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Particle confinement and adiabatic invariance

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The general problem of adiabatic invariance is discussed by using, as an example, the particle motion in a mirror magnetic trap. A new, resonant, adiabaticity parameter is introduced, which allows the reduction of the axisymmetric problem to the standard map, and which provides a sharp estimate for the chaos (instability) border in magnetic traps. Peculiarities of the particle diffusion out of the trap, due to the presence of the chaos border, are considered. The mechanism of slow Arnol'd diffusion in a non-axisymmetric field is explained, and some rough estimates are given, including the impact of weak noise (gas scattering).

1. Introduction

The problem of controlled nuclear fusion, aimed at the peaceful use of thermonuclear power, has stimulated a broad range of fundamental research in plasma physics. Among other problems, particle dynamics in electromagnetic fields has been intensively studied. On one hand, these fields are supposed to confine particles for a fairly long time within a bounded domain of space (the so-called 'magnetic traps'). On the other, they are expected to heat the same particles up to a very high thermonuclear temperature.

In these studies physicists have come across the first examples of a rather peculiar phenomenon, called *dynamical chaos*, that is a random (unpredictable) motion of a completely deterministic mechanical system whose equations of motion did not contain any random parameters or any noise (see, for example, Chirikov 1960; Lichtenberg & Lieberman 1983; Chirikov 1984; Zaslavsky 1985). Still earlier, similar phenomena were observed in numerical experiments (Goward 1953; Hine 1953) on particle dynamics in an accelerator (also a 'magnetic trap').

Apparently the first who actually dealt with dynamical chaos was English psychologist and anthropologist, Galton. In 1889 (almost a century ago!) he had developed his famous device, the Galton Board, to demonstrate statistical laws as well as statistical methods for studying mass phenomena, the methods he first introduced into psychology (Galton 1889). Galton himself had, apparently, no interest in the dynamics of his fairly simple mechanical model which has been termed later on as 'Lorentz's gas', and which nowadays is intensively studied, among others, by the Soviet mathematician Sinai and his disciples. Only 17 years later Poincaré conjectured that a highly irregular, chaotic motion in such systems was due to a strong local instability of the trajectories. It took more than half a century to develop this fairly simple physical insight into a rigorous theory. Now motion instability or, in modern language, a positive (non-zero) Kolmogorov–Sinai entropy is the most efficient criterion for dynamical chaos (Alekseev *et al.* 1981), although still not commonly accepted.

Back to plasma physics, I shall mention also the problems concerning the 'dynamics' of a magnetic line itself (Gelfand *et al.* 1963; Rosenbluth *et al.* 1966; Filonenko *et al.* 1967). These lines may be chaotic as well (a 'braided' magnetic field), which results in a considerable increase of the plasma transverse thermal conductivity. Particle diffusion in such a field (Rechester *et al.* 1978; Kadomtsev *et al.* 1978) provides a striking example of time-reversible chaos. Consider a toroidal magnetic trap (say, tokamak) and write the diffusion of magnetic lines as $\Delta r_B \propto s^{\frac{1}{2}}$ with s the distance along a line. Assuming that particle follows a certain particular magnetic line and neglecting the scattering completely we arrive at the standard diffusion estimate $\Delta r_{\mathrm{p}} \propto t^{\frac{1}{2}}$. Now, take the scattering into account, still neglecting all the finite Larmor radius effects. Then, the particle displacement along a line grows as $s \propto t^{\frac{1}{2}}$ only, because of random changes of the sign of its velocity as a result of scattering. Hence the transverse diffusion drops down to $\Delta r_{\mathrm{p}} \propto t^{\frac{1}{4}}$ (Rechester *et al.* 1978). Of course, the particle is going to leave its 'own' chaotic magnetic line eventually, yet the plasma thermal conductivity considerably decreases. It is an important practical consequence of the time reversibility of dynamical chaos.

2. Budker's problem

Now we are going to discuss in some detail a typical dynamical problem in the plasma theory, the particle motion in a mirror magnetic trap. Those devices are now called 'open' (for the magnetic lines, and we hope not for the plasma!). The problem was formulated in 1954 by Soviet physicist Budker who proposed, independently from American physicists, to make use of such a trap for controlled nuclear fusion (see Budker 1982). Precisely this problem was the beginning of the study of dynamical chaos in our institute, at that time in the depths of the Kurchatov Institute for Atomic Energy.

Particle confinement in an open trap is provided by the conservation of an adiabatic invariant, the magnetic moment of a particle, which is known, however, to be an approximate motion integral only. Thus, Budker's problem turned out to be a particular case of the general fundamental problem of *adiabatic invariance*, that is the invariance of the action variables under the special, adiabatic perturbation.

Previously, *pulsed*, or *single*, adiabatic perturbations, which are effective over a finite time interval only, had been mainly studied. As is well known, the principal condition for adiabatic invariance in this case is perturbation *slowness* as compared with the characteristic frequencies of a system. In the class of analytical functions the violation of adiabaticity is asymptotically (for $t \rightarrow \pm\infty$) exponentially small in the slowness parameter, and hence unimportant as a rule.

In Budker's problem we encounter a rather different situation when the adiabatic perturbation, due to a particle longitudinal libration, is a *stationary*, or *multiple* one. The smallness of the libration frequency as compared with the Larmor frequency (the slowness of adiabatic perturbation) does no longer guarantee the conservation of the adiabatic invariant because its small changes over each libration period may now accumulate. Here, as in many other problems in the theory of oscillations, the resonances (linear or nonlinear) between the particle gyration and libration are of primary importance.

A vague idea about the role of resonances in the problem of adiabatic invariance arose at the dawn of the quantum mechanics, where the action variables are of a special importance. Yet this question was only properly formulated and answered in 1928, for linear oscillations, by Soviet physicists Andronov, Leontovich and Mandelstam (see Mandelstam 1948). Remarkably, it proved to be sufficient simply to carefully look at the well-known Mathieu's equation and its solutions from the physical viewpoint. Indeed, the stop bands of parametric resonance do exist here for arbitrary small values of the slowness parameter

$$\epsilon = \Omega/2\omega_0 \tag{1}$$

near $\epsilon = 1/n$, where $n \neq 0$ is any integer, ω_0 the unperturbed frequency of a linear oscillator, and Ω the frequency of the harmonic parametric perturbation. Thus, in the present case, the necessary condition for adiabatic invariance is the *lack of resonances* in perturbation. Its slowness just helps in that the resonance width ($\Delta\omega_0$) as well as the maximal increment (γ) of the instability (in action) drop exponentially in ϵ:

$$\Delta\omega_0 \approx \gamma \approx \tfrac{1}{3}\Omega(\tfrac{1}{8}\mathrm{e}^2 g)^{1/\epsilon}. \tag{2}$$

Here $\mathrm{e} = 2.71\ldots$, and g is dimensionless perturbation amplitude: $\omega^2(t) = \omega_0^2(1-g\cos\Omega t)$ (see Chirikov 1984; 1986 for details).

The problem becomes much more complicated for the nonlinear oscillations of particle in a magnetic trap. First, the resonances now depend not only on the trap parameters but also on the motion initial conditions, which determine the oscillation frequencies. Moreover, a single resonance does not produce any instability at all because of the stabilization of resonant perturbation by nonlinearity. Yet, the interaction of several nonlinear resonances (if only two) results in a rather peculiar instability, dynamical chaos (see, for example, Lichtenberg & Lieberman 1983; Zaslavsky 1985).

3. Resonant perturbation theory for adiabatic problems

Consider, as a particular example, the axisymmetric trap whose magnetic field in some axis neighbourhood depends on the coordinate s along a magnetic line as

$$\omega(s) = \tfrac{1}{2}\omega_0[(\lambda+1)-(\lambda-1)\cos \pi s/L]. \tag{3}$$

Here ω_0 is the minimal magnetic field, and λ the mirror ratio. This model also describes a multimirror ($2L$ being the mirror spacing), or 'corrugated' magnetic field.

An important step from the beginning is the choice of unperturbed system. We require the magnetic moment to be constant,

$$\mu = v_\perp^2/2\omega \equiv \text{const.}, \tag{4}$$

where $v_\perp$ is the transverse component of particle velocity. That choice corresponds to the well-known Born–Oppenheimer approximation in quantum mechanics. The unperturbed hamiltonian becomes

$$H^0(p, s) = \tfrac{1}{2}v^2 = \tfrac{1}{2}p^2 + \mu\omega(s), \tag{5}$$

where $p = \dot{s}$ is the momentum. We use relativistic units $e = c = m(1-v^2)^{-\frac{1}{2}} = 1$. All the relations hold for an arbitrary particle velocity even though some of them have a 'non-relativistic' appearance.

Below we restrict ourselves for simplicity to two limiting cases

$$1/\lambda \ll \sin^2\theta_0 \ll 1, \tag{6a}$$

$$|1/\lambda - \sin^2\theta_0| \ll 1/\lambda. \tag{6b}$$

Here θ_0 is the pitch-angle at the field minimum: $\mu = v^2 \sin^2\theta_0/2\omega_0$. Case (a) corresponds to a small (harmonic) libration in the 'potential'

$$U(s) \approx \omega_0(1+s^2/s_p^2); \quad s_p = 2L/\pi\sqrt{(\lambda-1)} \ll L, \tag{7}$$

where s_p characterizes the spatial field scale. For (b) the motion is close to separatrix $H^0 = H_s^0 = \mu\lambda\omega_0$, or the loss cone $\theta_0 = \theta_s$, $\sin\theta_s = \lambda^{-\frac{1}{2}} \ll 1$, which separates the trapped $(\Delta\theta_0 = \theta_0 - \theta_s > 0)$ and untrapped trajectories. It is convenient to describe the distance from the separatrix by the dimensionless quantity

$$w(\mu) = \frac{2(H^0 - H_s^0)}{\mu(\lambda-1)\,\omega_0} \approx -4\sqrt{\lambda}\Delta\theta_0; \quad |w| \ll 1. \tag{8}$$

The unperturbed frequencies are:

$$\Omega(\mu) = \frac{\partial H^0}{\partial I} \approx \frac{\sqrt{(2\mu\omega_0)}}{s_p} \equiv \Omega_0(\mu),$$

$$\langle\omega(\mu, I)\rangle = \frac{\partial H^0}{\partial \mu} \approx \omega_0 + \frac{\pi}{2\sqrt{2}}\sqrt{\left(\frac{\lambda\omega_0}{\mu}\right)}\frac{I}{L} \approx \frac{\omega_0}{2\sin^2\theta_0}, \tag{9a}$$

$$\Omega(\mu) \approx \frac{\pi\Omega_0}{\Lambda(\mu)}; \quad \langle\omega(\mu)\rangle \approx \lambda\omega_0\left(1 - \frac{2}{\Lambda(\mu)}\right) \approx \lambda\omega_0. \tag{9b}$$

Here, $\Lambda(\mu) = \ln(32/|w|)$, I is longitudinal action, and $\langle\omega\rangle$ the Larmor gyrofrequency averaged over the libration period. Define the slowness parameter of the adiabatic perturbation as

$$\epsilon(\mu, I) = \frac{\Omega}{\langle\omega\rangle} \approx \frac{2\mu}{I} \approx \pi\frac{\rho_m}{L}\sqrt{\lambda}\sin^3\theta_0, \tag{10a}$$

$$\epsilon(\mu) \approx \frac{\pi\Omega_0}{\lambda\omega_0(\Lambda-2)} \approx \frac{\pi^2}{2}\frac{\rho_m}{L}\frac{\sqrt{\lambda}}{\Lambda}\sin\theta_0, \tag{10b}$$

where $\rho_m = v/\omega_0$ is the maximal Larmor radius. Notice that even for (a), *harmonic* libration (7), the latter is essentially *nonlinear* because its frequency $(9a)$ as well as the frequency ratio $(10a)$ depend on action variables μ, I. Both actions are related by the energy conservation $H^0 = \text{const.}$, whence $I \approx LH^0/\sqrt{(2\omega_0\mu)}$ for (a).

A specific difficulty of stationary adiabatic problems in the theory of nonlinear oscillations is that one cannot directly use here the powerful asymptotic methods in the perturbation theory, particularly, a fairly simple and efficient averaging method. This can be seen from estimate (2) for the linear problem. Even though the dependence on the perturbation amplitude is a power law, which can be obtained via a standard asymptotic series, the expansion in the small slowness parameter ϵ fails. On the other hand, at large perturbation amplitudes, which are typical for adiabatic processes, ϵ is the only small parameter. An example is the model under consideration (3) for $\lambda \gg 1$. The difficulty is still worse because for stationary oscillations the change in action variables consists of two quite different parts:

(i) the *quasiperiodic* variation, for instance, $\Delta\mu_q \sim \epsilon$, which is relatively big but non-cumulative and, hence, unimportant after all;

(ii) the *resonant* variation, $\Delta\mu_r \sim e^{-1/\epsilon}$ (over a libration period, see (12) below), which just results in motion instability and particle losses even though it is rather small compared with (i).

To overcome this difficulty, introduce a new 'good' adiabaticity parameter which, first, would describe the resonant cumulating variation $\Delta\mu_r$ only, and, second, would include the non-analytical exponential $\Delta\mu_r \sim e^{-1/\epsilon}$ from the beginning. Both objectives are naturally realized by the construction of a map over the libration period. Because of the symmetry of model (3) in respect to the field minimum, the map over half a period is sufficient. At any resonance ($m\langle\omega\rangle = 2n\Omega$), the quasiperiodic variations $\Delta\mu_q$ can be completely excluded, hence the term '*resonant perturbation theory*'. Actually, such an approach had been used already by Chirikov (1960) and was further developed later on (Chirikov 1978; see also Chirikov 1979, 1984).

To realize this approach one needs, first of all, to calculate $\Delta\mu_r$ by some direct integration. Curiously, this problem has been solved initially for the quantum equations of motion with subsequent transition to the classical mechanics (Dykhne *et al.* 1961), an amusing zigzag of cognition! A simple classical procedure for evaluating $\Delta\mu_r$ has been developed and applied by Hastie *et al.* (1968).

For the field (3), $\Delta\mu_r$ is exponentially small because of the (per period) singularity $\omega(s) = 0$ at $s = is_p$ (7). Because $s_p \ll L$ for $\lambda \gg 1$ ($\theta_0 \ll 1$) the change in μ occurs in a small neighbourhood of the field minimum. Introducing an new dimensionless variable

$$P = \sqrt{(\mu/\mu_{max})} = \sin\theta_0 \approx \theta_0 \ll 1 \tag{11}$$

we obtain in both cases ((*a*) and (*b*)) (Chirikov 1984)

$$\xi \equiv (\Delta P)_{max} \approx \frac{9\pi^2}{32}\frac{r_0\sqrt{\lambda}}{L}\frac{e^{-1/\epsilon_B}}{\epsilon_B}; \quad \epsilon_B = \frac{3\pi}{4}\frac{\rho_m\sqrt{\lambda}}{L} \ll 1 \tag{12}$$

for the amplitude of the variation of P, which we choose as a new 'good' adiabaticity parameter ξ. Here r_0 is the distance of the Larmor centre from the symmetry axis at a field minimum; $\epsilon_B \sim \epsilon$ the field 'smoothness' parameter (cf. (10)), and the last inequality is the applicability condition for (12).

4. The standard map

The resonant change in P depends on the Larmor phase ϕ at the field minimum: $\Delta P = \xi \cdot \sin\phi$. In turn, the ϕ variation is determined by the frequency ratio (10), or by the parameter of perturbation slowness: $\Delta\phi = \pi/\epsilon(P)$. This leads to a map $(P, \phi) \rightarrow (\bar{P}, \bar{\phi})$ over half a libration period

$$\bar{P} = P + \xi \sin\phi; \quad \bar{\phi} = \phi + \pi/\epsilon(\bar{P}), \tag{13}$$

which describes a long-term particle motion.

The cumulative non-adiabatic variation of μ is related to resonances at $P = P_n$ where $\epsilon(P_n) = \frac{1}{2}n$ with any integer n. The rate of accumulation is determined by the new parameter ξ, which may be called therefore the *resonant adiabaticity parameter* as well as the whole perturbation theory based upon it (§3). Notice that high harmonics of the libration frequency, which are required for the resonances at large n, are present even in the case (a) of harmonic oscillations due to the libration-modulated Larmor frequency.

Because ξ does include the principal non-asymptotic effect of adiabatic perturbation, one can use any standard asymptotic expansion in ξ, particularly, the efficient averaging method (Chirikov 1960, 1978, 1979, 1984).

Map (13) can be simplified still further by linearizing the second equation in P. Introducing a new variable $p = G(P)+G'(P)\,\Delta P$ with $G(P) = \pi/\epsilon$ we arrive at the so-called standard map

$$\bar{p} = p + K\sin\phi; \quad \bar{\phi} = \phi + \bar{p}, \tag{14}$$

which describes the local (in P) dynamics of model (13). These depend on the only parameter

$$K(P) = \xi|G'(P)| \approx \frac{81\pi^3}{128}\frac{r_0\sqrt{\lambda}}{L}\frac{e^{-1/\epsilon_B}}{\epsilon_B^2 P^4}, \tag{15a}$$

$$K(P) \approx \frac{27\pi^2}{64}\frac{r_0\lambda^2}{L}\frac{e^{-1/\epsilon_B}}{\epsilon_B^2|\Delta P|}. \tag{15b}$$

Here $\Delta P = P - P_s \approx \Delta\theta_0; P_s = \sin\theta_s$. We shall call K the *stability parameter* because there exists a critical value K_c separating bounded motion in $p(K \leqslant K_c)$ from unbounded motion ($K > K_c$). The latter just means the instability (non-adiabaticity) of particle motion that brings the particle onto the loss cone at $P = P_s = \lambda^{-\frac{1}{2}} \ll 1$. In terms of nonlinear resonances the critical K is determined by their 'overlapping' which results in an unbounded 'wandering' of trajectory in p (Chirikov 1979). On the other hand, the Kolmogorov–Arnol'd–Moser (K.A.M.) theory does rigorously guarantee (for $K \to 0$) the boundedness of p and μ oscillation, which results in the eternal confinement of a particle in the trap (Arnol'd 1963). In this sense the adiabatic invariant becomes an exact one. Notice that for some initial conditions the eternal oscillation of a particle is, nevertheless, chaotic. Thus we encounter the curious phenomenon of *chaotic adiabaticity*. On the other hand, there are domains of regular bounded oscillations for $K > K_c$ as well.

The precise K_c value is obtained in the theory of critical phenomena, which is a recent development of the K.A.M. theory. According to Greene (1979) and to MacKay & Percival (1985) the critical K lies within the interval

$$0.9716\ldots \leqslant K_c < \tfrac{63}{64} = 0.9843\ldots. \tag{16}$$

Such an uncertainty is of no importance for applications, of course, yet it leaves open some interesting questions in the theory of the critical structure (Chirikov & Shepelyansky 1986).

The elusively simple model (14) has turned out to be very popular in the studies on nonlinear and chaotic dynamics. Besides, as was shown above, the standard map has a direct bearing on some real physical problems. A new example is the dynamics of a comet under Jupiter's perturbation (Petrosky 1986).

Among the early researchers on model (14) was British physicist J. B. Taylor.

To the best of my knowledge, the standard map first appeared in the problem on electron dynamics in a new relativistic accelerator, the microtron, invented by Soviet physicist Veksler (1944). This was studied by Kolmensky (1960) and many others (see, for example, Kapitsa & Melekhin 1969). In all these papers the case of a stable regular acceleration only was considered. The main microtron domain corresponds to the standard map parameter K as follows

$$K = \frac{2\pi\omega V}{cB} = \frac{2\pi}{\sin\phi_s} \approx 6.59, \tag{17}$$

where ω, V, ϕ_s are the frequency, amplitude, and equilibrium phase of the accelerating voltage respectively, and where B is the magnetic field strength. The value (17) is much greater than the critical one. As a result, the regular acceleration domain is only about 1% of the phase plane, even for the optimal $\sin\phi_s \approx 0.95$. The other initial conditions give rise to chaotic electron motion, that is the microtron turns into a 'stochatron', upon some necessary changes in its design, of course. Such a chaotic electron acceleration has been observed recently in a 'plasma microtron' (Vaskov *et al.* 1984).

5. The chaos border and statistical anomalies

Back to Budker's problem we see from (15) that there is always a *chaos border* in the magnetic trap determined by the condition $K = K_c \approx 1$. In velocity space the border is a cone with vertex angle $\theta_b \approx P_b$, where

$$\frac{1}{\sqrt{\lambda}} \ll \theta_b \approx 2.1\left(\frac{r_0}{4}\right)^{\frac{1}{4}}\lambda^{\frac{1}{8}}\frac{e^{-\frac{1}{4}\epsilon_B}}{\epsilon_B^{\frac{1}{2}}} \ll 1, \tag{18a}$$

$$|\Delta\theta_b| \approx 4.2\frac{r_0}{4}\lambda^2\frac{e^{-1/\epsilon_B}}{\epsilon_B^2} \ll \frac{1}{\sqrt{\lambda}}. \tag{18b}$$

The inequalities recall the applicability conditions for simplifying assumptions of §3. One may say that the chaos border widens the adiabatic loss cone $\theta_s \approx \lambda^{-\frac{1}{2}}$.

A very intricate critical structure near the chaos border consists of alternating components of regular and chaotic motions whose spatial (in θ) scales indefinitely decrease towards the border while the temporal ones increase without limit. This leads to a long-time 'sticking' of a chaotic trajectory, and, hence, to a sharp drop of the diffusion rate. As a result, the statistical relaxation proceeds abnormally slowly according to some power, rather than exponential, law.

If, for example, the particles fill up all the chaotic component connected with the adiabatic loss cone homogeneously, the number of still confining particles $N(t)$ first decays exponentially in time because of an ordinary relatively fast diffusion off the chaos border. However, as soon as the relaxation process approaches the border its rate drops and, according to numerical experiments (Chirikov & Shepelyansky 1981; Karney 1983), proceeds approximately as follows:

$$N(t) \propto t^{-\frac{1}{2}}. \tag{19}$$

The time correlation in the chaotic motion with a border decays in the same way.

However, if the particles are injected near the adiabatic loss cone, i.e. relatively far from the chaos border, than $N(t)$ decays according to (19) *from the beginning* but for a different reason, namely homogeneous diffusion inside the trap. In this case $N(t)$ is proportional to the integral probability of Poincaré's recurrences into the initial state, at the loss cone. For different models this initial stage of the process was observed in some numerical experiments (Channon & Lebowitz 1980; Yamaguchi & Sakai 1986). However, as soon as the diffusion reaches the chaos border the decay proceeds as (Chirikov & Shepelyansky 1981)

$$N(t) \propto t^{-\frac{3}{2}}. \tag{20}$$

The latter régime was apparently observed by Indian physicists in a laboratory experiment on electron dynamics in a magnetic trap (Bora *et al.* 1980). The empirical value of the exponent in (20) was approximately -1.3 (Chirikov 1984).

A power-law correlation decay $C(\tau) \propto \tau^{-p}$ with the exponent $p<1$ (19) qualitatively transforms the diffusion related to that correlation. Formally, the diffusion rate becomes infinite. Actually, it means that the dispersion σ^2 of distribution function grows abnormally fast (Chirikov & Shepelyansky 1984):

$$\sigma^2(t) \propto t^{2-p} \approx t^{\frac{3}{2}}. \tag{21}$$

This occurs, for instance, in the chaotic component of the standard map in the presence of a domain of regular microtron acceleration (17) with the chaos border around, that is when the microtron behaves as a stochatron. In numerical experiments (Karney *et al.* 1982) the empirical 'diffusion rate' increased by a factor of about 100, even though the relative area of the regular microtron acceleration was as small as $A_s \approx 0.02$. The theory of critical phenomena leads to the estimate

$$\sigma^2 \approx \alpha A_s \tfrac{1}{2} K^2 t^{\frac{3}{2}}, \tag{22}$$

where $\alpha \approx 0.5$ from the numerical data (Karney *et al.* 1982).

6. Arnol'd diffusion and universal non-adiabaticity

Above we considered the axisymmetric magnetic trap where the particle motion is essentially two dimensional (two degrees of freedom). Any asymmetry of the magnetic field, making the motion three dimensional, greatly complicates the particle dynamics so that only rough estimates can be derived. In what follows we are going to discuss the three typical cases.

(a) *A strong asymmetry, the multiplet overlap*

Because of a particle drift with frequency Ω_d each of resonances $\langle\omega\rangle = 2n\Omega$ (§4) splits into a multiplet of m subresonances with spacing Ω_d and with

$$m \approx \frac{\Delta\langle\omega\rangle + 2n\Delta\Omega}{\Omega_d} \approx \left[\frac{\Delta\langle\omega\rangle}{\langle\omega\rangle} + \frac{\Delta\Omega}{\Omega}\right]\frac{\langle\omega\rangle}{\Omega_d} \equiv \nu\frac{\langle\omega\rangle}{\Omega_d}, \tag{23}$$

where $\Delta\langle\omega\rangle$, $\Delta\Omega$ are the full widths of frequency modulation on a *drift surface* (Chirikov 1978). Under condition $\nu \gtrsim 2\Omega/\langle\omega\rangle \approx 1/n \ll 1$ the neighbouring multiplets do overlap, and a usual global chaos sets in. The critical K value considerably diminishes

$$K_c(\nu) \approx \sqrt{(\pi m)}\,(\Omega_d/2\Omega)^2 \sim \epsilon\sqrt{\nu} \gtrsim \epsilon^{\frac{3}{2}} \ll 1. \tag{24}$$

In the last estimate $\Omega_d/\Omega \sim \epsilon = \Omega/\langle\omega\rangle$ is assumed, and the lower bound for $K_c(\nu)$ is determined by the multiplet overlap. For $K > K_c(\nu)$, the diffusion rate is about that at $\nu = 0(K > 1)$ as it depends on the spectral density of perturbation.

(*b*) *A moderate asymmetry, the modulational diffusion*

Let

$$\epsilon^2 \sim \Omega_d/\langle\omega\rangle \lesssim \nu \lesssim 1/n \sim \epsilon, \tag{25}$$

then multiplets do not overlap, yet they exist as $m \gtrsim 1$ (the left inequality). Under condition $K \gtrsim K_c(\nu)$ (24) the resonances within a multiplet do overlap and form a solid chaotic layer along a resonance $\langle\omega\rangle = 2n\Omega$, the layer width being $\Delta\omega \approx m\Omega_d \approx \nu\langle\omega\rangle$. In the absence of multiplet overlap the diffusion can still go along resonance $\langle\omega\rangle = 2n\Omega$ (for a simple model of such a *modulational diffusion* see, for example, Lichtenberg & Lieberman 1983; Vivaldi 1984).

According to (10*a*), modulational diffusion proceeds in both θ_0 and r_0 while

$$\frac{\langle\omega(r_0)\rangle}{\sin^3\theta_0} \approx \text{const.} \tag{26}$$

Because the magnetic field goes down with r_0 any decrease in θ_0 leads to an increase in r_0. Notice that in the axisymmetric trap the radial motion is forbidden by the conservation of angular momentum.

In our case the non-axisymmetric perturbation is a high-frequency one that is the detuning from the nearest resonance $\delta\omega \sim \Omega \gg \Delta\omega \approx \nu\langle\omega\rangle \sim \nu\Omega/\epsilon$ is much larger than the multiplet width. It is also adiabatic as the inverse frequency ratio is small. Hence, the effect of this perturbation is exponentially small too, and the corresponding resonant adiabaticity parameter ξ_M can be roughly estimated as

$$\xi_M \sim \nu \exp\left(-2\pi\frac{|\delta\omega|}{\Delta\omega}\right) \sim \nu\, e^{-a\epsilon/\nu} \gtrsim \epsilon^2\, e^{-a/\nu}; \quad a \sim 1. \tag{27}$$

For a more accurate evaluation see Vivaldi (1984); Chirikov *et al.* (1985).

With initial condition inside any multiplet the diffusion occurs, and the particle lifetime in the trap is of the order

$$\tau \sim \tau_0\,\xi_M^{-2} \approx \frac{\tau_0}{\nu^2}\, e^{2a\epsilon/\nu} \lesssim \frac{\tau_0}{\epsilon^4}\, e^{2a/\epsilon},$$

$$\tau_0^{-1} \sim \Omega\xi^2 \sim \Omega\epsilon_B^{-2}\, e^{-2/\epsilon_B}, \tag{28}$$

where τ_0 is the lifetime at $K > 1$ or $\nu \gtrsim \epsilon$. If one takes account of the dependence $\tau_0(\epsilon_B)$ ($\epsilon_B \sim \epsilon$, $\theta_0 \sim 1$) the lifetime $\tau(\epsilon)$ decreases with $1/\epsilon$ down to the minimal

$$\tau_{\min} \sim \frac{\Omega}{\epsilon^2\nu^2} \exp\left(4\sqrt{\frac{a}{\nu}}\right) \sim \frac{\Omega}{\epsilon^6}\, e^{a'/\epsilon} \quad (a' \sim 1)$$

at maximal $\epsilon \sim \sqrt{\nu}$ (25). The minimal critical value now is still less $K_c(\nu) \sim \epsilon\sqrt{\nu} \gtrsim \epsilon^2 \sim \nu$ (cf. (24)). The last estimate is the condition for a multiplet formation (25).

A considerable decrease in the instability threshold illustrates the general rule: the lower the modulation frequency Ω_M of perturbation, the easier it is for chaos to occur even though it may be confined within a multiplet, for example. The reason is very simple: upon splitting in a multiplet, the width of each resonance $\Delta\omega_r \propto \Omega_M^{\frac{1}{4}}$, and the overlap parameter $\Delta\omega_r/\Omega_M \propto \Omega_M^{-\frac{3}{4}}$ grows indefinitely as $\Omega_M \to 0$.

(c) *A weak asymmetry, the Arnol'd diffusion*

Is there any loss of particles for $\nu \ll \epsilon^2$ ($K \ll 1$) or for $K \ll K_c(\nu)$? A fine mechanism of the weak instability in many-dimensional nonlinear oscillations has been discovered by Soviet mathematician Arnol'd (1964), Kolmogorov's disciple. As further investigations revealed (Gadiyak *et al.* 1977; Chirikov 1979; Lichtenberg & Lieberman 1983) Arnol'd's mechanism leads to a slow diffusion along narrow chaotic layers around the destroyed separatrices of nonlinear resonances. This process has been termed *Arnol'd diffusion*. Its most interesting and important feature is *universality*, as chaotic layers exist at any arbitrarily weak perturbation because the frequency of phase oscillations at a resonance $\Omega_r \to 0$ towards the separatrix.

Here again we come across a typical example of '*inverse adiabaticity*' for stationary oscillations driven by high-frequency perturbation. The estimate (27) still holds for the resonant adiabaticity parameter ξ_A too, where now $\Delta\omega = 4\Omega_r$ is the full width of a nonlinear resonance. However, the width can be arbitrarily small as well as ξ_A. Because (see (14) and (15), $\theta_0 \sim 1$)

$$\Omega_r \sim \Omega\sqrt{K} \sim (\Omega/\epsilon_B)\, e^{-1/2\epsilon_B} \tag{29}$$

then for $\nu \ll \epsilon^2$ (no multiplet)

$$\xi_A \sim \nu \exp(f\epsilon_B\, e^{1/2\epsilon_B}); \quad f \sim 1. \tag{30}$$

The first estimate of the destruction ('splitting') of separatrix by a high-frequency perturbation was due to Poincaré (1899). Recently this problem has been studied by many authors (see, for example, Lichtenberg & Lieberman 1983; Zaslavsky 1985), including an accurate evaluation of the full width of a separatrix chaotic layer (Chirikov 1979; Escande 1985).

From (30), a rough estimate for the particle lifetime τ, similar to (28), can be obtained in the form

$$\tau\Omega\nu^2 \sim \exp(2f\epsilon_B\, e^{1/2\epsilon_B}). \tag{31}$$

A distinctive feature of the Arnol'd diffusion in a magnetic trap is the tremendous drop of its rate (*ca.* τ^{-1}) with a slowness parameter ϵ_B of the adiabatic perturbation (a double exponential).

At very small ϵ_B the diffusion is driven by some high-order resonances whose detuning $\delta\omega \ll \Omega$ while their strength is only(!) exponentially small. As a fairly simple analysis shows (Chirikov 1978; for details see Chirikov 1979) the latter mechanism considerably reduces the lifetime

$$\tau\Omega\nu^2 \sim \exp(b\, e^{1/2q\epsilon_B}); \quad q = N, \tag{32}$$

where $N = 3$ is the number of degrees of freedom of the particles and $b \sim 1$ only weakly depends on the parameters.

Arnol'd diffusion has been carefully studied on some simple models (Gadiyak *et al.* 1977; Chirikov 1979; Lichtenberg & Lieberman 1983; Petrosky 1984; Kaneko *et al.* 1985) including the case of high-order driving resonances, or Nekhoroshev's régime (Chirikov *et al.* 1979). A rigorous upper limit for the diffusion rate has been derived by the Soviet mathematician Nekhoroshev (1977), Arnol'd's pupil. The latter estimate can be reduced to the form of (32) with one important difference, namely, for $N \gg 1$ the exponent $q \sim N^2$ rather than $q = N$ in (32). This interesting question remains open. At any rate, in the numerical experiments on a simple model, made by V. V. Vecheslavov as a continuation of the previous work (Chirikov *et al.* 1979), the dependence like (32) was observed down to the extremely weak perturbation which would correspond to $\epsilon_B \approx 0.07$ and $\xi \sim 10^{-5}$.

Arnol'd diffusion was apparently observed also in the laboratory experiments on long-term confinement of electrons in magnetic traps (Ponomarenko *et al.* 1968; Ilyin *et al.* 1977). In any event, the dependence $\tau(B)$ had the characteristic shape of two plateaux with a sharp jump in between. The upper plateau at $B > B_c$ (the critical magnetic field) is determined by the residual gas scattering of an electron on angle $\Delta\theta \sim 1$. The lower plateau is caused by the same scattering but in the nearest chaotic layer of a nonlinear resonance that is on a substantially smaller angle. The latter results in the decrease of electron lifetime by one order of magnitude approximately. Hence, some external noise (scattering) is of great importance for the Arnol'd diffusion because otherwise it would take place for very special initial conditions only, namely, within the chaotic layers whose total relative measure is negligibly small (of the order of (32)). The same is true for the modulational diffusion too, yet the relative layer width here is much bigger ($\sim m\Omega_d/\Omega \sim \nu/\epsilon \gtrsim \epsilon$). One may say also that such a diffusion greatly amplifies the effect of the scattering or of any other noise.

Notice that the critical field $B_c = \omega_c$ is nearly independent of the asymmetry ν (see (12)):

$$\frac{2}{9\pi}\frac{L\omega_c}{\sqrt{\lambda v}} \approx \ln\ln(\tau\Omega\nu^2) - \ln b. \tag{33}$$

Here τ is of the order of the gas scattering time.

Thus, in spite of the universal Arnol'd diffusion the adiabatic invariance proves to be fairly precise generally, and even absolute (eternal) for most initial conditions.

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References

Alekseev, V. M. & Yakobson, M. V. 1981 *Physics Rep.* **75**, 287–325.
Arnol'd, V. I. 1963 *Usp. mat. Nauk* **18** (6), 91–192.
Arnol'd, V. I. 1964 *Dokl. Akad. Nauk SSSR* **156**, 9–12.
Bora, D., John, P. I., Saxena, Y. C. & Varma, R. K. 1980 *Plasma Phys.* **22**, 653–662.
Budker, G. I. 1982 *Collected Works*, pp. 72–90. Moskva: Nauka.
Channon, S. R. & Lebowitz, J. L. 1980 *Ann. N.Y. Acad. Sci.* **357**, 108–129.
Chirikov, B. V. 1960 *Plasma Phys.* **1**, 253–260.

Chirikov, B. V. 1978 *Fiz. plasmy* **4**, 512–541.
Chirikov, B. V. 1979 *Physics Rep.* **52**, 263–379.
Chirikov, B. V., Ford, J. & Vivaldi, F. 1979 *A.I.P. Conf. Proc.* **57**, 323–340.
Chirikov, B. V. & Shepelyansky, D. L. 1981 In *Proc. 9th Int. Conf. on Nonlinear Oscillations* (ed. Yu. A. Mitropolsky), vol. 2, pp. 421–424. Kiev: Naukova Dumka.
Chirikov, B. V. 1984 In *Topics in plasma theory* (ed. B. B. Kadomtsev), pp. 3–73. Moskva: Energoatomizdat.
Chirikov, B. V. & Shepelyansky, D. L. 1974 *Physica* D **13**, 395–400.
Chirikov, B. V., Lieberman, M. A., Shepelyansky, D. L. & Vivaldi, F. M. 1985 *Physica* D **14**, 289–304.
Chirikov, B. V. 1986 Asymptotic Methods in Adiabatic Problems. Preprint no. 86–22. Novosibirsk: Institute of Nuclear Physics. (In Russian.)
Chirikov, B. V. & Shepelyansky, D. L. 1986 The Chaos Border and Statistical Anomalies. Preprint no. 86–174. Novosibirsk: Institute of Nuclear Physics. (In Russian.)
Dykhne, A. M. & Chaplik, A. V. 1961 *Zh. eksp. teor. Fiz.* **40**, 666–669.
Escande, D. F. 1985 *Physics Rep.* **121**, 165–261.
Filonenko, N. N., Sagdeev, R. Z. & Zaslavsky, G. M. 1967 *Nucl. Fusion* **7**, 253–266.
Gadiyak, G. V., Izrailev, F. M. & Chirikov, B. V. 1977 In *Proc. 7th Int. Conf. on Nonlinear Oscillations, Berlin, 1975*, vol. II–1, pp. 315–323. Berlin: Akademie-Verlag. (In Russian.)
Galton, F. 1889 *Natural inheritance.* London.
Gelfand, I. M., Graev, M. I., Zueva, N. M., Mikhailova, M. S. & Morozov, A. I. 1963 *Dokl. Akad. Nauk SSSR* **148**, 1286–1289.
Goward, F. K. 1953 In *Lectures on the Theory and Design of an Alternating Gradient Proton Synchrotron*, pp. 19–30. Geneva: CERN.
Greene, J. M. 1979 *J. math. Phys.* **20**, 1183–1201.
Hastie, R. J., Hobbs, G. D. & Taylor, J. B. 1968 In *Proc. 3rd Int. Conf. on Plasma Physics and Controlled Nuclear Fusion Research*, vol. 1, pp. 389–402. Vienna: IAEA.
Hine, M. G. N. 1953 In *Lectures on the Theory and Design of an Alternating Gradient Proton Synchrotron*, pp. 69–82. Geneva: CERN.
Ilyin, V. D. & Ilyina, A. N. 1977 *Zh. eksper. teor. Fiz.* **72**, 983–988.
Kadomtsev, B. B. & Pogutse, O. P. 1979 In *Proc. 7th Int. Conf. on Plasma Physics and Controlled Nuclear Physics Research*, vol. 1, pp. 649–682. Vienna: IAEA.
Kaneko, K. & Bagley, R. J. 1985 *Phys. Lett.* A **110**, 435–440.
Kapitsa, S. P. & Melekhin, V. N. 1969 *The microtron.* Moskva: Nauka.
Karney, C. F. F. 1983 *Physica* D **8**, 360–380.
Karney, C. F. F., Rechester, A. B. & White, R. B. 1982 *Physica* D **4**, 425–438.
Kolomensky, A. A. 1960 *Zh. tekh. Fiz.* **30**, 1347–1354.
Lichtenberg, A. J. & Lieberman, M. A. 1983 *Regular and stochastic motion.* New York: Springer-Verlag.
MacKay, R. S. & Percival, I. C. 1985 *Communs math. Phys.* **98**, 469–504.
Mandelstam, L. I. 1948 *Collected Works*, vol. 1, pp. 297–304. Moskva: Akad. Nauk SSSR.
Nekhoroshev, N. N. 1977 *Usp. mat. Nauk* **32** (6), 5–66.
Petrosky, T. Y. 1984 *Phys. Rev.* A **29**, 2078–2091.
Petrosky, T. Y. 1986 *Phys. Lett.* A **117**, 328–332.
Poincaré, H. 1899 *Les méthodes nouvelles de la mécanique céleste*, vol. 3, §401. Paris.
Ponomarenko, V. G., Trajnin, L. Ya., Yurchenko, V. I. & Yasnetsky, A. N. 1968 *Zh. eksp. teor. Fiz.* **55**, 3–13.
Rechester, A. B. & Rosenbluth, M. N. 1978 *Phys. Rev. Lett.* **40**, 38–41.
Rosenbluth, M. N., Sagdeev, R. Z., Taylor, J. B. & Zaslavsky, G. M. 1966 *Nucl. Fusion* **6**, 297–300.
Vaskov, V. V., Gurevich, A. V., Karfidov, D. M. & Sergeichev, K. F. 1984 *Zh. eksp. teor. Fiz.* (Pisma) **40**, 101–103.
Veksler, V. I. 1944 *Dokl. Akad. Nauk SSSR* **43**, 346–348.
Vivaldi, F. M. 1984 *Rev. mod. Phys.* **56**, 737–754.
Yamaguchi, Y. & Sakai, K. 1986 *Phys. Lett.* A **117**, 387–393.
Zaslavsky, G. M. 1985 *Chaos in Dynamic Systems.* New York: Harwood.