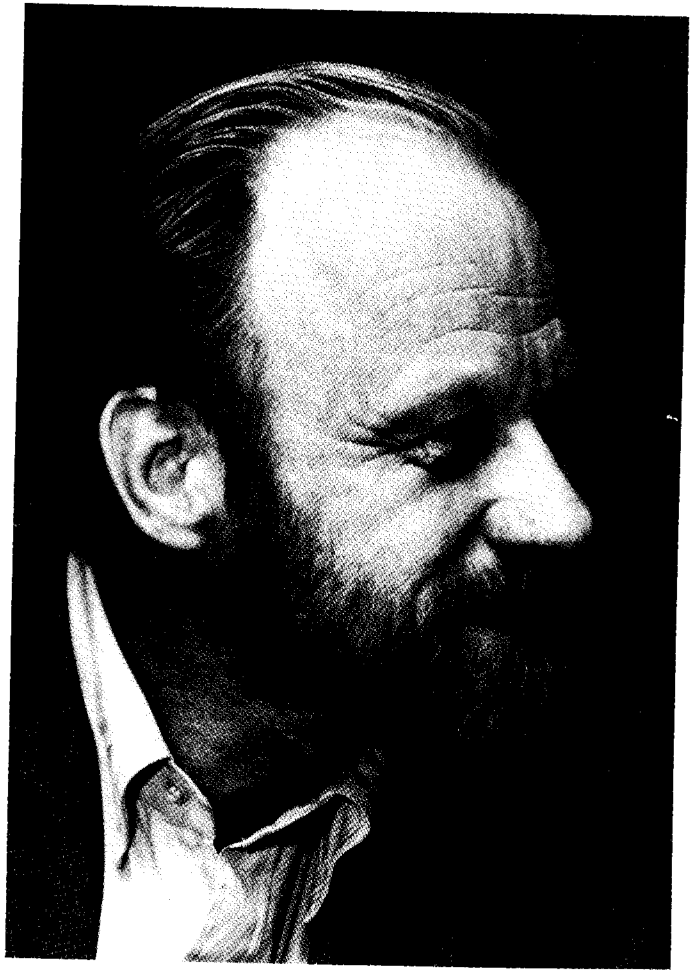
Analysis, et cetera

Edited by
Paul H. Rabinowitz
Eduard Zehnder



Analysis, et cetera

Research Papers Published in Honor of Jürgen Moser's 60th Birthday

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Dedication

This collection of papers is dedicated to Jürgen Moser on the occasion of his sixtieth birthday.

Jürgen Moser was born in Königsberg, Germany (now Kaliningrad) on July 4, 1928. There he attended the Wilhelms-Gymnasium, a school that included D. Hilbert among its former pupils. His childhood and youth were clouded by the Nazi regime and World War II. In 1947 he left the East Zone of Germany for Göttingen to study mathematics with F. Rellich. The return of C. L. Siegel to Göttingen in 1950 proved decisive in Jürgen's further mathematical development. Siegel's mathematical power, high standards and style impressed him deeply. In 1953 he visited New York University for the first time on a Fulbright scholarship. After a brief return to Göttingen as Siegel's assistant, in 1955 he moved to the United States. There began his long and fruitful affiliation with the Courant Institute interrupted only by a 3 year period from 1957-1960 at MIT. The stimulating atmosphere of the Courant Institute had a strong influence on him and on his growth as a mathematician. During this period, he did a great deal of important work and made many lasting friendships. In 1980 he left for the ETH Zürich, where he is now the director of the Forschungsinstitut für Mathematik.

Jürgen's personal qualities of generosity, modesty, and integrity leave lasting impressions on his friends and colleagues. His influential contributions to mathematics cover a wide range of topics and he possesses an exceptionally broad view of mathematics as a whole. This broad view is also reflected in the contributions to this book through which the authors express their appreciation, gratitude, and friendship.

KAM Integrability

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In 1892, almost a hundred years ago, Poincaré published his famous theorem [1] on the nonexistence of isolating analytical motion integrals (except energy) in a generic conservative Hamiltonian system (in an external static field, for an isolated system — except the integrals of Poincaré's group). This theorem, being formally correct, nevertheless caused a lot of confusion, at least among physicists who payed no attention to the importance of the term "analytical" (integrals). It seemed obvious — why should any singularities appear in a simple mechanical motion? In 1923, young Fermi even published a paper [2] where he ostensibly proved that the Poincaré theorem implied the ergodicity of motion (on an energy surface). However, as a physicist he has apparently never believed his own formal result, and by the end of his life he decided to check it via numerical simulation on one of the first computers [3]. The numerical experiment did not confirm the Fermi "theorem". The surprise of the authors was so great that they paid no attention to the clear signs of ergodicity in some runs (see, e.g Fig. 3 in Ref.[3]). Thus they had missed the phenomenon which has later been termed *dynamical chaos* (see Ref.[4]). Instead they discovered a remarkable stability of nonlinear oscillations which gave a strong impetus to the future development of powerful mathematical methods for "constructing" the whole families

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of completely integrable nonlinear equations (see, e.g., Ref.[5]) * The integrals in question are analytical, indeed, so all these completely integrable systems are exceptional in accordance with the Poincaré theorem.

The mystery of this theorem has been finally resolved in the fascinating KAM theory, one of whose creators was Professor Jürgen Moser. His decisive contribution to the theory was in the studies of nonanalytical perturbations, mappings including. He devoted many papers to the development of this theory as well as to its various applications (see, e.g., Ref.[7]).

According to the KAM theory, a sufficiently weak perturbation of a nonlinear system preserves the full set of its motion integrals for most initial conditions. The measure of the complementary set with unstable trajectories goes to zero with the perturbation, yet this set is everywhere dense. It consists of narrow chaotic layers along destroyed separatrices of the nonlinear resonances. A detailed description of such a structure is given, for instance, in Ref.[8].

Even though the problem is obviously improper, the theory guarantees, in the case of two degrees of freedom ($N = 2$), the eternal stability of motion in the sense of small variations of the unperturbed integrals over indefinite time intervals (including chaotic layers!). This is because chaotic layers are isolated in this case from each other by invariant tori while the instability of motion within a layer is sharply restricted by its negligible width.

The situation drastically changes in a many-dimensional system ($N > 2$) where chaotic layers form a single connected set, the everywhere dense network, or "web", comprising the whole energy surface. A chaotic trajectory within this set comes arbitrarily close to any point of the energy surface, yet it is not ergodic as it remains always on the set of a small measure! A priori, such an intricate structure of motion appears to be completely unlikely, at least for physicists. The KAM theory did help them considerably to develop intuition, and now the above generic picture seems to be quite natural and comprehensive in terms of nonlinear resonances and their interaction. The most important implication of this picture is a slow motion over the web [9] which proved to be chaotic and was termed the *Arnold diffusion* [8].

* In the development of those methods the integrability of another model — the Toda lattice — which had been discovered also in numerical experiments [6], played an essential role.

However, the problem remains essentially improper. One way to regularize it is to impose an external weak noise whose effect would be amplified by Arnold's diffusion for arbitrary initial conditions, the more so the weaker the noise [8]. Another way is in restriction of the motion time which converts the everywhere dense web into a finite-mesh grid.

In this paper we consider a different problem: what is the accuracy of approximate motion integrals in the KAM theory for arbitrary initial conditions? Following this approach we introduce a new concept of approximate integrability which we shall term *KAM integrability* [10].

Any motion integrals, even if only approximate, are of primary importance in physics. A classical example is the adiabatic invariants. It turns out that adiabaticity is closely related to KAM integrability [11]. As is well known by now (see also below) the chaotic layers are formed by a high-frequency perturbation, while the adiabatic invariance holds in case of a low-frequency one. Clearly, the two are different only in which of the interacting freedoms is treated as perturbor, and on which as perturbed. Hence, the KAM integrability may be called the *inverse adiabaticity*.

The variation of unperturbed motion integrals is proportional to the perturbation and generally is not very small for any initial conditions. However, such relatively big perturbations do not accumulate and can be calculated, theoretically, to a high accuracy. The accumulating variations, on the other hand, are caused by the diffusion only which is very slow and which does place indeed, the principal limit to the accuracy of KAM integrals. Another important characteristic of this accuracy is the width of chaotic layers to which the diffusion is confined without external noise. Both characteristics are interrelated: loosely speaking, the diffusion rate is proportional to the square of layer width (to the cube in presence of noise [12]). Precisely this dependence is going to be used below for evaluating the diffusion rate at a very weak perturbation. In this case the diffusion depends on high-order resonances in a very complicated way. Nevertheless, a fairly simple, though very rough , estimate for the diffusion rate was obtained in Ref.[12]. Our main objective below is the extension of this estimate onto a considerably weaker perturbation, and its comparison with the rigorous upper estimate in Ref.[13].

We confirm the exponential dependence on perturbation in the limit of sufficiently weak perturbations for both the diffusion rate and layer width. On the other hand, we have found some preliminary indications of the existence of a rather broad (in perturbation) domain with only a power-law decay for the diffusion rate. This interesting phenomenon requires further study.

In any event, the diffusion falloff is fairly sharp which proves once more the high precision and quality of the KAM integrals, and hence, their importance in physics.

1. Model

We make use of the same model as in Ref.[12] which is specified by the Hamiltonian

$$H(x_i, p_i) = \frac{p_1^2 + p_2^2}{2} + \frac{x_1^4 + x_2^4}{4} - \mu x_1 x_2 - \epsilon x_1 f(t). \qquad (1)$$

We introduce small dimensionless perturbation parameters

$$\tilde{\mu} = \frac{\mu}{a^2}; \qquad \tilde{\epsilon} = \frac{\epsilon f}{a^3}; \qquad \tilde{\nu} = \frac{\tilde{\epsilon}}{\tilde{\mu}} = \frac{\epsilon f}{\mu a}, \qquad (2)$$

where a is oscillation amplitude ($x_i \approx a_i \cos\theta_i$ related to the frequency $\dot{\theta}_i = \omega_i \approx \beta a$; $\beta = 0.8472\ldots$ (see Ref.[8]). The driving periodic force is chosen in the form

$$f(t) = \frac{\cos(\Omega t)}{1 - A_0 \cos(\Omega t)} \approx \sum_m \frac{2}{\sigma} e^{-\sigma m} \cos(m\Omega t). \qquad (3)$$

The latter expression holds for $\sigma = (1 - A_0^2)^{1/2} << 1$. If, moreover, $\Omega/\omega << 1$ the driving resonances $\omega = n\Omega$ form a dense net which increases slow diffusion.

The main coupling resonance $\omega_1 = \omega_2$ is taken as the guiding resonance along which the diffusion proceeds. At $\epsilon = 0$ the motion near resonance has a form of phase oscillation, i.e. the oscillation of the resonance phase $\psi = \Theta_1 - \Theta_2$ as well as of the amplitudes a_i. Approximately, it is described by the "pendulum" Hamiltonian [8]

$$H_1(\psi, P) \approx \frac{\beta^2}{a_0^2} P^2 - \frac{\mu a_0^2}{2} \cos\psi. \qquad (4)$$

Here P is momentum conjugate to ψ and $a_0 = \bar{a}_1 = \bar{a}_2 = const$. The stable equilibrium at $\psi = 0$ ($H_1 = \frac{-\mu a_0^2}{2}$) corresponds to the stable periodic trajectory at the resonance center. In its vicinity the small phase oscillation of frequency $\Omega_\mu = \beta\mu^{1/2}$ is harmonic. Smallness of this frequency as $\mu \to 0$ does determines the inverse adiabaticity of the driving perturbation with parameter ϵ.

The unstable equilibrium at $\psi = \pi$, ($H_1 = \frac{\mu a_0^2}{2}$) corresponds to the unstable periodic trajectory which is crossed by a separatrix surface (separatrix) — the boundary of a nonlinear resonance in the phase space. Any, arbitrarily weak, perturbation destroys ("splits up") the separatrix. In its vicinity a chaotic layer arises (Fig. 1) along which, i.e. in the direction perpendicular to the figure plane, the diffusion goes on if $\epsilon \neq 0$.

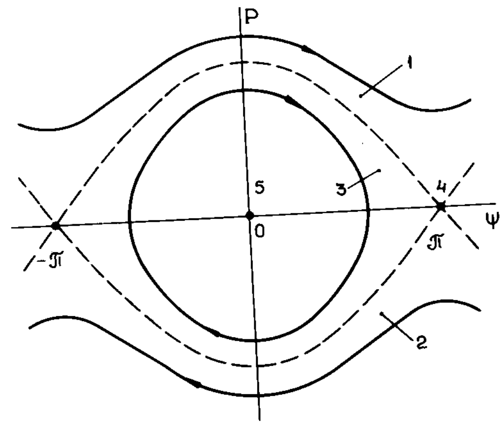

Fig. 1 Outline of a chaotic layer (see Eq.(4)): 1,2 are the domains of resonance phase ψ rotation in opposite senses; 3 same of ψ oscillation; 4,5 are unstable and stable periodic trajectories, respectively; arrows at layer edges indicate the direction of motion; unperturbed separatrix is shown by dashed curve.

We make use of a canonical mapping $(x_i, p_i) \longrightarrow (\bar{x}_i, \bar{p}_i)$ generated by the function

$$G(x_i, \bar{p}_i) = x_1\bar{p}_1 + x_2\bar{p}_2 + H(x_i, \bar{p}_i) \qquad (5)$$

as the numerical algorithm. Its accuracy grows with the number of steps per oscillation period $N_1 = \frac{2\pi}{\beta a}$, and it proves to be sufficiently high provided $N_1 \gtrsim 30$ ($a \lesssim 1/4$) [8,12].

2. Primary resonances

Arnold diffusion is only possible if there are at least three resonances — a guiding one, and two driving. With one driving resonance the diffusion goes at some angle to the layer, and thus is restricted by the same mechanism as the layer width.

If the perturbation is moderate, i.e. is not too weak, it suffices to take account of primary, or first-order resonances only. In model (1) these are the guiding resonance $\omega_1 = \omega_2$, and a couple of close driving resonances $\omega_1 = m\Omega$.

On the other hand, the perturbation should not be too strong to avoid global chaos due to resonance overlap. The driving resonances alone do overlap under condition [8]: $\epsilon \gtrsim a\Omega^2/2\pi\beta^2 f_m$, or

$$\lambda \equiv \frac{|\delta\omega|}{\Omega_\mu} \lesssim \pi\left(\frac{\tilde{\nu}_m}{2}\right)^{1/2} = \lambda_1.$$

Here λ is the *adiabaticity parameter* which plays the principal role in the problem of KAM integrability; $|\delta\omega| \approx \Omega/2$ is the maximal detuning with respect to the neighboring driving resonances; $f_m \approx 2\sigma^{-1}\exp(-\sigma m)$ are amplitudes of driving force (3), and $\tilde{\nu}_m = \epsilon f_m/\mu a$ (see Eq.(2)). At $\lambda \ll \lambda_1$ the diffusion rate (in energy) grows up to (see, e.g., Ref. [15]):

$$D_1 \equiv \frac{\overline{(\Delta H)^2}}{t} \approx \frac{\pi}{4}\cdot\frac{(\epsilon f_m a\omega)^2}{\Omega} \tag{6}$$

Taking account of the coupling resonance, the overlap border increases to

$$\lambda_2 \approx \pi\left[\left(\frac{\tilde{\nu}_m}{2}\right)^{1/2} + \frac{1}{2}\right] = \lambda_1 + \frac{\pi}{2}. \tag{7}$$

The diffusion is confined within a narrow chaotic layer of the coupling resonance only if $\lambda \gtrsim \lambda_2$. Introduce a dimensionless layer width $w = \left(\frac{2H_1}{\mu a_0^2}\right) - 1$. On the unperturbed separatrix $w = 0$, while in the resonance center $w = -2$. According to Ref. [18] the half-width of chaotic layer is

$$w_s \approx 4\pi\tilde{\nu}_m\lambda^2 e^{-\pi\lambda/2} \tag{8}$$

provided $\tilde{\nu}_m \lesssim 1$. In the middle between two driving resonances ($|\delta\omega| \approx \Omega/2$ all three domains of the chaotic layer (1, 2, 3, Fig. 1) have equal width. Otherwise, one of the external domains (1 or 2, ψ rotation) is much more narrow. To the contrary, the width of internal domain (3, ψ oscillation) would be twice as much. The explanation is as follows [8]. Changes in w by the driving perturbation occur around $\psi \approx 0$,that is, they follow with a period

$$T(w) \approx \frac{1}{\Omega_\mu}\ln\frac{32}{|w|}, \tag{9}$$

which is the period of ψ rotation and the half-period of ψ oscillation. In asymmetric case ($|\delta\omega| \ll \Omega/2$) the perturbation is operative only on one half-period of oscillation, and only for one direction of rotation. Hence, the perturbation period for oscillation doubles, and this increases the layer width [8].

That the chaotic layer is exponentially narrow (8) mainly depends on the adiabaticity parameter $\lambda \gg 1$. In turn, λ is only related to that part of the perturbation which determines the guiding resonance (parameter μ).

A simple equation

$$w_s \approx \lambda W \tag{10}$$

relates the layer width to the separatrix splitting $W = (\Delta w)_{max}$ which, in turn is equal to the maximal change in w over period T (9).

The splitting of the separatrix, and the formation of an intricate homoclinic structure was known already to Poincaré who even obtained the first exponential estimate for W [1] (Sections 226 and 397). Subsequently, this problem was studied in many papers (see, e.g., Refs. [7], [16], [17]). The concept of a chaotic layer near a separatrix was first developed in Ref.[18] and then thoroughly investigated in refs. [8],[19],[20]. The results of numerical simulations in the latter papers (especially in Ref. [19]) are well in agreement with a simple estimate like Eq.(8) provided λ is not too big.

The diffusion rate along the coupling resonance (in energy) is also exponentially small [8] (cf. Eq. (6)):

$$\mathrm{D_H} \approx \frac{\pi^2}{3}\frac{(\epsilon \mathrm{f} a\omega)^2}{\Omega_\mu^2 \mathrm{T_a}} \mathrm{e}^{-\pi\lambda}. \qquad (11)$$

Here T_a is the average period of motion in a chaotic layer (see Eq. (9)):

$$\Lambda = \Omega_\mu \mathrm{T}_a = \ln\frac{32\mathrm{e}}{w_s} = \Omega_\mu \mathrm{T}(w_s) + 1. \qquad (12)$$

Comparing Eqs.(11) and (8), we arrive at our basic relation:

$$\tilde{\mathrm{D}} \equiv \frac{\mathrm{D_H}\Omega_\mu}{(\epsilon \mathrm{f_m} a\omega)^2} \approx C\frac{\tilde{w}_s^2}{\Lambda\lambda^4}; \qquad \tilde{w}_s = \frac{w_s}{\tilde{\nu}_m}; \qquad C = \frac{1}{48} \qquad (13)$$

between the dimensionless diffusion rate $\tilde{\mathrm{D}}$ and the reduced layer width $\tilde{w}_s$. Notice that Eq. (13) actually holds around $\delta\omega \approx \Omega/2$ only. Otherwise, w_s depends on the closer driving resonance (i.e. on smaller $|\delta\omega|$) while $\mathrm{D_H}$ does so for larger $|\delta\omega|$.

3. High-order resonances

Arnold diffusion was observed first in numerical experiments [21], and then was studied in detail in Refs.[8,22]. As was noted already in Ref.[8] the diffusion rate considerably exceeded the simple estimate (11) when $\lambda \gtrsim 5$. Qualitatively, this was explained by the effect of high-order resonances, the higher the resonance, the larger λ becomes. Even though their amplitudes are very small they form a much more dense net as compared to primary driving resonances. This leads to a decrease in detuning: $\delta\omega \to \widetilde{\delta\omega} < \delta\omega$, and, hence, to a poor adiabaticity: $\lambda \to \tilde{\lambda} < \lambda$. Clearly, the motion structure in this region is extremely complicated, so that any analytical theory can, at best, provide a very rough order-of-magnitude estimate only. One was obtained in Ref.[8] (see also Ref. [23]), namely:

$$\tilde{\mathrm{D}} \sim \mathrm{D_0} \exp\left(-A\lambda^{1/M}\right). \qquad (14)$$

Here M is the number of linearly independent (incommensurable) unperturbed frequencies which form the high-order resonances, $\mathrm{D_0}$ and A assumed to be constant. Unlike Eq. (11) we now take $\lambda = \Omega/2\Omega_\mu$ its maximal value on primary resonances assuming $\widetilde{\delta\omega}$ to depend weakly on the original $\delta\omega$. According to Eq. (14) an effective adiabaticity parameter (cf. Eq. (11))

$$\tilde{\lambda} \approx \frac{\widetilde{\delta\omega}}{\Omega_\mu} \approx \frac{A}{\pi}\lambda^{1/M} \lesssim \lambda, \qquad (15)$$

the latter inequality being the condition for applicability of estimates (14,15).

The theory, developed in Ref. [8], shows that the basic relation (13) does hold for an arbitrary resonance set. However, the constant C generally depends on the system's parameters, and may change considerably. Eqs. (13-15) then imply

$$\tilde{w}_s \sim \left(\frac{\Lambda \mathrm{D_0}}{C}\right)^{1/2} \tilde{\lambda}^2 \mathrm{e}^{-\pi\lambda/2}. \qquad (16)$$

Earlier, in paper [13], a rigorous upper estimate was obtained which can be transformed to type (14) with the parameter [8]

$$M = M_N = \frac{(3N-1)N}{4} + 2 > n, \qquad (17)$$

where N is the number of freedoms for a conservative Hamiltonian system. Even for $N = 2$, parameter $M_N = 4.5$ considerably exceeds the value $M = 2$ found in Ref.[12]. This is in no contradiction with the upper bound (17), of course. Yet, a more effective estimate is desirable.

In ref. [12] the diffusion rate $\mathrm{D_H}$ was directly measured on a supercomputer CRAY-1. Nevertheless, it was only possible to reach $\lambda \approx 10$ due to a rapid decrease in the diffusion rate. In the present work a different technique is used, namely, we measure the width of a chaotic layer w_s, and then calculate $\mathrm{D_H}$ from Eq. (13). As a result we have reached $\lambda \approx 50$ on a personal computer! The flip side of the coin will be discussed in Section 6.

4. Numerical experiments

We employ the numerical algorithm, described in Section 1, and the following values of parameters (in the main series of experiments):

$$\epsilon/\mu = 0.01; \quad \Omega = 0.0346; \quad A_0 = 0.995\ (\mathrm{f_m} \approx 11.5);$$
$$a = 0.225; \quad \omega/\Omega \approx 5.5; \quad |\delta\omega| \approx \Omega/2; \quad \tilde{\nu}_m \approx 0.5.$$

Variables μ and λ run over a broad range:

$$200 \le \frac{1}{\sqrt{\mu}} \le 2500; \qquad 4 \le \lambda \le 50. \tag{18}$$

For $\lambda \gtrsim 20$, double precision (20 decimal places) is used. The initial conditions are chosen within the chaotic layer.

In the method employed, everything is calculated from the only measured quantity, the period $T(w)$ (9). To suppress big fluctuations in chaotic motion and, thus, to minimize errors in T, a special averaging of resonance phases ψ is applied [14].

Given $T(w)$ the quantity w is calculated from Eq. (9). The difference in successive w values is satisfactorily described by a simple relation [8]:

$$\Delta w \approx W \sin\varphi; \qquad W^2 \approx 2\overline{(\Delta w)^2}, \tag{19}$$

where φ is some high-order-resonance phase, random in a chaotic layer. The actual width of the chaotic layer w_m is calculated from the minimal period $T(w_m)$ which satisfactorily agrees with average period T_a (12): $\langle \Omega_\mu (T_a - T(w_m)) \rangle = 1.14$. To find the full width w_s a correction is introduced according to Eq. (4.49) in Ref.[8], namely:

$$\frac{w_s}{w_m} \approx 1 + \left(\frac{100}{n}\right)^{0.4} \approx 2 \tag{20}$$

Here $n = t/T_a$ is the mean number of periods over the total motion time t; typically, $n \approx 100$ is chosen. All quantities obtained are averaged over 10 trajectories to suppress big fluctuations which are characteristic for the chaotic motion with chaos border (at layer edges) [24].

The diffusion rate is calculated from Eq. (13) where $\tilde{\lambda} = w_s/W$ is substituted for λ. Parameter C , assumed to be constant, is found from the same Eq. (13) using the data of Ref. [12] in the range $\lambda \approx 3.7 \div 8$. Averaging over 11 points provides

$$\langle \log C \rangle = 0.55 \pm 0.27; \qquad C \approx 3.6, \tag{21}$$

where the dispersion of decimal logarithm values is given. No systematic variation of C with λ is observed. Notice that the C value

exceeds that in Eq. (13) by a quite big factor of about 200, the reason for which is not completely clear (see below).

The results of the main series of measurements with $a = 0.225$ ($\delta\omega = 0.0156$) are shown in Fig. 2. The straight line represents the dependence (14) with parameters $D_0 \approx 2.0$, and $A \approx 5.60$ obtained by the least square fit of numerical data. According to Ref. [12], $D_0 \approx 26$ and $A = 7.9$ which gives an idea as to the accuracy of estimate (14). However, the scattering of points in Fig. 2 appears to be surprisingly small. This may be related to the fact that only one parameter μ is varying.

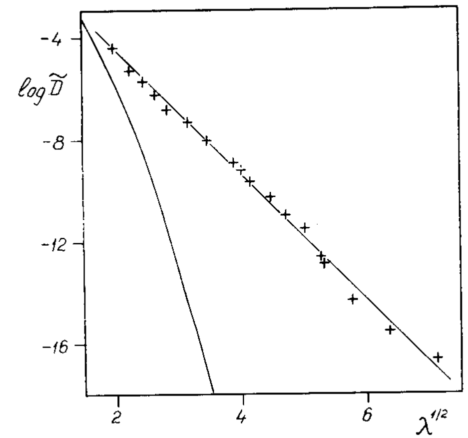

Fig. 2 Main results of numerical experiments (+) for model (1): $\lambda = \Omega/2\Omega\mu$; straight line is estimate (14) with $M = 2$; curve shows the effect of primary resonances (11); logarithm is decimal.

In Fig. 3 the dependence of the diffusion rate on another parameter – reduced detuning $\delta = 1 - (2|\delta\omega|/\Omega)$ – is shown. The value $\delta = 0$ corresponds to an average between the two driving resonances, while $\delta = 1$ falls just on one of them. The horizontal line represents Eq. (14) with the parameters fitted in Fig. 2.

In Fig. 3 the data for two values of $1/\sqrt{\mu}$: 600 and 300 are shown. In the first case, besides the main force (3), two other types of driving perturbation are represented:

i) a two-frequency force with $\Omega_1 = 5\Omega$ and $\Omega_2 = 6\Omega$; $\Omega_2/\Omega_1 = 6/5$, and with the same amplitudes as in Eq. (3);

ii) the force with two independent frequencies $\Omega_2/\Omega_1 = 1.2381966\ldots$

All three versions are in a good agreement. They reveal a significant dependence of diffusion rate on detuning $\delta\omega$. This is most striking in a narrow interval $\delta \lesssim 10^{-3}$ which covers a low-order resonance $2\omega = \Omega_1 + \Omega_2$. A similar drop in $\widetilde{\mathrm{D}}$ also occurs at $\delta \approx 1$. The rest of dependence shows the accuracy of estimate (14) with a constant A.

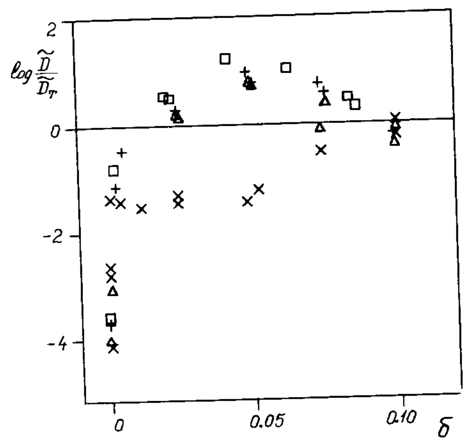

Fig. 3 diffusion rate $\widetilde{D}$ (13) vs. reduced detuning $\delta = 1 - 2|\delta\omega|/\Omega$, $\widetilde{D}_T$ is estimate (14) with $\delta\omega = \Omega/2$. $1/\sqrt{\mu} = 300$: (x) force (3). $1/\sqrt{\mu} = 600$: (+) force (3); ($\triangle$) $\Omega_2/\Omega_1 = 6/5$; ($\square$) independent Ω_1, Ω_2.

The low diffusion rate at $\delta \approx 0$ also puts the upper bound for a possible background which turns out to be about 4 orders of magnitude below the diffusion, independent of λ. We mention that for $\epsilon = 0$ the fictitious diffusion rate calculated from Eq. (13) drops by 11 orders of magnitude. Notice that a finite width of chaotic layer in this case is real due to the interaction with other coupling resonances $m_1\omega_1 = m_2\omega_2$, $m_1 \neq m_2$. On the other hand, the "real" additional diffusion caused by the discreteness of the numerical algorithm (5) is completely negligible. According to estimate (14) its $\widetilde{\mathrm{D}} \sim 10^{-70}$ (!) due to a very big value of adiabaticity parameter $\lambda = \pi/\beta\sqrt{\mu} \approx 800$ at $1/\sqrt{\mu} = 500$.

In Fig. 4 the dependence of the effective adiabaticity parameter $\tilde{\lambda} = w_s/W = w_s \left(2\overline{(\delta w)^2}\right)^{-1/2}$ on $\lambda = \Omega/2\Omega_\mu$ is depicted. The asymptotic relation (15) – the horizontal line– is reached, within fluctuations, at $\lambda \gtrsim 15$. Notice that in the whole range $\lambda \lesssim 10$, studied in Ref. [12], the relation (15) is satisfied poorly. This may be a reason for the quite big C value obtained from the data in Ref. [12].

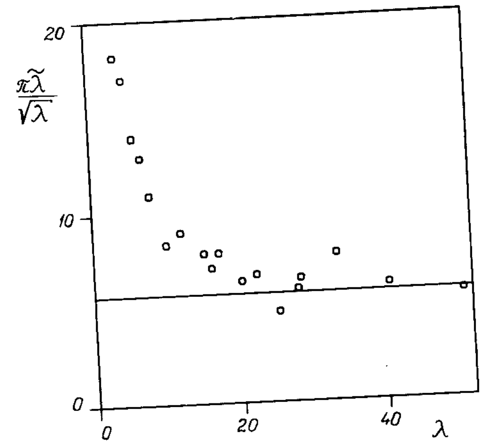

Fig. 4 Effective adiabaticity parameter $\tilde{\lambda}$ vs. λ for the main data (Fig. 2). Horizontal line is Eq. (15) with $M = 2$ and fitted $A = 5.60$ from Fig. 2.

Our results confirm the value $M = 2$ for the main parameter estimate (14). It was obtained in Ref. [12] and explained there by the presence of two independent frequencies, Ω and ω (in coupling resonance $\omega_1 = \omega_2 = \omega$). This explanation is further confirmed by a sharp decrease in the diffusion rate at $\delta = 0$ (Fig. 3) when the two frequencies become commensurable ($\omega/\Omega = 11/2$) and $M = 1$.

Thus, the theoretical value of M seems to be confirmed. If so, one would expect a considerable increase in $\widetilde{\mathrm{D}}$ for two independent frequencies Ω_1, Ω_2 of the driving perturbation as $M = 3$ in this case. However, this is not observed (Fig. 3,$\square$). In the next Section we attempt to resolve this contradiction.

5. A weak adiabaticity?

We assume that the parameter A in estimate (14), surmised to be constant, is actually growing with M. Suppose that it grows linearly, i.e. $A = BM$ where B is constant. Then, at $\lambda = 12$ (Fig. 3, □) the diffusion rate calculated for $M = 2$ and $M = 3$, is nearly the same. Moreover, the data in Fig. 2 imply the value $B = 2.80$ which is close to $B = \pi$ for $M = 1$ (see Eq. (11)).

In the case of the modified estimate $\widetilde{D}(\lambda) \sim D_0 \exp(-BM\lambda^{1/M})$ the curves $\widetilde{D}(\lambda)$ for different M do intersect. This implies that the diffusion rate for some $\widetilde{M} < M$ may happen to be bigger than for the actual M. Such an enhanced diffusion can be caused by the driving resonances formed by a smaller number $(\widetilde{M})$ of unperturbed frequencies. For example, at sufficiently small λ, the diffusion is always driven by primary resonances (11), and hence $\widetilde{M} = 1$ for any M. Therefore, one should find such $\widetilde{M}(\lambda) \leq M$, for each λ, which provides the highest diffusion rate. In this way we arrive at the dependence $\widetilde{D}(\lambda)/D_0$ in the form of successive functions $\exp(-B\widetilde{M}\lambda^{1/\widetilde{M}})$ with different $\widetilde{M} \leq M$. Since our estimates are fairly rough we may smooth over that broken curve. To this end we consider the dependence $\widetilde{M}(\lambda)$ to be continuous, and derive it from the local condition $\partial\widetilde{D}(\lambda, \widetilde{M})/\partial\widetilde{M} = 0$, whence $\widetilde{M} = \ln\lambda$. Substituting the latter relation into $\widetilde{D}(\lambda, \widetilde{M})$ we arrive at a fairly simple estimate

$$\widetilde{D} \sim D_0\lambda^{-Be}, \tag{22}$$

where $e = 2.71\ldots$. That slow decay of the diffusion rate – *the weak adiabaticity* – persists, however, while $\widetilde{M} < M$ only, i.e. for $\lambda < \lambda_M = e^M$. Subsequently, the exponential dependence is recovered.

Thus, if our hypothesis is true, the final estimate for the diffusion rate becomes

$$\widetilde{D} \sim \begin{cases} D_0\lambda^{-Be}; & \lambda \lesssim e^M \\ D_0 \exp(-BM\lambda^{1/M}); & \lambda \gtrsim e^M \end{cases} \tag{23}$$

Notice that both curves $\tilde{D}(\lambda)$ are tangent to each other at $\lambda = e^M$.

In Fig. 5 the data of Fig. 2 (+) are represented in a log-log scale. The straight line is the power law (23) with fitted parameters $D_0 \approx$

1.6; $B \approx 2.84$. The latter value is very close to $B \approx 2.80$ previously obtained. The curves in Fig. 5 show exponential dependence (23) for $M = 1; 2; 3;$. Squares correspond to minimal detunings $\delta < 10^{-3}$ (see Fig. 3) with expected $M = 1$ (Section 4).

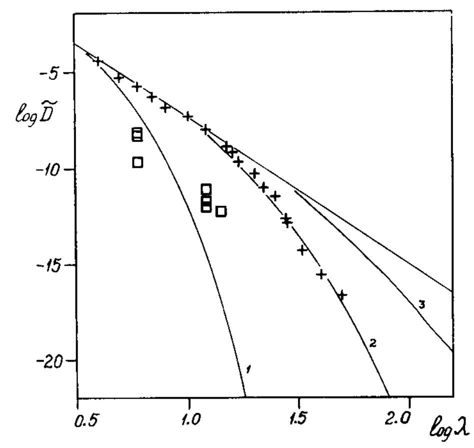

Fig. 5 Conjectured weak adiabaticity (23): straight line is power law; curves correspond to exponentials with $M = 1, 2, 3$ as indicated; (+) data form Fig. 2 ($M = 2$); (□) $\delta = 0$ (Fig. 3, $M = 1$).

We also measured the diffusion rate for two independent frequencies Ω_1, Ω_2 (expected $M = 3$) and $\lambda = |\Omega_1 - \Omega_2|/2\Omega_\mu = 35.7$ $(1/\sqrt{\mu} = 1500)$. At this $\lambda > e^3 \approx 20$, estimates (23) with $M = 2$ and $M = 3$ differ by 3 orders of magnitude. Indeed, the measured values of $-\log\tilde{D}$ all lie in the interval 13.8+9.6 (at average, $-\langle\log\tilde{D}\rangle = 11.9$) while Eq. (23) gives 14.3 ($M = 2$) or 11.8 ($M = 3$). A considerable dispersion in $\tilde{D}$ is apparently related to the dependence on detuning $\delta = 0.00084 + 0.085$ (cf. Fig. 3). At $\delta = 0$ the diffusion rate drops down to $-\log\tilde{D} = 16.8$ which is comparable to the estimate for $M = 2$.

We understand, of course, that the consideration and data given above are only preliminary indications toward the existence of a domain of weak adiabaticaity (22). Nor does the hypothetical relation $A = BM$ follow directly from a simple theory in Ref. [8] . Instead, it requires a more accurate evaluation of the density of high-order resonances. To summarize, this interesting question remains as yet open.

6. Concluding remarks

The main result of our studies is the confirmation of the exponential estimate (14) in a broad range of values of the adiabaticity parameter λ (Fig. 2). In our opinion, the most important problem to be solved would be the relation between the simple empirical estimate (14) and the rigorous estimate from above in Ref. [13] (see Eq. (17)). Why do they differ? Could it be related to the fact that the fairly large values of λ studied in this paper are still not big enough? Generally, we may also ask if the domain of applicability of estimate (14) is restricted in λ from above?

On the other hand, we certainly know that this domain is bounded from below, and not only by the resonance overlap (Section 2). Already in paper [12], the deviation from dependence (14) at $\lambda \lesssim 4$ was observed. This could be roughly explained by a change in M value from 2 ($\lambda \gtrsim 4$) to 1 ($\lambda \lesssim 4$). In this paper we put forward the hypothesis which extends such a behavior to arbitrary M (Section 5). Surprisingly, this leads to the conclusion of the existence of the domain of "weak adiabaticity" where the diffusion rate falls off as some power of the adiabaticity parameter only (23). The domain width in λ rapidly grows with M, and it certainly deserves further thorough studies.

Penetration into the region of large λ in this work proved to be possible due to a new method for evaluation of the diffusion rate via the width of the chaotic layer. How powerful it might seem, the method has its own limitations. Particularly, it fails (in the present form at least) as soon as a modulational chaotic layer appears whose width is irrelevant to the diffusion rate [14,25]. More precisely, the applicability of the new method is restricted by the condition $\lambda > \lambda_{\rm mod}$ where $\lambda_{\rm mod}$ is some critical value at which a modulational layer is formed. Another disadvantage of the method is in its poor accuracy due to unknown factor C which may change considerably (cf. Eqs. (21) and (13)).

Even in the domain of weak adiabaticity the long-term variation of motion integrals rapidly drops with perturbation which confirms a high accuracy of the KAM integrals, and emphasizes again their importance in physics. At the same time the estimates obtained may

happen to be helpful in those special applications where the diffusion, no matter how slow, turns out to be significant as, for example, in the dynamics of an asteroid or a heavy particle in the storage ring (see. e.g., Ref[26]).

We are happy to acknowledge Professor Moser's great contribution to the international collaboration which is so vital in this, as well as in others, fields of research.

References

1. H. Poincaré. *Les méthodes nouvelles de la mécanique céleste.* Paris, 1892, Sections 81-83.
2. E. Fermi. Phys. Z. **24** (1923) 261.
3. E. Fermi, J. Pasta and S. Ulam. *Studies of nonlinear problems I.* Report LA-1940, Los Alamos, 1955.
   Collected Works of E. Fermi. Univ. of Chicago Press, 1965, Vol.2, p. 978.
4. F. M. Izrailev and B. V. Chirikov. Dokl. Akad. Nauk SSSR **166** (1966) 57.
5. N. Zabusky. J. Comp. Phys. **43** (1981) 196.
6. J. Ford, S. D. Stoddard and J. S. Turner. Prog. Theor. Phys. **50** (1973) 1547.
7. J. Moser. *Stable and random motions in dynamical systems with special emphasis on celestial mechanics.* Princeton Univ. Press, 1973.
8. B. V. Chirikov. Phys. Reports **52** (1979) 263.
9. V. I. Arnold. Dokl. Akad. Nauk SSSR **156** (1964) 9.
   Stability problem and ergodic properties of classical dynamical systems. Proc. Int. Congress of Mathematicaians, Moscow, (1966), p. 387 (in Russian).
10. B. V. Chirikov. *The nature and properties of the dynamical chaos.* Preprint INP 82-152, Novosibirsk, 1982.
11. B. V. Chirikov. Proc. R. Soc. Lond. A **413** (1987) 145.
12. B. V. Chirikov, J. Ford and F. Vivaldi. *Some numerical studies of Arnold diffusion in a simple model.* A.I.P. Conf. Proc., 1979, N 57, p.323.
13. N. N. Nekhoroshev. Usp. Mat. Nauk **32**. 6 (1977) 5.
14. F. Vivaldi et al. *Modulational diffusion in nonlinear oscillator systems.* Proc. 9th Int. Conf. on nonlinear oscillations, Kiev,

1981, Vol. 2, p. 80.
15. B. V. Chirikov, D. L. Shepelyansky. Zh. Tekh. Fiz. **52** (1982) 238.
16. V. K. Melnikov. Dokl. Akad. Nauk SSSR **144** (1962) 747.
17. V. F. Lazutkin. *Separatrix splitting in the standard family of area-preserving mappings.* Topics in Mathematical Physics. Leningrad, 1986, N 12 (in Russian).
18. N. N. Filonenko, R. Z. Sagdeev and G. M. Zaslavsky. Nuclear Fusion 7 (1967) 253.
G. M. Zaslavsky, N. N. Filonenko. Zh. Eksper. Teor. Fiz. **54** (1968) 1590.
19. B. V. Chirikov. Lecture Notes in Physics **179** (1982) 29.
20. D. F. Escande. Phys. Reports **121** (1985) 165.
21. G. V. Gadiyak, F. M. Izrailev, B. V. Chirikov. *Numerical experiments with universal instability in nonlinear oscillator systems (Arnolds's diffusion.* Proc. 7th Int. Conf. on nonlinear oscillations, Berlin, 1975, Vol.II-1, p. 315.
22. J. L. Tennyson, M. A. Lieberman and A. J. Lichtenberg. *Diffusion in near-integrable Hamiltonian systems with three degrees of freedom.* A.I.P. Conf. Proc., 1979, N 57, p. 272.
23. B. V. Chirikov. Fiz. Plasmy **4** (1978) 521.
24. B. V. Chirikov and D. L. Shepelyansky. Physica D **13** (1984) 395.
25. B. V. Chirikov, M. A. Lieberman, D. L. Shepelyansky and F. M. Vivaldi. Physica D **14** (1985) 289.
26. M. Month and J. C. Herrera. Eds. *Nonlinear dynamics and the beam-beam interaction.* A.I.P. Conf. Proc., 1979, N 57.

Anderson Localization and KAM-Theory

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It is our great pleasure to dedicate this paper to Jürgen Moser. One of us (Ya. S.) had warm personal contacts with Jürgen for more than 25 years. We wish Jü. Moser many remarkable results, great successes in music, good health, etc.

1. Introduction.

Four years ago, one of us (Ya. S.) had a long discussion with J. Fröhlich concerning his paper with T. Spencer [**1**] on Anderson localization for discrete Schrödinger operators with random potentials. Jurg argued that their procedure resembled KAM-theory in many respects. There seemed to be two problems with this. The first was connected with the absence of subsequent changes of variables, which was certainly a formality. The second one concerned an analogy to the