_D \propto t^{-1/(\alpha-2)}$, while for $s \to \infty$ the flux (4.14) becomes independent of x.

If the initial density at the layer edge is less than that at equilibrium, the relaxation corresponds to a negative (i.e. toward the edge) flux ($C > 0$), and to the boundary conditions:

$$\varphi(0,t) = 0; \quad \varphi(\infty,t) = \beta C \approx f_0 = 1; \quad \beta = \frac{\Gamma\left(\frac{\alpha-1}{\alpha-2}\right)}{(\alpha-2)^{\frac{\alpha}{\alpha-2}}} \qquad (4.15)$$

where equilibrium distribution f_0 is assumed to be constant, and $\Gamma(z)$ is the gamma function. Asymptotically as $t \to \infty$, and except the diffusion region $\sim x_D$, the relaxation proceeds as follows

$$|\varphi(x,t) - \varphi(x,\infty)| \to \int_t^{\infty} |Q|\,dt = \frac{\alpha-2}{\beta \cdot t^{\frac{1}{\alpha-2}}} \qquad (4.16)$$

In the opposite case a similar positive flux sets in ($C < 0$), and Eq.(4.16) remains unchanged. Thus, the slow diffusion ($\alpha > 2$) near the chaos border results in a power-type relaxation.

Since the time correlation of a pair of functions depends on the relaxation for one of them, we would expect, generally, the same power law (4.16) for the correlation as well. The latter may be faster though, if the relaxing function is close to equilibrium one near the border already from the beginning.

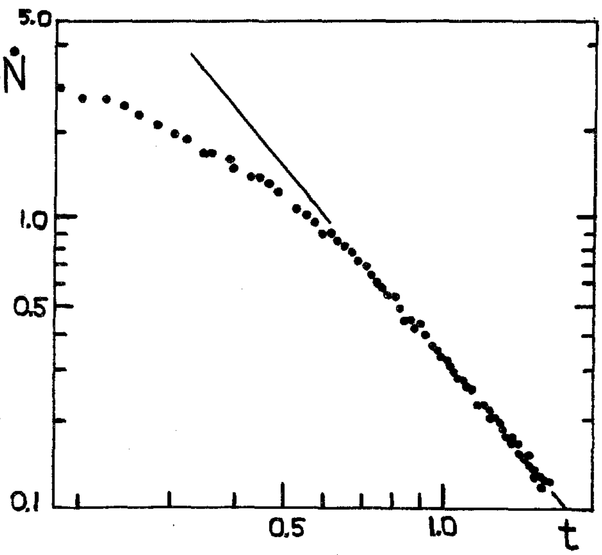

Fig. 3 Electron current out of magnetic trap (arbitrary units) vs. time (in msec); straight line: $\dot{N} \propto t^{-2.2}$

There is an interesting experiment on the behavior of electrons in a magnetic trap [20] which appears to confirm a power-type relaxation. The authors [20] observed a nonexponential dependence on time for the electron current $J(t) = -e\dot{N}$ out of the trap, due to a chaotic motion of electrons in inhomogeneous magnetic field, and did fit it by a doubly exponential function. On the other hand, the chaotic region of that electron motion is known to always have the border [4]. If one rescales the data [20] in the log-log plot, as shown in Fig. 3, they perfectly fit, for a sufficiently large time, the power dependence $\dot{N} \propto t^{-q}$ with exponent $q \approx 2.2$. This is to be compared to the flux (4.14): $Q \propto t^{-\frac{\alpha-1}{\alpha-2}}$, whence $\alpha \approx 2.83$. Remarkably, this value is not far away from that for the standard map ($\alpha \approx 2.55$, see (4.5)). It indicates some universal behavior near the chaos border.

For further studies of this behavior the Poincaré recurrencies proved to be very useful [14].

b. Poincaré recurrencies

Consider separatrix map (3.3), and follow a single trajectory while it crosses successively the symmetry line $y = 0$. The motion time interval between two successive crossings we shall call the recurrence time τ. As motion proceeds the distribution of τ values tends to a limiting function $F(\tau)$ defined as the probability for a recurrence to occur later than τ. Obviously, $F(1) = 1$ (for the map), and generally $F(\tau) \to 0$ as $\tau \to \infty$. An exception from the latter is, for example, the asymptotic motion (2.4) along the unperturbed pendulum separatrix. Note that in case of the motion with discrete spectrum (quasiperiodic or almost periodic motions) $F(\tau) \equiv 0$ at any τ greater than some τ_1, while in chaotic motion $F(\tau) \neq 0$ for all τ. Poincaré recurrencies do not imply, thus, quasiperiodicity as is stated sometimes.

In stochastic layer motion the asymptotic behavior of $F(\tau)$ as $\tau \to \infty$ relates to the structure of the layer border. Such an approach was actually used in [21] where the power dependence

$$F(\tau) \sim \tau^{-p}; \quad \tau \geq 1 \qquad (4.17)$$

has been found with $p = 1/2$. As was pointed out in [14], it corresponds to the free homogeneous diffusion until the layer border is reached, that is for $\tau \lesssim \lambda^2$. At larger $\tau \gg \lambda^2$ the dependence $F(\tau)$ approximately remains of a power-type but the exponent p changes; according to numerical data [14], the mean p for various λ is $\langle p \rangle \approx 3/2$. Besides, apparently irregular variations of $p(\tau)$ are present which do not depend on trajectory and, hence, relate to the border structure rather than to fluctuations in motion.

c. Scaling

As was mentioned above, there are numerical indications suggesting some universal behavior near the chaos border in the phase space. Now we are going to consider a theoretical model for this alleged universality. That the resonance structure determining transition to chaos is hierarchic has been known already since quite long ago (see, e.g. [22,4]). Yet, only in the pioneering work due to Greene [10] that structure has been exploited to evaluate a critical perturbation for the standard map. Hierarchic and scaling behavior at the transition to chaos was further studied extensively in many papers (see, e.g. [11,13] and references therein). A distinctive feature of our problem (see also [23]) is in that the perturbation strength here is not a parameter, as for standard map, but rather a function of dynamical variables (mainly, momentum y for separatrix map (3.3)). This leads just to a chaos border in the phase space rather than to a critical perturbation strength.

Assume the following scaling hypothesis: near the chaos border any two of dynamical variables (v, u) are interrelated by a power dependence:

$$v \propto u^{p_{uv}} \tag{4.18}$$

where p_{uv} is scaling parameter, and $p_{uv} \cdot p_{vu} = 1$. Choosing one variable (u) as the fundamental scaling unit we have

$$v \propto u^{p_v} \tag{4.19}$$

Such a scaling hypothesis is essentially identical to that in the fluctuation theory of phase transitions [24] which leads to some similarity of these two problems. However, important distinctions should not be missed. The scaling in phase transitions is continuous and essentially statistical (fluctuation scaling), while in our problem scaling is discrete (see below), and does relate to both chaotic as well as purely regular components of motion on both sides of the chaos border. What makes the two problems similar is a crucial impact of an infinite sequence of scales (continuous or discrete) upon the behavior at transition.

Transform (x, θ) variables in such a way as to provide: $x \propto |\omega(x) - \omega_b|$ near the border, $2\pi\omega(x)$ being the motion frequency of system (3.3) under consideration, and $\omega_b = \omega(0)$ the frequency at the border x = 0. Hence: $p_x = p_\omega$, or, choosing $(\omega - \omega_b)$ as the fundamental scaling unit ($p_\omega \equiv 1$), p_x = 1. Note that in original variables the exponent p_x would depend on θ (see [11]). The measure of chaotic component $\mu \propto x$ since at the border the resonances are just about to overlap in all scales (comp. Fig. 2), whence p_μ = 1.

To proceed further we need to relate these scales to that of time. It can be done via the overlap parameter s (3.8). The width $(\Delta\omega)_q$ of a high order resonance $\omega_q = r/q$ depends on its phase oscillation frequency Ω_q as [4,5]: $q\,(\Delta\omega)_q \sim \Omega_q$, while the resonance spacing $\delta\omega_q \sim q^{-2}$. The latter follows from the total number of resonances, within a given interval of ω, which is proportional to q^2. In a more formal way it is also implied from the best approximation of a given irrational number (ω_b in our case) by the convergents of the continued fraction representation [25]:

$$\left|\omega_b - \frac{r}{q}\right| \sim \frac{C(\omega_b)}{q^2} \qquad (4.20)$$

Hence, at the chaos border

$$s_q = \frac{(\Delta\omega)_q}{\delta\omega_q} \sim q^2(\Delta\omega)_q \sim q\,\Omega_q \sim 1 \qquad (4.21)$$

The overlap parameter s_q is related to the Greene residue [10]: $R_q \sim s_q^2$. For standard map with $|K| = K_{cr}$, which corresponds to the chaos border in map (3.3), $R_q \to 1/4$ as $q \to \infty$ [10] in accordance with estimate (4.21).

Suppose that a given scale is essentially determined by some resonance ω_q. Then, the associated time scale would be $T_q \sim \Omega_q^{-1}$, and $(\Delta\omega)_q \propto |\omega_q - \omega_b|$. Whence, $p_T = p_q = -1/2$. The scaling for diffusion rate near the border is, hence, $D \propto (\Delta\omega)_q^2/T_q \propto x^2/T \propto x^{2.5}$, and the diffusion parameter $\alpha = 5/2$ which is close to the numerical values given above.

Since resonance width $(\Delta\omega)_q \propto V_q^{1/2}$, where V_q is the corresponding Fourier amplitude of the limiting perturbation in the Hamiltonian (see below), the scaling (4.21) implies $V_q \propto q^{-4}$, i.e. the perturbation has two continuous derivatives only. This is precisely the critical smoothness of perturbation for the map [26,4]. It means the following: If the initial perturbation $V^0(\theta)$ is an analytic function, its Fourier amplitudes, as is well known, fall off exponentially, like $V_q^0 \propto \exp(-\sigma^0 q)$, for example. However, as we proceed to higher approximations the amplitudes grow, or parameter σ decreases [27]: $\sigma^0 \to \sigma(K)$. At critical perturbation the dependence V_q on q becomes, as everything else, of power-type, that is [10] $\sigma(K_{cr}) = 0$. On the other hand, as is also known [26], the initial perturbation needs not to be analytic for a chaos border to exist, instead it suffices for $V^0(\theta)$ to be only smooth, that is $V_q^0 \propto q^{-p_0}$ provided $p_0 > p_{cr}$. Otherwise, the motion is chaotic for any non-zero perturbation strength.

As was mentioned above, the scaling near the chaos border is discrete. It means that there exists a denumerable sequence of principal scales which is determined by a sequence of resonances $\omega_{q_n} = r_n/q_n$ converging to the border: $r_n/q_n \to \omega_b$ as $q_n \to \infty$. The resonance sequence depends on arithmetical properties of irrational ω_b, for example, on its representation as a continued fraction: $\{\omega_b\} = [b_1, b_2, \ldots, b_n, \ldots]$ where $b_i \geq 1$ are integers, and brackets denote the fractional part. According to Greene's conjecture [10], ω_b is the "golden mean", i.e. $\{\omega_b\} = g_1 = [1, 1, \ldots, 1, \ldots] = (\sqrt{5}-1)/2 \approx 0.618.$

It is not known whether this is true for the standard map but generally it does not hold [12,23]. A much weaker hypothesis that ω_b has a "golden tail", i.e. $\{\omega_b\} = [b_1, \dots, b_n, 1, \dots, 1, \dots]$ seems plausible. The main problem is to match the arithmetic of ω_b to the critical value of K which depends on X (comp. [23]). Apparently, the discrete scaling accounts for p(τ) variations mentioned above.

Finally, let us estimate the contribution to Poincaré recurrences from internal chaos borders of resonance stochastic layers. That there are many such layers within the main layer is immediately seen in Fig. 2 from a low equilibrium density near the border. It also follows from the limiting value of Greene residue R = 1/4 which means that the resonance centers near the border are not destroyed.

Let the time scale of a given resonance be T_q. Then the mean sojourn time in its region of measure $\mu_q \propto X_q$ is, due to ergodicity, $N_q T_q / t \propto X_q$, where N_q is the number of entries into this region, and t is the total motion time. Assume the universal distribution of Poincaré recurrences $F(\tau) \propto \tau^{-p}$ with some, unknown so far, p. Particularly, this implies the probability $F_q \propto (T_q/\tau)^p \propto (q/\tau)^p$ $(\tau \gtrsim q)$ for any internal chaos border of a resonance stochastic layer. Then, the contribution to Poincaré recurrences in the main layer from a particular resonance would be

$$F^{(q)} \propto N_q F_q / N \propto \frac{X_q q^{p-1}}{\tau^p} \cdot \frac{t}{N} \propto \frac{q^{p-3}}{\tau^p} \qquad (4.22)$$

where $N \propto t$ is the total number of recurrences. Now we need to sum up the contributions of all undestroyed resonances which do retain their stochastic layers. The number of those resonances can be estimated as follows. Define the border zone $X_z(q)$ as $\sigma(X_z) \cdot q \sim 1$ where $\sigma(X)$ is the exponential factor of the perturbation Fourier amplitudes introduced above. Assuming a linear dependence $\sigma(X) \propto X$ near the border we arrive at the scaling $X_z \propto q^{-1}$ for the border zone size. The latter implies that for a given q just one resonance gets into this zone, so we are to merely sum up contributions (4.22) over q :

$$F' \sim \sum_q^{\tau} F^{(q)} \propto \frac{1}{(p-2)\tau^2}, \; p \neq 2; \qquad F' \propto \frac{\ln \tau}{\tau^2}, \; p = 2$$

From universality $F'(\tau) \sim F(\tau)$, and p = 2. First of all, this would imply that the main contribution to Poincaré recurrences were not due to the diffusion near the main layer border but from a laby-

rinth of infinite hierarchies of internal chaos borders where the trajectory spends most of its recurrence time. If confirmed, it would
tory spends most of its recurrence time. If confirmed, it would
also mean that near the chaos border the above scaling hypothesis holds only approximately, to logarithmic accuracy. This also would change the behavior of both relaxation as well as correlation near the chaos border as compared to estimates in Section 4a based upon the diffusion equation (4.10). In any event, a power-type relaxation inevitably leads to big fluctuations in motion which are clearly seen, for example, in Fig. 2.

Certainly, the problem of the chaos border structure needs and deserves further studies.

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REFERENCES

1. V.I.Arnold and A.Avez, Ergodic Problems of Classical Mechanics, Benjamin (1968).
2. I.P.Kornfeld, Ya.G.Sinai, S.V.Fomin, Ergodic Theory, Nauka, 1980 (in Russian).
3. V.M. Alekseev and M.V. Yakobson, Physics Reports, 75, 287 (1981).
4. B.V.Chirikov, Physics Reports, 52, 263 (1979).
5. A.J. Lichtenberg and M.A. Lieberman, Regular and Stochastic Motion, Springer-Verlag (1982).
6. H. Poincaré, Les méthods nouvelles de la méchanique céleste, Vol. II (1893), Sections 225-232; Vol. III (1899), Section 401.
7. N.N. Bogoliubov and Yu.A. Mitropolsky, Asymptotic Methods in the Theory of Nonlinear Oscillations, Hindustan Publ. Corp., Dehli, 1961.
8. V.K. Melnikov, Dokl. Akad. Nauk SSSR, 144, 747 (1962) (in Russian).
9. L.P. Shilnikov, Mat. sbornik, 77, 461 (1968) (in Russian).
10. J.M. Greene, J. Math. Phys. 20, 1183 (1979).
11. L.P.Kadanoff, Phys. Rev. Lett. 47, 1641 (1981); S.J. Shenker and L.P. Kadanoff, J. Stat. Phys., 27, 631 (1982).
12. B.V. Chirikov, F.M. Izrailev, D.L. Shepelyansky, in Soviet Scientific Reviews, Section C, Vol. 2 (1981), p. 209.
13. D.F. Escande, Large-Scale Stochasticity in Hamiltonian Systems, Intern. Conf. on Plasma Physics, Göteborg (1982).
14. B.V. Chirikov, D.L. Shepelyansky, Statistics of the Poincaré Recurrences and the Structure of Stochastic Layer of a Nonlinear Resonance, Preprint 81-69, Institute of Nuclear Physics, Novosibirsk (1981) (in Russian).
15. C. Grebogi and A.N. Kaufman, Phys. Rev., A24, 2829 (1981).
16. E.M. Lifshits, L.P. Pitaevsky, Physical Kinetics, Nauka (1979) (in Russian).
17. J.L. Lebowitz, in Statistical Mechanics, New Concepts, New Problems, New Applications, Univ. of Chicago Press (1972).
18. R. Balescu, Equilibrium and Nonequilibrium Statistical Mechanics, Wiley, New York (1975), Appendix.

19. G.E. Norman, L.S. Polak, Dokl. Akad. Nauk SSSR, 263, 337 (1982) (in Russian).
20. D. Bora, P.I. John, Y.C. Saxena and R.K. Varma, Plasma Physics, 22, 653 (1980).
21. S.R. Channon and J.L. Lebowitz, Ann. N.Y. Acad. Sci., 357, 108 (1980).
22. J.M. Greene, J. Math. Phys., 9, 760 (1968).
23. J.M. Greene, in Nonlinear Dynamics and the Beam-Beam Interaction, A.I.P. Conf. Proc., N°57 (1979), p. 257.
24. A.Z. Patashinskii and V.L. Pokrovskii, Fluctuation Theory of Phase Transitions, Pergamon (1979).
25. A.Ya. Khinchin, Continued Fractions, Fizmatgiz, Moscow (1961) (in Russian).
26. J. Moser, Stable and Random Motions in Dynamical Systems, Princeton Univ. Press (1973).
27. V.I. Arnold, Usp. mat. nauk, 18, N°6, 91 (1963)(in Russian).