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Nonlinearity with Disorder

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Linear Chaos

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Abstract. The controversial phenomenon of quantum (wave) chaos is reviewed using a simple analogy with classical linear waves in cavities. Estimates for the main statistical properties of wave dynamics are evaluated and discussed. The transient nature of wave chaos is emphasized and explained in detail.

The main purpose of my talk is to present a fairly new conception of the so-called quantum chaos. Formally, it is a part of quantum dynamics, rather surprising but not at all exotic. On the contrary, the quantum chaos turns out to be a typical phenomenon which was overlooked until recently. The understanding is now gradually coming from the classical mechanics where the phenomenon of dynamical chaos was well ascertained and studied in great detail (see, e. g., Ref. [1]). For this reason I am going first to remind you of the classical dynamical chaos (Section 2).

In what follows I restrict myself to the conservative (energy-preserving) Hamiltonian (nondissipative) dynamics as a more fundamental one. To illustrate the general theory I will use simple models of classical and quantum billiards as well as waveguides and cavities.

1. Simple Models

Consider free motion of a particle in the domain of d dimensions surrounded by a (d–1)-dimensional perfectly reflecting wall. For $d=2$ the model is called billiard, and it was extensively studied by many authors (see, e. g., Refs [2, 3]). For special shapes of the wall the motion is completely integrable, or regular, which means that there is the full set of d motion integrals in involution (see, e. g., Ref. [4]). The simplest example is a rectangular parallelepiped with sides a_i. Then motion integrals are momenta $|p_i|$ or the actions

$$n_i = \frac{a_i\ |p_i\ |}{\pi}. \tag{1}$$

The corresponding Hamiltonian $H(\theta_i, n_i)$ in action-angle variables is given by

$$H^2 = \pi^2 \sum_i \frac{n_i^2}{a_i^2} + m^2, \tag{2}$$

where m is particle's mass, and we put the speed of light $c=1$.

The completely integrable motion is quasi-periodic, i. e. it has discrete Fourier spectrum with d basic frequences $\omega_i = \pi^2 n_i / Ha_i^2$ (for model (2)). This is a nonlinear oscillation since $d\omega_i / dn_i = \pi^2 / Ha_i^2 \neq 0$. Being regular (by definition!) the motion is unstable in phases θ_i just because of nonlinearity: $\delta\theta_i \sim t\delta n_i$. Yet, this unavoidable instability is weak, only linear in time.

Now, introduce a small perturbation, that is deform the wall in such a way to completely destroy integrability and to convert the motion into chaotic one. This always can be done [2, 3], and we shall characterize the perturbation strength by a small angle $\varepsilon \ll 1$ between two surfaces of the wall, unperturbed and corrugated.

In quantum mechanics the same model is described by the linear Schrödinger equation. For boundary condition $\Psi = 0$ (infinitely high potential wall) the actions n_i are now integers.

Instead of quantum Ψ waves we may consider any classical linear waves (sound, electromagnetic, etc.) inside a cavity with appropriate boundary conditions. Then the classical problem (e. g., Eq. (2)) corresponds, as is well known, to the limit of geometrical optics with the rays as dynamical trajectories. The Hamiltonian is now the local dispersion relation (see, e. g., Ref. [5]):

$$H \equiv \omega\,(k_l, x_l) = \omega\,(n_i, \theta_i) \tag{3}$$

where x_l are Cartesian coordinates, wave numbers $k_l = p_l$, and we put $\hbar = 1$. Particularly, for electromagnetic waves in a medium with refraction index $n(x_l)$ the Hamiltonian

$$\omega(k_l, x_l) = \frac{k}{n(x_l)}\,. \tag{4}$$

Finally, we consider a straight waveguide in vacuum with the perturbation independent of longitudinal coordinate x_3. Then $k_3 = \text{const}$, and the dispersion relation is the same as Hamiltonian (2)

$$\omega^2 = k_1^2 + k_2^2 + m^2\,. \tag{5}$$

It describes "heavy" photons with "mass" $m = |k_3|$.

2. Nonlinear Ray Chaos

As is well known the classical dynamical chaos is only possible in nonlinear systems, that is those whose equations of motion are nonlinear [1]. Such are, for example, the simple models described above, but not a linear (harmonic) oscillator. On the other hand, the main condition for chaos is linear (local) instability of motion discribed by the linearized equations of motion [1]. Moreover, the instability must be strong (exponential) and not a linear one characteristic for a regular motion (Section 1).

The rate of local instability is given by the Lyapunov exponent Λ of the linearized equations of motion. In a billiard or cavity

$$\Lambda \sim \ln \frac{a}{R}, \tag{6}$$

where $R \sim \alpha / \varepsilon$ is the local curvature radius of the wall corrugation with characteristic linear size α. The quantity

$$h \sim \Lambda d \sim d \cdot \ln \frac{a}{R} \tag{7}$$

is called metric entropy, or KS-entropy (after Kolmogorov and Sinai). It is the most important characteristic of chaotic motion in the modern ergodic theory (see, e. g., Ref. [6]). Roughly, the condition

$$h \geq \Lambda > 0 \,; \qquad \frac{a}{R} \sim \frac{\varepsilon\, a}{\alpha} \geq 1 \,. \tag{8}$$

is necessary for chaos to occur.

I will not go into details of this condition. Rigorous results can be found in Refs [2, 3]. Extensive studies of ray chaos in cavities were performed by Abdullaev and Zaslavsky [7] (see also Refs [5, 8]). For my purposes the rough but transparent estimates are sufficient to expose the nature of dynamical chaos.

Let me just mention that for a big perturbation parameter $\varepsilon \gtrsim 1$ the chaos occurs even for $\alpha \sim a$. Such is a very popular billiard model, the "stadium": two semicircles connected by two straight lines of arbitrary finite length [3]. Again, to simplify the presentation I consider small perturbation $\varepsilon \to 0$, hence $\alpha \to 0$ (8). The perturbation in this case is like tiny ripples.

Now, what is the role of nonlinearity in chaos? The point is that instability (8) is necessary but not sufficient condition for chaos. The other condition is boundedness of the motion, at least in some unstable variables. For example, unstable motion of a linear system (the so-called hyperbolic motion) is unbounded and perfectly regular. Nonlinearity makes the energy surfaces closed and compact even for an unstable motion. In a billiard, for example, the motion is trivially restricted by the wall.

So, what is really required for chaos is the boundedness of the motion whatever the cause could be. As to the nonlinearity it depends on the description chosen. As is well known there is an equivalent description in terms of the phase space density, or distribution function, $f(x_1, p_1, t)$ which obeys linear Liouville equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [H, f] = 0 \,, \tag{9}$$

where [,] are Poisson brackets. This form of dynamical equation is especially convenient for comparison mith quantum mechanics. Of course, Eq. (9) does not change the boundedness of the motion. The density f itself is even the motion integral (9).

How the local instability is described in this presentation? Consider the characteristic function of a small domain that is the initial density concentrated there. In case of the complete chaos the local motion instability results in indefinite expansion of the domain over some $(d-1)$-dimensional manifold and, what is more important, in also indefinite contraction on the complementary manifold, also of $d-1$ dimensions. Notice the energy conservation and zero Lyapunov exponent along a trajectory, which reduces the expansion/contraction dimensions by 2.

Now, if the motion is bounded (in any representation) the expanding manifold has eventually to mix up and to form the so-called foliation with ever increasing number of sheets whose width is exponentially decreasing. This highly intricate structure of the fine-grained (exact) phase density is necessary to provide the time reversibility of density evolution in agreement with trajectory reversibility. However, any coarse-graining (averaging) of the density results in the loss of this reversibility in apparent contradiction with the reversible dynamics.

Note, that the evolution of the coarse-grained density is the same in both directions of time because the only dynamical difference is interchange of the two manifolds, expanding and contracting. Hence, there is no need for the conception of "time arrow".

The time evolution of some coarse-grained density, say, $\bar{f}(n_i, t)$ (usually averaged over phases θ_i) is described by the kinetic equation which, in principle, can be completely derived from the equations of motion or from Liouville equation (9) without any statistical hypothesis. Note, that coarse-graining as well as the use of the phase density are not hypotheses, but a particular method of adequate description to reveal the essential features of the motion.

In case of small perturbation the kinetic equation takes the form of the Focker–Planck–Kolmogorov (FPK) equation that is a diffusion equation. In our examples of ray dynamics the diffusion rate in spatial angles φ_i and in actions n_i is

$$D_\varphi \sim \varepsilon^2 \frac{\mathrm{v}}{a}; \qquad D_n \sim (\varepsilon n)^2 \frac{\mathrm{v}}{a}, \tag{10}$$

where we dropped sup i ($a_i \sim a$, $\varphi_i \sim \varphi$), and where

$$\mathrm{v} \sim a\dot{\theta} \sim a\frac{\partial\omega}{\partial n}; \qquad \frac{\mathrm{v}}{a} \sim \Omega \tag{11}$$

are the velocity of a "particle" and its unperturbed oscillation frequency, respectively.

The diffusion leads to the statistical relaxation

$$\bar{f}(n_i, 0) \to \bar{f}_s(n_i) = \delta(H_0(n_i) - E) \equiv \bar{f}_c . \tag{12}$$

Here $\bar{f}_s$ is the steady-state, or statistical equilibrium density, E is the energy, and H_0 is unperturbed Hamiltonian of the completely integrable system. The latter relation (12) means that $\bar{f}_s$ is microcanonical distribution fe on an energy surface. This is called ergodicity, which is the weakest statistical property of motion.

In simple dynamical systems like our models density $\bar{f}_s \neq f_c$ is typically different from microcanonical one. The important statistical property is relaxation itself, no matter what would be the final state $\bar{f}_s$. Loosely speaking, the relaxation is equivalent to a continuous motion spectrum. The exponential instability is a sufficient condition, because the discrete spectrum can provide the linear instability at most (see above). Yet, the exponential instability is much stronger a property as compared with the mixing and relaxation. On the other hand, the latter seems to be quite enough to develop a relevant statistical theory. The point is that relaxation, or correlation decay, provides the property of statistical independence, which is the ultimate basis of the probability theory. These considerations, not yet completely understood, are most important in analyzing the conception of wave chaos below (Section 3).

Coming back to our simple models let me mention an obvious estimate for the relaxation time t_c in case of motion ergodicity. From Eqs (10, 11) we have

$$t_c \sim \frac{1}{D\varphi} \sim \frac{n^2}{D_n} \sim \varepsilon^{-2} W^{-1}. \tag{13}$$

This is the principal result in classical, or ray, dynamics to compare below with quantum, or wave, mechanics.

3. Linear Wave Chaos: Statistical Relaxation in Discrete Spectrum

The motion of a quantum particle in billiard is described by Schrödinger equation. This and other quantum equations are linear with respect to quantum dynamical variable, the Ψ-function, which represents the whole closed quantum system. Moreover, all those equations have purely discrete spectrum for any bounded motion. Hence, a common belief that dynamical chaos is impossible in quantum mechanics. But this conclusion is in apparent contradiction with the fundamental correspondence principle, which requires the complete transition to the classical limit, chaos including.

The quantum evolution equations are wave equations, and they are formally equivalent to any other (classical) wave equations like sound, elastic, electromagnetic etc. ones. Of course, the latter must be linear. For nonlinear wave equations there is no problem with dynamical chaos (see, e. g., Ref. [1]). This is why the quantum chaos is also called the wave chaos [9]. For classical waves it is especially clear that the ray dynamics contains nothing beyond the wave equation, and the nonlinear ray chaos must be present somehow in linear wave equations.

To the best of my knowledge, nobody considered so far how the ray chaos is represented and/or modified in the classical wave equations, for example, in cavities or waveguides.

Before discussing this central topic of my talk let us consider first the following question: why the wave (particularly, quantum) spectrum is discrete? The reference to linearity of the equations would be superficial. Indeed, the linear Liouville equation can have a continuous spectrum as well, which means that there are no nonsingular eigenfunctions. The latter is explained just by the arising of indefinitely dense foliations described above, which is obviously an aperiodic process.

It would be more correct to say that linear equations may be qualitatively different. But what is the nature of the difference, and how to recognize it in a given equation seems still to remain an open question.

A constructive solution of this problem could be the attempt either to determine the type of spectrum (discrete or continuous) or to find the ray approximation described by some ordinary differential equation.

The latter is apparently impossible for the Liouville equation. In case of a wave equation the rays, or characteristics, are embedded in the "phase space" of double dimensions as compared to the wave space. In the limit of geometrical optics or of classical mechanics all variables of the former are independent whereas for the wave equation each pair of variables is interconnected by a Fourier transform, and, hence, is restricted by the uncertainty relation.

To describe this situation in a different way one can say that the ray phase space (k_i, x_i) is discrete, the size of a cell being of the order of unity $(\Delta k_i \cdot \Delta x_i \sim 1)$. This is true for any finite $k \to \infty$ and qualitatively different from the very ray limit with its continuous phase space. Hence, the transition to the limit is singular and the implementation of the correspondence principle is far from trivial.

Our solution of this problem [10] (see also review [11]) is based upon introducing characteristic time scales of the wave dynamics on which the latter is close to the ray dynamics, the scales increasing indefinitely towards the ray limit $k \to \infty$. Let me show how this approach works in our simple models.

The most important relaxation time scale t_R is determined by the mean density ρ of frequency (energy) eigenvalues

$$t_R \sim \rho = \Delta^{-1}, \tag{14}$$

where Δ is the average eigenvalue spacing. This is a direct implication of the uncertainty relation $\Delta t \cdot \Delta\omega \sim 1$. Indeed, while $t \sim \Delta t \lesssim \rho$ the discreteness of spectrum is not resolved as $\Delta\omega \sim t^{-1} \gtrsim \Delta$. Hence, the perturbation can and actually does act as one with a continuous spectrum. Particularly, this produces diffusion and relaxation in the discrete spectrum!

Then, an important parameter of wave dynamics is the ratio (see Eq. (13))

$$\lambda = \frac{t_R}{t_e}, \tag{15}$$

which I call the ergodicity parameter. If $\lambda \gtrsim 1$ the ergodic microcanonical steady state $\bar{f}_e$ is reached (12) like in the ray limit. For $\lambda \lesssim 1$ we would expect a different steady state which will be discussed below.

Now, let us estimate parameter λ for our models. To this end, we observe that the total number of eigenvalues up to $n_i \sim n$ is roughly $N(\omega) \sim n^d$. Hence, density ρ and relaxation time scale t_R are

$$t_R \sim \rho = \frac{dN}{d\omega} \sim n^{d-1} \left| \frac{\partial\omega}{\partial n} \right|^{-1}. \tag{16}$$

In combination with Eq. (13), we obtain

$$\lambda \sim \varepsilon^2 n^{d-1} = (\varepsilon n)^2 n^{d-3}. \tag{17}$$

Note, that λ does not depend on dispersion relation $\omega(n_i)$ that is on a particular mechanics of waves.

But the condition $\lambda \geq 1$ is still not sufficient for the wave ergodicity. The point is that the change in action n per collision with the wall is $\Delta n_1 \sim \varepsilon n$. Since in wave mechanics all n_i are integer the transitions between unperturbed states n_i, hence, any diffusion is only possible if

$$\Delta n_1 \sim \varepsilon n \geq 1. \tag{18}$$

This condition allows another simple interpretation if we compare the scattering angle ε of a ray with the minimal diffraction angle $\mu \sim \lambda\!\!\!^{-}/a \sim 1/n$, where $\lambda\!\!\!^{-} \sim a/n$ is the wave length. Then, condition (18) means that diffraction can be neglected: $\mu \leq \varepsilon$. This is, of course, the well-known condition for geometrical optics to hold. Yet, this is also not sufficient, generally, because there is another condition (17). The relation between them depends on spatial dimensions. If $d \geq 3$, usual condition (18) is crucial for ergodicity. However, for $d = 2$ the ergodicity depends on λ only while the diffraction parameter $\varepsilon n \geq n^{1/2}$ must be very big for $n \gg 1$.

In any event the wave dynamics becomes ergodic for sufficiently large n if ray dynamics is chaotic. This implies that wave eigenfunctions are also ergodic in agreement with Shnirelman theorem announced in 1974 and proved in 1990 [12]. From Eqs (17) and (18) the ergodicity border in n or in ω is

$$n_e(\omega_e) \sim \begin{cases} \varepsilon^{-2}; & d=2, \\ \varepsilon^{-1}; & d \geq 3. \end{cases} \tag{19}$$

The transition to the ray limit depends on two parameters, wave number $n \to \infty$ and time interval $t \to \infty$. The complexity of transition is in that the result depends on the order of two limits. If, first, we take formally the limit $n \to \infty$, then for any finite $t \to \infty$ the wave dynamics becomes the ray dynamics. However, if we fix n, no matter how large, the wave behaviour remains close to the ray diffusion while $t \leq t_R(n)$ only. Hence, the ray limit is singular, indeed.

Above the ergodicity border (19), as $n \to \infty$, almost every eigenfunction, in Wigner's representation, approaches microcanonical distribution $\bar{f}_e$ (12) [12]. Moreover, the fluctuations in chaotic eigenfunctions were shown to be Gaussian [20] (see also Ref. [10]). Yet, they are not completely random. The most important elements of their microstructure are so-called "scars", that is some enhancements above the average microcanonical density along unstable periodic rays. The scars had been discovered in quantum billiard models by Heller [21] and were subsequently confirmed by many others (see, e. g., Ref. [22]). The question how to reconcile scars with Shnirelman's theorem is not completely clear so far. Apparently, all the scars are of the minimal width comparable with an elementary wave cell (Section 3). If so, the scars would not affect any quasi-classical integral relation which is Shnirelman's wave ergodicity.

4. Wave Chaos: Diffusion Localization

A more interesting question is what happens if $n \leq n_c$ or $\omega \leq \omega_c$? For $d \geq 3$ the answer is very simple: nothing happens as all the transitions are suppressed, so that the initial state of wave field doesn't change at all. It is also called perturbative localization, because the wave perturbation theory is applicable just under condition $\varepsilon n \leq 1$. In this case the eigenfunctions remain close to the unperturbed ones.

For $d = 2$ the problem is much more difficult. The first guess could be as follows. The diffusion proceeds during time interval $t_R \sim \rho$, so that the angular size of the wave packet becomes $(\Delta\varphi)^2 \sim D_\varphi t_R \sim \lambda \leq 1$.

However, there are at least two questions. First, would the diffusion stop for $t \geq t_R$? This plausible conjecture was well confirmed numerically, indeed [10, 11]. Second, would the density ρ remain unchanged if $\Delta\varphi \leq 1$? Apparently, it will not as now only a part of the unperturbed states contributes to ρ. A plausible estimate for new density and relaxation time scale is

$$\tilde{\rho} \sim \rho\,\Delta\varphi \sim \tilde{t}_R\,. \tag{20}$$

Then, from the diffusion law $(\Delta\varphi)^2 \sim D_\varphi\,\tilde{t}_R$, we obtain

$$(\Delta\varphi)_s \sim \varepsilon^2 n; \quad (\Delta n)_s \sim D_n' \sim l_s\,, \tag{21}$$

which is much smaller than the first guess. In the second estimate (21) $D_n' \sim (\varepsilon n)^2$ is the diffusion rate per collision with wall. This estimate was also well confirmed numerically in somewhat different but related models [10, 11].

The size l_s is called also the localization length as it characterizes suppression, or localization, of the ray diffusion. Note, that condition (18) must be satisfied, otherwise the perturbative localization occurs with minimal

$$(\Delta\varphi)_p \sim \frac{1}{n}; \quad (\Delta n)_p \sim 1. \tag{22}$$

For $\varepsilon n \geq 1$ a very peculiar steady state is formed whose size is given by estimate (21). As in the ray limit this steady state is a result of statistical (diffusion) relaxation $\bar{f}(n, 0) \to \bar{f}_s(n)$ but it differs from the limiting microcanonical distribution $f_c(n)$ in two major ways.

First, the wave steady state depends on the initial state as the former is a result of diffusion localization of the latter. Hence, the steady state is always spread around the initial state. Second, the steady state is composed of a finite number M of eigenfunctions owing to the discreteness of the wave phase space. Roughly, $M \sim l_s$ (21), hence, statistical fluctuations are finite, $\sim M^{-1/2} \sim (\varepsilon n)^{-1}$. One can say also that wave phase density $\bar{f}(n, t)$ represents always a finite ensemble of $\sim M$ systems [11]. Fluctuations of the classical electromagnetic field in wave-guides were discussed, e. g., in Ref. [16], without any relation to the wave chaos though.

This remains true for an ergodic steady state as well, for any λ and d, when $M \sim n^{d-1}$. This estimate is inferred from the only restriction $\omega(n_i) \approx$ const for the ergodic steady state and eigenfunctions. In turn, it implies a finite width $\Delta\omega$ of the energy shell occupied by an ergodic state. Using the last estimate and Eq. (16), we obtain

$$\Delta\omega \sim \Omega\,. \tag{23}$$

As $n \to \infty$ the relative width $\Delta\omega/\omega \to 0$ because $\omega(n) \to \infty$ in our units ($\hbar = 1$), and the energy shell becomes an energy surface of the ray dynamics. Estimate (23) is also valid for nonergodic steady states.

Even on the relaxation time scale t_R the wave behaviour is qualitatively different from the limiting ray dynamics in that the former is perfectly stable whereas the latter is strongly unstable. It was proved in numerical (computer) experiments via time reversal in both cases [13]. For the ray dynamics the diffusion is immediately restored due to a fast growth of computation errors while for waves the "antidiffusion" proceeds down to the initial state which is recovered with surprisingly high accuracy.

At a first glance, it seems to contradict to the correspondence principle. The resolution is in that there exists another characteristic time scale t_{BZ} on which the wave dynamics is also unstable, yet $t_{BZ} \ll t_R$. It was discovered and explained by Berman and Zaslavsky [14] (see also Refs [10, 11]). The instability manifests itself in rapid spreading of a narrow wave packet which follows for a while the beam of rays according to the Ehrenfest theorem. In case of exponential instability the corresponding time scale is very short. Roughly (see Eq. (7)),

$$t_{BZ} \approx \frac{\ln n}{\Lambda} \ll t_R\,. \tag{24}$$

Nevertheless, $t_{BZ} \to \infty$ as $n \to \infty$ in accordance with the correspondence principle. Yet, this transition, as well as that for t_R discussed above, is singular, being also a double limit in n and t.

A nice picture of the initially unstable wave evolution can be found in Ref. [15] where the formation of the foliation in phase space, described above in Section 2, is clearly seen.

5. The Nature of Wave and Quantum Chaos

The principal distinction of the wave chaos is its transient character. In other words, the wave dynamics remains close to the ray chaotic motion on a finite time scale only [10]. Moreover, particular statistical properties of the wave evolution correspond to rather different time scales. The strongest ones related to the exponential instability persist on the shortest time scale (24) which is almost independent on integer wave numbers n. Yet, it is important that such significant statistical behaviour as diffusion and relaxation continues much longer, on time scale (16), thus providing a relevant statistical description of wave dynamics. This is very important in many applications as it allows for a fairly simple statistical representation of the essential features of otherwise highly complicated phenomena.

From mathematical point of view the problem of wave chaos requires a fundamental generalization of the contemporary ergodic theory [6] which was developed for the ray chaos, and which is essentially asymptotic in time ($|t| \to \infty$). Note the considerable simplification of the theory due to that asymptotic approach.

To emphasize the finite-time nature of the wave chaos we call it pseudochaos as distinct from the true ray chaos. The ultimate origin of wave chaos limitations is in the discrete phase space as explained above (Section 3).

The same is true, by the way, for the computer simulation of any dynamical systems. Moreover, in the digital computer any quantity is "quantized" that is represented by an integer number, whereas in the wave mechanics only the product of each pair of canonically conjugated variables does so. As a result any dynamical trajectory in computer eventually becomes periodic as compared with almost periodic evolution of waves. Note, that it is the effect of round-off "errors" which are not random at all. This should be taken into account in computer simulations of ray dynamics. The shortest time scale (24) is of the order of length n_m (the number of bits) of computer word (mantissa), and it is negligible in most cases. Fortunately, the relaxation time scale (16) grows like some power of the maximal computer number ($\sim 2^{n_m}$) which is typically not a serious restriction (for details see, e. g., Refs [10, 17]).

Coming back to wave dynamics I would like to stress that even though from formal mathematical point of view all linear wave equations have similar properties the physics of quantum waves is fundamentally different. While most classical linear waves are simply a low-amplitude approximation (an important exception is electromagnetic waves) the linear quantum mechanics is as yet the most fundamental and universal theory. Hence, the "true" classical chaos is but a limiting pattern, very important in the theory but never realized, strictly speaking, in nature.

On the other hand, the evolution of Ψ wave is only a part, and a simpler one, of the quantum dynamics as a physical theory. The other part, much more difficult and vague, is the quantum measurement with its misterious Ψ collapse. It is not excluded that the latter is the most spectacular example of the true quantum chaos (for discussion see, e. g., Refs [11, 18, 19]).

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