A theory of quantum diffusion localization

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The quantum localization of chaotically diffusive classical motion is reviewed, using the kicked rotator as a simple but instructive example. The specific quantum steady state, which results from statistical relaxation in the discrete spectrum, is described in some detail. A new phenomenological theory of quantum dynamical relaxation is presented and compared with the previously existing theory.

I. INTRODUCTION

In this article we discuss a new development in the theory of quantum diffusion localization. This surprising phenomenon, which was initially discovered in numerical experiments[1] and has been further studied from many different perspectives (see, e.g., reviews in Refs. 2 and 3 and references therein), is interesting in many respects, two of which we mention here. First, it extends the very unusual classical phenomenon of "deterministic chaos" into the more fundamental realm of quantum mechanics. Second, "localization" sets a quantum limit on classical chaos and thus shows, much to our (initial) surprise, that quantum mechanics is far more regular and less chaotic than classical mechanics. However, it should be stressed that this latter conclusion is essentially related to a new philosophy in the study of quantum dynamics, adopted (sometimes explicitly but more commonly implicitly) by many researchers in this field; namely, that the whole of quantum dynamics is divided into two (unequal!) parts: (i) the proper quantum dynamics as described by the time evolution of the wave function $\psi(t)$, and (ii) the quantum measurement process, which is (as yet) the only link between the micro- and the macroworld, and the only way for us, macroscopic creatures, to study the mysterious quantum world.

Of course, the second part is much more intricate than the first, as there is so far no dynamical description of the quantum measurement process, only the "Copenhagen convention," which provides the macroscopic interpretation for ψ. That such an interpretation is not the whole story is suggested by a suspicion that the "collapse," which is the central core of the Copenhagen interpretation of quantum measurement, may apparently occur without any explicit measurement by an "observer." The most spectacular (albeit perhaps imaginary) example may be the birth of the Universe in quantum cosmology.[4] But there are less extreme examples. A much more mundane but important possibility is that every "event" around us, as we understand it, is a result of some continuously implicit ("hidden") quantum measurement. Thus we badly need a (dynamical) theory of quantum measurement to resolve these issues. Indeed, recently a revival of interest in that sort of theory is apparent (see, e.g., Ref. 5). But it is a hard nut! Even some modification of the basic equations in quantum mechanics seems to be required. If so, essentially new physics will appear.

In contrast, the first part of the problem of quantum dynamics, as described above, is relatively "simple" and essentially mathematical, since well-established quantum evolution equations—e.g., the Schrödinger equation—are available. We need "only" to solve them and to analyze the solutions. In what follows, we concentrate on this simpler part of the problem, as do most of the researchers in the field. The physical aspect of this part of the problem lies particularly in developing new qualitative concepts in quantum dynamics, among which "quantum diffusion" and "localization" are especially important. Let me just mention that a more general concept of "quantum chaos" seems also to play a crucial role in the second part of the problem, the quantum measurement process.

In our problem, quantum chaos represents a very peculiar quasiclassical transition; that is, an asymptotic relation between classical and quantum mechanics. Generally, this transition is highly intricate and singular, as was stressed especially by Berry (see, e.g., Ref. 6). For a physicist, the fundamental guide here is the correspondence principle, which asserts that classical physics is not a separate theory but rather is a limit of quantum physics.

This article is organized as follows. In Sec. II a simple model—the "kicked rotator"—which we will use for the presentation of quantum chaos and its localization is described. A brief review of its classical and quantum properties is given. A more detailed discussion of the peculiar quantum steady state, which results from diffusion localization, is presented in Sec. III. The microstructure of chaotic eigenstates—the so-called "scars"—due to the instability localization is briefly reviewed in Sec. IV. The existing theory of diffusion suppression is outlined in Sec. V, with special emphasis on a number of unresolved questions. The central part of this article is Sec. VI, in which we attempt to solve some of these open questions.

II. A SIMPLE QUANTUM MODEL

To begin, let us consider a simple classical model—the kicked rotator—which is described in classical mechanics by the "standard map" (see, e.g., Ref. 3)

$$\bar{n}=n+k\sin\theta;\ \bar{\theta}=\theta+T\bar{n}. \tag{1}$$

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Here, $\{n,\theta\}$ are the action-angle variables, and k,T—the strength and period of perturbation—are the parameters of the model.

This simple model, which seems almost trivial at first glance, has turned out to be very rich; this has made it very popular in studies of both classical and quantum dynamics and chaos. In addition, this model describes approximately a number of real physical problems. For definiteness, let us consider one specific example; namely, the photoelectric effect in Rydberg (highly excited) hydrogen-like atoms.[7] In the classical approximation, this effect can be described by the following map over a Kepler period of the electron around the nucleus[8] (see also Ref. 3):

$$\bar{E}=E+k\sin\theta;\quad \bar{\theta}=\theta+2\pi\omega(-2\bar{E})^{-3/2}. \tag{2}$$

Here, $E<0$ is the electron's energy; ω and θ are the electric field frequency and phase when the electron is at the perihelion; and

$$k\approx 2.6(\epsilon/\omega^{2/3}) \tag{3}$$

is the perturbation of the field strength ϵ in atomic units ($e=m=\hbar=1$). The map (2) is readily reduced to the standard map (1) by linearizing the second equation in (2) in E around some E_0; this leads to the parameter T being given by

$$T=6\pi\omega(-2E)^{-5/2}. \tag{4}$$

Thus the standard map of type (1) describes the motion of more complicated models locally (for E near E_0) in action n.

For a sufficiently strong perturbation ($K\equiv kT>K_c\approx 1$) an unbounded (in n) chaotic component arises in which the motion is diffusive with the constant rate:

$$D\equiv\langle(\Delta n)^2\rangle/\tau=(k^2/2)C(K). \tag{5}$$

Here, τ is the number of map's iterations, and the function $C(K)$ describes the dynamical correlation of phase θ. In particular, $C(K)\to 0$ if $K\to K_c$ from above, and $C(K)\to 1$ for $K\to\infty$. One finds that $C(K)$ can be approximated by

$$C(K)=\begin{cases}1-2J_2(|K|)+2J_2^2(|K|); & |K|\geqslant 4.5,\\ 0.6(|K|-K_c)^3/K^2; & |K|\leqslant 4.5,\end{cases} \tag{6}$$

except for narrow windows near $|K|=2\pi m$, with integer $m>0$, where the so-called "superfast diffusion" occurs with $\langle(\Delta n)^2\rangle\sim\tau^{3/2}$.

The coarse-grained distribution function $f(n,\tau)$ (averaged over θ) obeys the diffusion equation

$$\frac{\partial f}{\partial t}=\frac{D}{2}\frac{\partial^2 f}{\partial n^2}, \tag{7}$$

which has as a Green's function an expanding Gaussian distribution

$$G(n,\tau)=(2\pi D\tau)^{-1/2}*\exp(-n^2/2D\tau). \tag{8}$$

The diffusion is unbounded, and hence, no steady state is formed.

The particular initial condition, $G(n,0)=\delta(n)$, used in Eq. (8) (and below) is not a restriction for the following reasons. First, in a map, diffusion makes sense for integer $\tau\geqslant 1$ only, and, as $K\geqslant 1$, for $|\Delta n|\geqslant 1$. Second, the map (1) is periodic not only in θ but also in n. Hence, to the above accuracy, the diffusion is homogeneous in n.

It is interesting to mention that even in a continuous dynamical system the diffusion is bounded from below by the time of correlation decay. In the simplest case (but not always), this time is of the order of the inverse Lyapunov exponent. The latter characterizes the rate of the local instability of motion, which is the mechanism for dynamical chaos. In this initial stage of motion, the distribution function spreads out exponentially in time, rather than as a power law as in the later stages of diffusion. This completes our discussion of the classical kicked rotator.

What happens in quantum mechanics, which is believed to be a more fundamental theory including the classical mechanics as the limiting case? To put it another way: does classical chaos survive quantization? This was the motivation of our first paper on quantum chaos.[1] Specifically, we studied the model of quantized kicked rotator described by the quantized standard map:

$$\bar{\psi}=\hat{R}\hat{F}\psi,\qquad \hat{F}=\exp(-ik\cos\theta);\quad \hat{R}=\exp(-iTn^2/2), \tag{9}$$

where the operator $\hat{F}$ describing the kick is given in the coordinate representation while the operator $\hat{R}$ describing the rotation (of quantum phases) is in momentum representation, and $\hbar=1$.

The equations as well as the whole picture of motion are completely different in classical and quantum mechanics [cf. Eqs. (1) and (9)]. It is not at all clear how the evolution in both cases could be similar, as required by the correspondence principle for the quasiclassical region. This region is determined for the model in question by the conditions: $k\to\infty$, $T\to 0$, and the classical parameter $K=kT=\text{const}$. Notice that the quantum number n is not a quasiclassical parameter in the kicked rotator because of the periodicity of this model (see above) in n. Despite the increasing spacings between the unperturbed energy levels ($E_{n+1}-E_n=n+1/2$), the perturbation (a "kick") couples about $2k$ levels, independent of n [see Eq. (9)].

The first numerical experiments showed that, indeed, diffusion persists in the quantum model, and its rate is close to the classical one.[1] Further studies confirmed that the quantum diffusive relaxation follows the classical process in many details.[9] Not only does the oscillation of the diffusion rate due to the classical dynamical correlation (6) persist in the quantum model, but also a superfast diffusion at $K\approx 2\pi m$ ($m>0$ integer) is also observed.

However, the quantum diffusion was found to be dynamically stable,[10] in contrast with the strong, exponential instability of motion in the classical limit. This quantum peculiarity is closely related to a slow correlation decay to be discussed below. One can also say that the quantum chaos is a "second-rate chaos," and we term it "pseudochaos."

The most important quantum limitation of chaos is the decrease of the diffusion rate in time, discovered already in the first numerical experiments[1] and subsequently confirmed by many researchers (see, e.g., Ref. 11). Within statistical fluctuations, the diffusion completely stops in some time τ_R, and a very specific quantum steady state is generally formed that has no counterpart in the classical limit. This process is called "quantum localization of classical diffusion," or just "quantum diffusion localization."

III. DIFFUSION LOCALIZATION AND THE QUANTUM STEADY STATE

The quantum steady state is characterized by the following main quantities: (i) the size l_s, or the localization length; (ii) the mean energy, $E_s=\langle n^2\rangle/2\sim l_s^2$; (iii) the fluctuations in energy, $\epsilon=\langle(\Delta E)^2\rangle^{1/2}/E_s$; and (iv) the relaxation (diffusion) time scale τ_R, or the time of formation of quantum steady state.

The theory of quantum chaos[2,3,12] provides the following estimates for all these quantities:

$$l_s\sim\tau_R\sim D\sim\rho;\quad E_s\sim D^2;\quad \epsilon\sim D^{-1/2}. \tag{10}$$

Here, ρ is the mean level density for the "operative" quasienergy eigenfunctions. The new concept of operative eigenfunctions is very important in the whole theory and means those that are actually present in the initial quantum state $\psi(n,0)$. The density ρ is finite just because of the localization of all the eigenfunctions, which is inevitably related to the diffusion localization.

The physical meaning of the first set of estimates in Eq. (10) is very simple, and it is related to the uncertainty principle; namely, the system does not resolve a discrete spectrum unless the energy uncertainty $\sim 1/t\leqslant 1/\rho$ becomes less than the mean level spacing. The estimate for E_s is also straightforward.

Now, a finite fluctuation ϵ is the principal difference between the quantum steady state and the classical equilibrium distribution. It is also related to the discrete spectrum of a finite density ρ and, hence, a finite number of the operative eigenfunctions $N\sim l_s$. In a chaotic motion all N quantum phases are decoupled statistically which results in fluctuation $\epsilon\sim N^{-1/2}$. This implies that the quantum distribution function

$$f_q(n,\tau)=|\psi(n,\tau)|^2 \tag{10a}$$

corresponds roughly to a finite classical ensemble of $N\sim D$ systems. Hence, quantum $f_q(n,\tau)$ is always recurrent, unlike the classical $f_{\rm cl}(n,\tau)$, which for chaotic motion would never come back to the initial state. In other words, in the quantum case both $\psi(n,\tau)$ and $f_q(n,\tau)$ behave like a classical trajectory rather than the phase density $f_{\rm cl}(n,\tau)$. Notice that the relaxation time scale τ_R has nothing to do with the Poincaré recurrence time, which is much longer and which sharply depends on the recurrence domain.

Another peculiarity of the quantum steady state is that it depends on the initial state, since the former is a result of the diffusion localization of the latter. Hence, the steady state is always spread around the initial state. Of course, such diffusive spreading is possible only if the size of the initial state, $l_0\ll l_s$ (for the Green's function, for example). This condition guarantees the statistical relaxation to the quantum steady state. An extended initial state ($l_0\geqslant l_s$) may happen to be close to one of the eigenfunctions or to the steady state itself, which would suppress the relaxation.

To develop a more accurate description of the quantum steady state we need, first of all, the exact definitions of the quantities in Eq. (10). This is relatively simple for the eigenfunction localization length l because the asymptotic exponential shape of localized eigenfunctions is well confirmed by now both numerically and analytically:

$$\varphi_m(n)\approx(1/l^{1/2})\exp(-|n-m|/l). \tag{11}$$

Using this definition one can find numerically the relationship between l and D. The result is[9,13]

$$l=D/2. \tag{12}$$

The size l_s of the quantum steady state is a more difficult question. Straightforward calculation using Eq. (11) gives for the Green's function[13]

$$G_s\approx\frac{1}{2l_s}\exp\left(-\frac{2|n|}{l_s}\right)\left(1+\frac{2|n|}{l_s}\right), \tag{13}$$

with the same localization length $l_s=l$. But this is certainly incompatible with the numerical results[13]

$$l_s\approx D\approx 2l. \tag{14}$$

A qualitative explanation of this discrepancy is related to large fluctuations in eigenfunctions about the simple exponential dependence (11). However, the question requires further study (see Ref. 3).

The predicted deviation by Eq. (13) from the simple exponential dependence is also very difficult to check numerically, again due to large fluctuations. Meanwhile, the additional factor

$$(1+2|n|/l_s) \tag{14'}$$

in Eq. (13) doubles mean energy E_s in the quantum steady state:

$$E_s\approx D^2/2. \tag{15}$$

This seems to agree with numerical results[2] but the accuracy of the latter is not very good. Besides, Eq. (13) describes the asymptotic ($|n|\to\infty$) dependence only, while E_s is essentially determined by the initial part of the distribution ($|n|\sim l_s$). We shall come back to this question below.

The most difficult problem concerns the relaxation time scale τ_R. To determine it with any accuracy, we need to know the transient behavior that describes the transition from the Gaussian [cf. Eq. (8)] to the exponential distribution [cf. Eq. (13)]:

$$\frac{\exp(-n^2/2D\tau)}{(2\pi D\tau)^{1/2}}\to\frac{(1+2|n|/l_s)\exp(-2|n|/l_s)}{2l_s}. \tag{16}$$

This brings us to the central issue of the article. But before discussing this, we should briefly mention the interesting microstructure of the chaotic eigenstates.

IV. INSTABILITY LOCALIZATION AND "SCARS"

The large spatial fluctuations in a chaotic eigenstate, mentioned above, are not completely random but reveal the structure of classical periodic trajectories. This interesting phenomenon was discovered by Heller in numerical experiments with the quantum stadium billiard[14] and was subsequently confirmed by many others (see, e.g., Refs. 15), particularly in quantum maps. The microstructure was observed in the form of enhancements along classical periodic trajectories in both configurational and phase space. Such enhancements were termed "scars" by Heller.[14]

The general theory of scars in conservative systems with arbitrary number of freedoms N was developed by Berry[6,16] (see also Ref. 17). He made use of the Wigner function (W), which is the quantum counterpart of the classical fine-grained phase space density. Notice that W is generally not positive definite.

Within a scar, W forms complicated diffraction fringes, rapidly oscillating and rather extended along the energy surface. The relative width of the central fringe contracts with the quantum number n as $\sim n^{-1/2}$. In this sense the scars have an essentially quantum structure that vanishes in the quasiclassical region. However, this transition to the classical limit is not a trivial one, as the fringe amplitude does not depend on n. To get rid of scars, one needs a coarse-grained (averaged) density $\overline{W}$, which is also called the Husimi distribution, and which is positive definite. Then the averaged density of a scar vanishes as $n^{-(N-1)}$.

According to Berry's theory, the Wigner chaotic eigenfunction can be approximately represented as a sum over classical periodic trajectories:

$$W(x)\approx\frac{dE}{dx}\delta[E-H(x)] \times\left[1+\sum_s 2(2\pi\hbar)^{N-1}\exp\left(-\frac{N-1}{2}\overline{\Lambda}_s T_s\right) \times\cos\left(\frac{S_s}{\hbar}+\gamma_s\right)\delta(X_s)\right]. \tag{17}$$

Here, $x=(p,q)$ is a point in $2N$-dimensional phase space, while $X=(P,Q)$ describes a $2(N-1)$-dimensional Poincaré section transverse to a periodic trajectory at $X=0$. The periodic trajectory is characterized by an action S and a quasiclassical phase as well as by a mean instability rate $\overline{\Lambda}$. Each term in the sum (17) represents a scar which, by the way, can be of either sign; that is, it may produce either a bump or a dip in the phase space density W.

A possible physical interpretation of this formal theory is as follows. One can consider Eq. (17) as an expansion in the basis of certain "coherent" states, the "scars"

$$W_s=(1/T_s)\delta(X_s)\delta[E-H(x)];\quad\int W_s\,dx=1, \tag{18}$$

which are localized on periodic trajectories. A peculiar property of such coherent states is that they are stationary; that is, they do not move in phase space, nor do they spread. The mechanism of localization is essentially the same as that for the diffusion discussed in the previous section, but now it involves the exponential spreading of a narrow wave packet prior to diffusion, mentioned in Sec. II. The difference is in the level density, which for a scar is $\rho_s\sim T_s^{-1}$. Hence, the time scale for the localization of the instability is T_s, and this explains the exponential factor in the expansion (17).

The small quasiclassical factor $\hbar^{N-1}$ in Eq. (17) is due to the maximal localization of scars in X, which is within a quantum cell of size $\hbar$.

Oscillation tails $\delta(X)$ of unknown length overlap to produce somehow the average ergodic (microcanonical) distribution $\sim\delta[E-H(x)]$ [see Eq. (17)], as well as the Gaussian fluctuations predicted in Ref. 18 and observed in numerical experiments.[2] The total number of separated scars is $\sim\hbar^{1-N}$. Since the number of periodic trajectories with $T_s<T$ grows roughly as $\exp[(N-1)\overline{\Lambda}T]$ (see Ref. 19), the longest period T_m of basis scars is given by the estimate

$$\overline{\Lambda}T_m\sim|\ln\hbar|, \tag{19}$$

which coincides with the so-called Berman–Zaslavsky time scale.[20] This is the time interval for a wave packet to spread over the whole energy surface. The scars with longer periods $T_s\geqslant T_m$ are not separated from each other; that is, even their central fringes essentially overlap and hence they are crucially modified. As a crude approximation, one can simply drop these higher terms, and this would be another way to cope with the divergence of the series in Eq. (17). After this digression, we now return to the main problem of quantum diffusion localization.

V. THE BERMAN–IZRAILEV THEORY OF QUANTUM DIFFUSION SUPPRESSION

The first attempt to explain the transient behavior from classical-like diffusion to quantum localization was made in Ref. 21 (see also Ref. 22). The main idea is that residual correlations, even fairly small ones but on the whole relaxation time scale τ_R, will eventually result in the complete suppression of the diffusion. The authors managed to derive the explicit dependence of the mean energy on time:

$$E(\tau)=E_s\{1-1/[1+(\tau/\tau_R)]^{\beta}\}. \tag{20}$$

There are three parameters to be determined from numerical data. One is just the desired time scale τ_R; E_s is the mean energy of the steady state, and a new parameter β characterizes the "repulsion" of quasienergy levels:

$$p(s)\sim s^{\beta};\quad s\to 0. \tag{21}$$

Here, $p(s)$ is the probability density and s is the level spacing. Equation (20) seems to agree with numerical data for $\beta=0.3$.[21] This rather small value corresponds to the so-called "intermediate level statistics" that were introduced

and studied in detail by Izrailev.[23] In the present problem, the statistics in Eq. (21) describe the operative eigenfunctions only.

If we assume the expression (15) for E_s, the two parameters in Eq. (20) are related by $\tau_R = \beta D$ from the classical diffusion law $E \approx D\tau/2$ for $\tau \ll \tau_R$. This is in agreement with estimate (10), and it apparently solves the problem of the relaxation time scale τ_R.

However, in Ref. 24 a different law for the diffusion suppression was derived, using an analogy to Anderson localization in solids (see Refs. 25 and 26). The main difference from Eq. (20) is in the exponent, $\beta \to 1$, which implies a much faster suppression of diffusion. The origin of this discrepancy is as yet not clear.

In any case, the saturation shown in Eq. (20) of the energy growth $E(\tau) \to E_s$ is not the whole story. The transformation of the initial Gaussian distribution into the final exponential one is still to be explained [see Eq. (16)].

VI. A NEW PHENOMENOLOGICAL THEORY OF QUANTUM LOCALIZATION

This theory, proposed in Ref. 3, was aimed, first of all, at explaining the transformation of the Green's function from Gaussian to exponential [cf. Eq. (16)]. Let us begin with the general diffusion equation (see, e.g., Ref. 27):

$$\frac{\partial G(n,\tau)}{\partial \tau} = \frac{1}{2}\frac{\partial^2}{\partial n^2}(GD) - \frac{\partial}{\partial n}(GA). \tag{22}$$

The second term describes a "drift"

$$A = \frac{\langle \Delta n \rangle}{\tau} = \frac{dD}{dn} + B. \tag{23}$$

By introducing Eq. (23) into Eq. (22) we obtain

$$\frac{\partial G}{\partial \tau} = \frac{1}{2}\frac{\partial}{\partial n} D \frac{\partial G}{\partial n} - \frac{\partial}{\partial n} BG. \tag{24}$$

The last term usually represents the effect of some dissipation. In our problem there is no dissipation, yet there exists what is sometimes called "backscattering," that is, the reflection of the ψ wave propagating in momentum space.

To explain this backscattering, let us make use of a deep analogy between the dynamical problem in question and Anderson localization in solids. As mentioned above, this fruitful analogy was discovered in Ref. 25 and further developed in Ref. 9. The two problems are related by a Fourier transform from momentum to coordinate space or vice versa. In the original Anderson theory, the spatial potential was assumed to be random, which implies that it has a continuous spectrum. Then, for any energy of a propagating particle, there exists a resonant harmonic of the potential which provides an effective reflection of the ψ wave, or backscattering. This results in the localization of all eigenfunctions in one-dimensional problem.

Notice that the potential in a solid corresponds to the rotation operator $\hat{R}$ in the dynamical problem (9) while the kick operator $\hat{F}$ represents "free" motion of a particle. In the exact analogy the random potential would require random quantum phases $Tn^2/2$ in the standard map [see Eq. (9)]. These phases are obviously not random, so an attempt to use Anderson's theory for proving the diffusion localization[25] fails. However, the analogy can be applied in the opposite direction: The diffusion localization with nonrandom phases implies that the random potential is only sufficient but not a necessary condition for the Anderson localization. Indeed, it was proved later that even a quasiperiodic potential with only two spatial frequencies can provide localization (see, e.g., Ref. 28).

Now, we can determine the backscattering function B in Eq. (24) by requiring the steady state to be exponential as in Eq. (13) with $l_s = D$. For simplicity we neglect the factor

$$(1 + 2|n|/l_s) \tag{24a}$$

and consider first a constant diffusion rate (see below). The answer is very simple, namely,

$$B = \mp 1; \quad n \gtrless 0. \tag{25}$$

We assume $G(n,0) = \delta(n)$, so that $G(N,\tau)$ is always symmetric about $n = 0$, and we consider below only $n \geqslant 0$.

The mechanics of the backscattering is still to be understood. Certainly, it is not a single-kick effect, since this is perfectly symmetric. To get an idea on the effect of backscattering, consider the equations for moments $m_1 = \langle n \rangle (n > 0)$ and $m_2 = \langle n^2 \rangle$:

$$\dot{m}_1 = \tfrac{1}{2} DG(0,\tau) - 1, \quad \dot{m}_2 = D - 2m_1. \tag{26}$$

On the other hand, the ratio

$$m_2/m_1^2 \equiv 2/\gamma^2 \tag{27}$$

remains almost constant in the transition from the Gaussian to exponential distribution:

$$\gamma = 2^{1/2}\pi^{-1/4} \approx 1.06 \to 1. \tag{27a}$$

Neglecting this small variation and introducing a new variable $\xi = 2\gamma E^{1/2}/D$ and new time $\tau = \gamma^2 \tau/D$, we arrive at the equation

$$\xi\dot{\xi} = 1 - \xi, \tag{28}$$

with the solution

$$-t = \xi + \ln(1 - \xi); \quad \xi(0) = 0. \tag{29}$$

Initially, as $\tau \to 0$ we have $\xi^2 \approx 2t$, or $E \approx D\tau/2$, independent of γ; this is just the classical diffusion. Asymptotically, as $\tau \to \infty$, the solution $\xi \to 1$, or $E \to E_s \approx D^2/4\gamma^2$. If $\gamma = 1$, this differs from the value (15) by a factor of 2. This difference may be related to our neglect of the factor

$$(1 + 2|n|/l_s) \tag{29a}$$

in the steady-state distribution and can be corrected by choosing $\gamma = \gamma_c = 1/2^{1/2} \approx 0.71$. The relaxation $\xi \to 1$ is exponential

$$\xi = 1 - e^{-t-\xi} \approx 1 - e^{-t-1} \tag{30}$$

with a characteristic time

$$\tau_R = \tau/t = D/\gamma^2 \geqslant D. \tag{31}$$

This is again in agreement with a rough estimate (10) but much larger than in Eq. (20) ($\tau_R = \beta D \approx 0.3D$).

A preliminary comparison of some numerical data with the new relaxation dependence (30) reveals a reasonable agreement in a fairly large range of $\Delta(E/E_s) \approx 25$, the average fitting parameter being $\bar{\gamma} \approx 0.75 \approx \gamma_c$. However, the empirical E_s is only about a half of the expected value $E_s = D^2/4\gamma^2$ with this $\bar{\gamma}^2 \approx 1/2$.

One way to check the proposed theory is to consider other, more difficult problems, for example, the case of a variable rate

$$D = D_0 n^{2\alpha} \tag{32}$$

with some constant α. The general expression for the steady state is obtained from the diffusion equation [Eq. (24)] with $B = -1$ $(n > 0)$ in the form

$$\ln G = -2 \int \frac{dn}{D(n)} = \begin{cases} -2n^{1-2\alpha}/(1-2\alpha)D_0; & \alpha \neq \frac{1}{2}, \\ -(2/D_0)\ln n; & \alpha = \frac{1}{2}. \end{cases} \tag{33}$$

In agreement with previous results[13] the critical value of the parameter is $\alpha_c = \frac{1}{2}$. For $\alpha < \alpha_c$ the localization remains exponential, while for $\alpha > \alpha_c$ delocalization occurs since $G(n) \to \text{const}$ as $n \to \infty$. In the borderline case $\alpha = \alpha_c = \frac{1}{2}$, the steady-state distribution is a power law

$$G_c \sim n^{-2/D_0} \tag{34}$$

and localization occurs only for sufficiently small $D_0 < 2$, when $G_c(n)$ is normalizable.

The diffusion equation (24) with backscattering [cf. Eq. (25)] leads to an exponential relaxation [cf. Eq. (30)] which is qualitatively different from the power law (20) in the existing theory. Is it possible to derive Eq. (20) from Eq. (24)? It turns out that it is! One needs only allow some variation in time of the diffusion rate $D(\tau)$. Indeed, assume $m_2(\tau)$ is given by Eq. (20) with E_s approaching m_s. Then from the second Eq. (26), we have

$$D(\tau) = \frac{m_s}{\tau_R} \frac{\beta}{(1+\tau/\tau_R)^{1+\beta}} + 2m_s^{1/2}\left(1 - \frac{1}{(1+\tau/\tau_R)^{\beta}}\right)^{1/2} \tag{35}$$

assuming $m_1^2 = m_2$ (27). If, moreover, $m_s = 4\tau_R^2/\beta^2$ so that $D(0) = D(\infty)$, the rate $D(\tau)$ has an oscillation only with $r = D_{\max}/D_{\min} \approx 1.4$ (with $\beta = 0.3$)[21] and $r = 1.2$ [$(\beta = 1)$[24]].

To summarize, the proposed theory seems to have the prospect of being true but, of course, a much more extensive comparison with numerical data is required.

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[1] G. Casati, B. V. Chirikov, J. Ford, and F. M. Izrailev, *Lecture Notes in Physics*, Vol. 93 (Springer-Verlag, Berlin, 1979), p. 334.
[2] B. V. Chirikov, F. M. Izrailev, and D. L. Shepelyansky, Physica D **33**, 77 (1988).
[3] B. V. Chirikov, "Time-Dependent Quantum Systems," in *Proceedings of the Les Houches Summer School on Chaos and Quantum Physics* (Elsevier, New York, 1990).
[4] M. Gell-Mann and J. Hartle, "Quantum Mechanics in the Light of Quantum Cosmology," in *Proceedings of the 3rd International Symposium on the Foundations of Quantum Mechanics in the Light of New Technology*, Tokyo, 1989.
[5] J. A. Wheeler and W. Zurek, *Quantum Theory and Measurement* (Princeton U.P., Princeton, NJ, 1983); T. Dittrich and R. Graham, Phys. Rev. A **42**, 4647 (1990); I. Percival, "Quantum records" (unpublished) (1990); S. Weigert, "Chaos and QND Measurements," Phys. Rev. A (to appear); J. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge U.P., Cambridge, 1987).
[6] M. Berry, "Some Quantum-to-Classical Asymptotics," in *Proceedings of the Les Houches Summer School on Chaos and Quantum Physics* (Elsevier, New York, 1990).
[7] G. Casati *et al.*, Phys. Rep. **154**, 77 (1987).
[8] G. Casati, I. Guarneri, and D. L. Shepelyansky, Phys. Rev. A **36**, 3501 (1987); IEEE J. Quantum Electron. **24**, 1420 (1988).
[9] D. L. Shepelyansky, Physica D **28**, 103 (1987).
[10] D. L. Shepelyansky, Physica D **8**, 208 (1983); G. Casati *et al.*, Phys. Rev. Lett. **56**, 2437 (1986).
[11] G. Casati *et al.*, Phys. Rev. A **34**, 1413 (1986).
[12] B. V. Chirikov, F. M. Izrailev, and D. L. Shepelyansky, Sov. Sci. Rev. C **2**, 209 (1981).
[13] B. V. Chirikov and D. L. Shepelyansky, Radiofizika **29**, 1041 (1986).
[14] E. Heller, Phys. Rev. Lett. **53**, 1515 (1984).
[15] R. Jensen *et al.*, Phys. Rev. Lett. **63**, 2771 (1989).
[16] M. Berry, Proc. R. Soc. London Ser. A **423**, 219 (1989).
[17] E. B. Bogomolny, Physica D **31**, 169 (1988).
[18] M. Berry, J. Phys. A **10**, 2083 (1977).
[19] Ya. G. Sinai, *Introduction to Ergodic Theory* (Princeton U.P., Princeton, NJ, 1976).
[20] G. P. Berman and G. M. Zaslavsky, Physica A **91**, 450 (1978).
[21] G. P. Berman and F. M. Izrailev, "On the dynamics of correlation functions and diffusion limitations in the region of quantum chaos," preprint 497, Kirensky Institute of Physics, Krasnoyarsk (1988).
[22] F. M. Izrailev, Phys. Rep. **196**, 299 (1990).
[23] F. M. Izrailev, Phys. Lett. A **134**, 13 (1988); J. Phys. A **22**, 865 (1989).
[24] D. Cohen, "Diffusion in the Quantum Damped, Kicked Rotator Problem" (unpublished) (1990).
[25] S. Fishman *et al.*, Phys. Rev. A **29**, 1639 (1984).
[26] V. L. Berezinsky, Zh. Eksp. Teor. Fiz. **65**, 1251 (1973).
[27] A. Lichtenberg and M. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1983).
[28] V. A. Chulaevsky and Ya. G. Sinai, Commun. Math. Phys. **125**, 91 (1989).