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Poster Session and Program

Chaotic Quantum Systems

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Abstract: New ideas concerning the peculiar phenomenon of quantum chaos are presented with special emphasis on a number of unsolved problems and current apparent contradictions.

1 Introduction

This lecture[1] is primarily addressed to mathematicians with the main purpose to explain new physical ideas in the so-called *quantum chaos* which since recently attracts ever growing interest of many researchers [1-5],[10].

The breakthrough in understanding of this phenomenon has been achieved, particularly, due to a new philosophy accepted, explicitly or more often implicitly, in most studies of quantum chaos. Namely, the whole physical problem of quantum dynamics is separated into two different parts: (*i*) the proper quantum motion described by a specific dynamical variable $\psi(t)$ which obeys, e.g., the Shrödinger equation, and (*ii*) the quantum measurement including ψ collapse which, as yet, has no dynamical description. In this way one can single out the vague problem of the fundamental randomness in quantum mechanics which is related to the second part only, and which in a sense is foreign to the proper quantum system. The remaining first part is then perfectly fits the general theory of dynamical systems.

The importance of quantum chaos is not only in that it represents a new unexplored field of nonintegrable quantum dynamics with many applications but also, and this is most interesting for fundamental science, in reconciling the two seemingly different dynamical mechanisms for the statistical laws in physics.

Historically, the first mechanism is related to the *thermodynamic limit* $N \to \infty$ in which the completely integrable system becomes chaotic [6]. A

[1] Here is the abridged version, for the full text see [33].

natural question, what happens for large but finite number of freedoms N, has still no rigorous answer but the new phenomenon of quantum chaos, at least, presents an insight into this problem too. We call this mechanism, which is equally applicable in both classical and quantum mechanics, the *traditional statistical mechanics* (TSM).

The second (new) mechanism is based upon the strong (exponential) local instability of motion characterized by positive Lyapunov's exponent $\Lambda > 0$ [6],[7]. It is not at all restricted to large N, and is possible, e.g., for $N > 1$ in a Hamiltonian system. However, this mechanism has been operative, until recently, in the classical mechanics only. We term this the *dynamical chaos* as it does not require any random parameters or any noise in the equations of motion.

The quantum system bounded in phase space has a discrete energy (frequency) spectrum and, in this sense, is always completely integrable similar to the finite-N TSM. Yet, the fundamental correspondence principle requires the transition to the classical mechanics, including the dynamical chaos, in the *classical limit* $q \to \infty$, where q is some quasi-classical parameter, e.g., the quantum number n (the action variable, $\hbar = 1$). Again, a natural physical conjecture is that for finite but large q there must be some chaos similar to finite-N TSM. Yet, in a chaotic quantum system the number of freedoms N does not need to be large as well as in the classical chaos. The quantum counterpart of N is q, both quantities determining the number of frequencies which control the motion. Thus, mathematically, the problem of quantum chaos is the same as that of the finite-N TSM.

The main difficulty in both problems (especially for mathematicians) is in that they suggest some chaos in discrete spectrum which is completely contrary to the existing theory of dynamical systems and the ergodic theory where the discrete spectrum corresponds to the opposite limit of regular motion.

The ultimate origin of the quantum integrability is discreteness of the phase-space (but not, as yet, of the space-time!) or, in modern mathematical language, the noncommutative geometry of the former.

As an illustration I will make use of the simple model described classically by the *standard map* (SM) [7],[8]:

$$\bar{n} = n + k \cdot \sin\theta; \qquad \bar{\theta} = \theta + T \cdot \bar{n} \,, \tag{1.1}$$

with action-angle variables n, θ, and pertubation parameters k, T. The quantized standard map (QSM) is given by [9],[10]

$$\bar{\psi} = \exp(-ik \cdot \cos\hat{\theta}) \cdot \exp\left(-i\frac{T}{2}\hat{n}^2\right)\psi \,, \tag{1.2}$$

where momentum operator $\hat{n} = -i\partial/\partial\theta$. To provide the boundedness of motion we consider SM on a torus of circumference (in n)

$$L = \frac{2\pi m}{T} \tag{1.3}$$

with integer m to avoid discontinuities. The quasi-classical transition corresponds to quantum parameters $k \to \infty$, $T \to 0$, $L \to \infty$ while classical parameters $K = kT = \text{const}$, and $m = LT/2\pi = \text{const}$.

QSM models the *energy shell* of a conserved system which is the quantum counterpart of the classical energy surface.

In the studies of dynamical systems, both classical and quantal, most problems unreachable for rigorous mathematical analysis are treated "numerically" using computer as a universal model. With all obvious drawbacks and limitations such "numerical experiments" have very important advantage as compared to the laboratory experiments, namely, they provide the complete information about the system under study. In quantum mechanics this advantage becomes crucial as in laboratory one cannot observe (measure) the quantum system without a radical change of its dynamics.

2 Definition of Quantum Chaos

The common definition of the classical chaos in physical literature is the *strongly unstable motion*, that is one with positive Lyapunov's exponents $\Lambda > 0$. The Alekseev — Brudno theorem then implies that almost all trajectories of such a motion are unpredictable, or random (see [11]). A similar definition of quantum chaos fails because, for the bounded systems, the set of such motions is empty due to the discreteness of the phase space and, hence, of the spectrum.

The common definition of quantum chaos is *quantum dynamics of classically chaotic systems* whatever it could happen to be. Logically, this is most simple and clear definition. Yet, in my opinion, it is completely inadequate from the physical viewpoint just because such a chaos may turn out to be a perfectly regular motion as, for example, in case of the *perturbative localization* [12]. In QSM this corresponds to $k \leq 1$ when all quantum transitions are suppressed independent of classical parameter K which controls the chaos.

I would like to define the quantum chaos in such a way to include some essential part of the classical chaos. The best definition I have managed to invent so far reads: *the quantum chaos is statistical relaxation in discrete spectrum* .This definition is certainly in contradiction with the existing ergodic theory as the relaxation (particularly, correlation decay) requires the mixing, hence, a continuous spectrum. In what follows I will try to explain a new, modified, concept of mixing which is necessary to describe the peculiar phenomen of quantum chaos.

3 The Time Scales of Quantum Dynamics

Already the first numerical experiments with QSM revealed the quantum diffusion in n close to the classical one under conditions $K \geq 1$ (classical stability border) and $k \geq 1$ (quantum stability border) [9]. Further studies confirmed this conclusion and showed that the former followed the latter in all details but on a *finite time interval* only [10],[13]. The latter fact was the clue to understanding the dynamical mechanism of the diffusion, which is apparently an aperiodic process, in discrete spectrum. Indeed, the fundamental uncertainty principle implies that the discreteness of the spectrum is not resolved for sufficiently short time interval. Whence, the estimate for the *diffussion (relaxation) time scale* :

$$t_R \sim \varrho_0 \leq \varrho \,. \tag{3.1}$$

Here ϱ is the density of (quasi)energy levels, and ϱ_0 is the same for the *operative eigenstates* which are actually present in the initial quantum state $\psi(0)$. In QSM the quasi-energies are determined mod $2\pi/T$, hence, $\varrho = LT/2\pi = m$ is a classical parameter (1.3). As to ϱ_0, it depends on the dynamics and is given by the estimate [10],[13]:

$$\frac{\varrho_0}{T} \sim \frac{t_R}{T} \equiv \tau_R \sim D \equiv \frac{\langle(\Delta n)^2\rangle}{\tau} \leq \frac{m}{T} \,. \tag{3.2}$$

Here τ is discrete map's time (the number of iterations), and D is the classical diffusion rate. This remarkable expression relates an essentially quantum characteristic (τ_R) to the classical one (D). The latter inequality in Eq. (3.2) follows from that in Eq. (3.1), and it is explained by the boundedness of QSM on a torus.

In the quasi-classical region $\tau_R \sim k^2 \to \infty$ (see Eq.(1.1)) in accordance with the correspondence principle. Yet, the transition to the classical limit is (conceptually) difficult to understand (and still more to accept) as it involves two limits ($k \to \infty$ and $t \to \infty$) which do not commute. The second limit is related to the existing ergodic theory which is asymptotic in t. Meanwhile the new phenomenon of the quantum chaos requires the modification of the theory to a finite time which is a difficult mathematical problem still to be solved.

Besides relatively long time scale (3.2) there is another one given by the estimate [14],[10]

$$t_r \sim \frac{\ln q}{\Lambda} \to \frac{T|\ln T|}{\ln(K/2)} \,, \tag{3.3}$$

where q is some (large) quasi-classical parameter, and where the latter expression holds for QSM. It is called some time the *random time scale* since here the quantum motion of a narrow wave packet is as random as classical trajectories according to the Ehrenfest theorem. This was well confirmed in a number of numerical experiments [15]. The physical meaning of t_r is in fast spreading of a wave packet due to the strong instability of classical motion.

Even though the random time scale t_r is very short it grows indefinitely in the quasi-classical region ($q \to \infty$, $T \to 0$), again in agreement with the correspondence principle.

Big ratio t_R/t_r implies another peculiarity of quantum diffusion: It is dynamically stable as was demonstrated in striking numerical experiments [16].

4 The Quantum Steady State

As a result of quantum diffusion and relaxation some steady state is formed whose nature depends on the *ergodicity parameter*

$$\lambda = \frac{l_s}{L} \approx \frac{D}{L} \,. \tag{4.1}$$

If $\lambda \gg 1$ the quantum steady state is close (at average) to the classical statistical equilibrium which is described by ergodic phase density $g_{cl}(n)$ =const where n is continuous variable. In quantum mechanics n is integer, and the quantum phase density $g_q(n,\tau)$ in the steady state fluctuates according to the Gauss law approximately [17],[5], the ergodicity meaning

$$g_q(n) = \overline{|\psi_s(n,\tau)|^2} = \frac{1}{L} \,, \tag{4.2}$$

where the bar denotes time averaging.

According to numerical experiments the ergodicity does not depend on initial state which implies that all eigenfunctions $\phi_m(n)$ are also ergodic at average with Gaussian fluctuations [17],[5]:

$$\langle |\phi_m(n)|^2 \rangle = \frac{1}{L} \,. \tag{4.3}$$

This is always the case sufficiently far in the quasi-classical region as $\lambda \sim \sim k^2/L \sim Kk/m \to \infty$ with $k \to \infty$ ($K = kT$ and $m = LT/2\pi$ remain constant) in accordance with old Shnirelman's theorem [18].

Finite fluctuations (4.2) show that a single chaotic quantum system, described by $\psi_s(n,\tau)$, represents, in a sense, finite statistical ensemble of $M \sim L$ "particles". The fluctuations can result in partial recurrences toward the initial state but the recurrence time is much longer as compared to the relaxation time scale τ_R and sharply depends on the recurrence domain.

If $\lambda \ll 1$ the quantum steady state is qualitatively different from the classical one. Namely, it is localized in n within the region of size l_s around the initial state if the size of the latter $l_0 \ll l_s$. Numerical experiments show that the phase space density, or the *quantum statistical measure* , is approximately exponential [10],[13]

$$g_s(n) \approx \frac{1}{l_s} \exp\left(-\frac{2|n|}{l_s}\right); \qquad l_s \approx D \tag{4.4}$$

for initial $g(n,0) = \delta(n)$. The quantum ensemble is now characterized by $M \sim l_s \sim k^2$ "particles".

The relaxation to this steady state is called *diffusion localization* , and it is described approximately by the diffusion equation [19],[33]

$$\frac{\partial g}{\partial \tau'} = \frac{1}{2}\frac{\partial}{\partial n} D \frac{\partial g}{\partial n} \pm \frac{\partial g}{\partial n} \tag{4.5}$$

for initial $g(n,0) = \delta(n)$ where the signs correspond to $n \gtrless 1$, and where new time

$$\tau' = \tau_R \ln\left(1 + \frac{\tau}{\tau_R}\right) \tag{4.6}$$

accounts for the discrete motion spectrum [20]. The last term in Eq. (4.5) describes "backscattering" of ψ wave propagating in n which eventually results in the diffusion localization. The fitting parameter $\tau_R \approx 2D$ was derived from the best numerical data available (see Ref. [21] where a different theory of diffusion localization was also developed).

5 Intermediate Statistics

Statistical properties of ergodic states are well described by the random matrix theory (RMT) [22]. Particularly, the statistics of nearest-neighbour level spacings s is given by the Wigner — Dyson distribution

$$p(s) = As^\beta e^{-Bs^2} , \tag{5.1}$$

where constants A and B are found from normalization and condition $\langle s \rangle = 1$, and where level repulsion parameter $\beta = 1$; 2 or 4 depending on system's symmetry. This is to compare with the Poisson distribution for integrable systems

$$p(s) = e^{-s} \tag{5.2}$$

with $\beta = 0$ (independent levels). Extension of these statistics onto the quasi-energy levels was made in [23],[5].

The impact of diffusion localization on the level statistics is described by the *Izrailev distribution* [24]

$$p(s) \approx As^\beta \exp\left(-\frac{\pi^2}{16}\beta s^2 - \left(B - \frac{\pi\beta}{4}\right)s\right) , \tag{5.3}$$

which is also called *intermediate* (between (5.1) and (5.2)) *statistics* . Repulsion parameter β is now continuous and may take any value in the whole interval (0, 4). The *limiting statistics* ($\lambda = l_s/L \to \infty$) corresponds, for a given l_s, to small L which in RMT is matrix's size. This shows that RMT is a local theory which holds true within the localization length l_s only.

Repulsion parameter in (5.3)

$$\beta \approx \beta_H = \exp(\langle H \rangle - H_e) \approx \frac{2l_H}{L} \,, \tag{5.4}$$

where $\langle H \rangle$ is the mean entropy of eigenstates

$$H = -\sum_n |\phi(n)|^2 \ln |\phi(n)|^2 \,, \tag{5.5}$$

$l_H = \exp(H)$ is another localization length and $H_e \approx \ln(L/2)$ is the entropy of ergodic state. No explanation of a simple relation (5.4) has been given as yet, nor for the important scaling $\beta_H(\lambda)$ obtained numerically [5].

The discovery of intermediate statistics stimulated the development of a new RMT which makes use of band random matrices (BRM) [25] with the scaling parameter

$$\lambda_r = \frac{b^2}{L} \,, \tag{5.6}$$

where $2b$ is the band width. In the limit $L \to \infty$ this scaling has been proved recently in Ref. [26].

6 Concluding Remarks

In conclusion I would like to briefly mention a few important results for unbounded quantum motion. In SM it corresponds to $L \to \infty$. First, there is an interesting analogy between localization in momentum space and the celebrated Anderson localization in disordered solids. It was discovered in [27] and essentially developed in [28]. The analogy is based upon (and restricted by) the equations for eigenfunctions. The most striking (and less known) difference is in the absence of diffusion regime in $1D$ solids [29]. This is because the energy level density of the operative eigenfunctions in solids

$$\rho_0 \sim \frac{l dp}{dE} \sim \frac{l}{u} \sim t_R \,, \tag{6.1}$$

which is the localization (relaxation) time scale, is always of the order of the time interval for a free spreading of the initial wave packet at caracteristic velocity u.

Another similarity between the two problems is in that the Bloch extended states in periodic potential correspond to a pecular quantum resonance in QSM for rational $T/4\pi$ [9],[10]. An interesting open question is the dynamics for irrational Liouville's (transcendental) $T/4\pi$. As was proved in [30] the motion can be unbounded in this case unlike a typical irrational value. In [33] the conjecture is put forward, supported by some semiqualitative considerations, that depending on a particular Liouville's number the broad range of motions is possible, from purely resonant one ($|n| \sim \tau$) down to complete localization ($|n| \leq l$).

If quantum motion is not only unbounded but its rate in unbounded variables is exponential, the "true" chaos (not restricted to a finite time scale) can occur. A few exotic examples together with considerations from different viewpoints can be found in [10],[31]. However, such chaos does not seem to be a typical quantum dynamics.

The final remark is that the quantum chaos, as defined in Section 2, comprises not only quantum systems but also any linear, particularly classical, waves [32]. So it is essentially the *linear wave chaos* . Moreover, a similar mechanism works also in completely integrable nonlinear systems like Toda lattice, for example. From mathematical point of view all these new ideas require reconsideration of the existing ergodic theory. Perhaps, better to say, that a new ergodic theory is needed which, instead of benefiting from the asymptotic approximation ($|t| \to \infty$ or $N \to \infty$), could analyze the finite-time statistical properties of dynamical systems. This is the most important conclusion from first attempts to comprehend the quantum chaos.

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