# The Problem of Quantum Chaos

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Abstract: The new phenomenon of quantum chaos has revealed the intrinsic complexity and richness of the dynamical motion with discrete spectrum which had been always considered as most simple and regular one. The mechanism of this complexity as well as the conditions for, and the statistical properties of, the quantum chaos are explained in detail using a number of simple models for illustration. Basic ideas of a new ergodic theory of the finite-time statistical properties for the motion with discrete spectrum are discussed.

# 1. Introduction: the theory of dynamical systems and statistical physics

The purpose of these lectures is to provide an introduction into the theory of the so-called quantum chaos, a rather new phenomenon in the old quantum mechanics of finite-dimensional systems with a given interaction and no quatized fields. The quantum chaos is a "white spot" far in the rear of the contemporary physics. Yet, in opinion of many physicists, including myself, this new phenomenon is, nevertheless, of a great importance for the fundamental science because it helps to elucidate one of the "eternal" questions in physics, the interrelation of dynamical and statistical laws in the Nature. Are they independently fundamental? It may seem to be the case judging by the striking difference between the two groups of laws. Indeed, most dynamical laws are time-reversible while all the statistical ones are apparently not with their notorious "time arrow". Yet, one of the most important achievements in the theory of the so-called dynamical chaos, whose part is the quantum chaos, was understanding that the statistical laws are but the specific case and, moreover, a typical one, of the nonlinear dynamics. Particularly, the former can be completely derived, at least in principle, from the latter. This is just one of the topics of the present lectures.

Another striking discovery in this field was that the opposite is also true! Namely, under certain conditions the dynamical laws may happen to be a specific case of the statistical laws. This interesting problem lies beyond the scope of my lectures, so I just mention a few examples. These are Jeans' gravitational instability, which is believed to have been responsible for the formation of stars and eventually of the celestial mechanics (the exemplary case of dynamical laws!); Prigogine's "dissipative structures" in chemical reactions; Haken's "synergetics"; and generally, all the so-called "collective instabilities" in fluid and plasma physics (see, e. g., Ref. [1-3]). Notice, however, that all the most fundamental laws in physics (those in quantum mechanics and quantum field theory) are, as yet, dynamical and, moreover, exact (within the boundaries of existing theories). To the contrary, all the *secondary laws*, both statistical ones derived from the fundamental dynamical laws and vice versa, are only approximate.

By now the two different, and even opposite in a sense, mechanisms of statistical laws in dynamical systems are known and studied in detail. They are outlined in Fig. 1 to which we will repeatedly come back in these lectures. The two mechanisms belong to the opposite limiting cases of the general theory of Hamiltonian dynamical systems. In what follows we will restrict ourselves to the Hamiltonian (nondissipative) systems only as more fundamental ones. I remind that the dissipation is introduced as either the approximate description of a many-dimensional system or the effect of external noise (see Ref.[103]). In the latter case the system is no longer a pure dynamical one which, by definition, has no random parameters.

The first mechanism, extensively used in the traditional statistical mechanics (TSM), both classical and quantal, relates the statistical behavior to a big number of freedoms  $N \to \infty$ . The latter is called thermodynamic limit, a typical situation in macroscopic molecular physics. This mechanism had been guessed already by Boltzmann, who termed it "molecular chaos", but was rigorously proved only recently (see, e. g., Ref. [4]). Remarkably, for any finite N the dynamical system remains completely integrable that is it possesses the complete set of N commuting integrals of motion which can be chosen as the action variables I. In the existing theory of dynamical systems this is the highest order in motion. Yet, the latter becomes chaotic in the thermodynamic limit. The mechanism of this drastic transformation of the motion is closely related to that of the quantum chaos as we shall see.

The second mechanism for statistical laws had been conjectured by Poincare at the very beginning of this century, not much later than Boltzmann's one. Again, it took half a century even to comprehend the mechanism, to say nothing about the rigorous mathematical theory (see, e.g., Refs.[4-6]). It is based on a strong local instability of motion which is characterized by the Lyapunov exponents for the linearized motion. The most important implication is that the number of freedoms N is irrelevant and can be as small as N = 2 for a conservative system, and even N = 1 in case of a driven motion

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Figure 1: The place of quantum chaos in modern theories: action-angle variables  $I, \theta$ ; number of freedoms N; Lyapunov's exponent  $\Lambda$ ; quasiclassical parameter q; Planck's constant  $\hbar$ . Two question marks indicate the problems in a new ergodic theory nonasymptotic in N and |t|.

that is one whose Hamiltonian explicitly depends on time. In the latter case the dependence is assumed to be regular, of course, for example periodic, and not a sort of noise.

This mechanism is called dynamical chaos. In the theory of dynamical systems it constitutes another limiting case as compared to the complete integrability. The transition between the two cases can be described as the effect of "perturbation"  $\varepsilon V$  on the unperturbed Hamiltonian  $H_0$ , the full Hamiltonian being

$$H(I,\theta,t) = H_0(I) + \varepsilon V(I,\theta,t)$$
(1.1)

where  $I, \theta$  are N-dimensional action-angle variables. At  $\varepsilon = 0$  the system is

completely integrable, and the motion is quasiperiodic with N basic frequencies

$$\omega(I) = \frac{\partial H_0}{\partial I} \tag{1.2}$$

Depending on initial conditions (I(0)) the frequencies may happen to be commensurable, or linearly dependent, that is the scalar product

$$m, \omega(I) = 0 \tag{1.3}$$

where m is integer vector.

This is called *nonlinear resonance*. The term nonlinear means the dependence  $\omega(I)$ . The interaction of nonlinear resonances (because of nonlinearity) is the most important phenomenon in nonlinear dynamics. The resonances are precisely the place where chaos is born under arbitrarily weak perturbation  $\varepsilon > 0$ . Hence the term *universal instability* (and chaos) of nonlinear oscillations [6]. The structure of motion is generally very complicated (fractal), containing an intricate mixture of both chaotic and regular motion components which is also called *divided phase space*. According to the Kolmogorov—Arnold—Moser (KAM) theory, for  $\varepsilon \to 0$ , most trajectories are regular (see, e. g., Ref. [7]). The measure of the complementary set of chaotic trajectories is exponentially small (~  $\exp(-c/\sqrt{\varepsilon})$ ), hence the term *KAM integrability* [8]. Yet, it is everywhere dense as is the full set of resonances (1.3). A very intricate structure!

Even though the mathematical theory of dynamical systems looks very general and universal it actually has been built up on the basis of, but of course is not restricted to, the classical mechanics with its limiting case of the dynamical chaos. The quantum mechanics as described by some dynamical equations, for example, Schrödinger's one, for a specific dynamical variable  $\psi$  well fits the general theory of dynamical systems but turns out to belong to ... the limiting case of regular, completely integrable motion.

This is because the energy (frequency) spectrum of any quantum system bounded in phase space is always discrete and, hence, its time evolution is almost periodic. The ultimate origin of this quantum regularity is discreteness of the phase space itself inferred from the most fundamental uncertainty principle which is the very heart of the quantum mechanics. In modern mathematical language it is called *noncommutative geometry* of the phase space. Hence, the full number of quantum states within a finite domain of phase space is also finite. Then, what about chaos in quantum mechanics?

On the first glance, this is no surprise since the quantum mechanics is well known to be fundamentally different as compared to the classical mechanics. However, the difficulty, and a very deep one, arises from the fact that the former is commonly accepted to be the universal theory, particularly, comprising the latter as the limiting case. Hence, the correspondence principle which requires the transition from quantum to classical mechanics in all cases including the dynamical chaos. Thus, there must exist a sort of quantum chaos!

Of course, one would not expect to find any similarity to classical behavior in essentially quantum region but only sufficiently far in the quasiclassical domain. Usually, it is characterized formally by the condition that Planck's constant  $\hbar \to 0$ . I prefer to put  $\hbar = 1$  (which is the question of units), and to introduce some (big) quantum parameter q. Generally, it depends on a particular problem, and may be, for instance, the quantum (level) number. The quasiclassical region then corresponds to  $q \gg 1$  while in the limit  $q \to \infty$ the complete rebirth of the classical mechanics must occur somehow.

Notice that unlike other theories (of relativity, for example) the quasiclassical transition is rather intricate. Actually, this is the main topic of these lectures. Thus, the quantum chaos we are going to discuss is essentially a quasiclassical phenomenon in finite (essentially few-dimensional) systems with bounded motion. These restrictions are very important to properly understand the place of the new phenomenon - quantum chaos - in the general theory of dynamical systems, and to distinguish the former from the old mechanism for statistical laws in infinite systems  $N \to \infty$ . The latter nature is sometimes well hidden in a particular model as, for example, the nonlinear Schrödinger equation (Lecture 8).

The number of papers devoted to the studies of quantum chaos and related phenomena is rapidly increasing, and it is practically impossible to comprise everything in this field. In what follows I have to restrict myself to some selected topics which I know better or which I myself consider as more important. The same is true for references. I apologize beforehand for possible omissions and inaccuracies. Anyway, I refer in addition to a number of recent reviews [9-14], and to these proceedings.

My presentation below will be from a physicist's point of view even though the whole problem of quantum chaos, as a part of quantum dynamics, is essentially mathematical.

The main contribution of physicists to the studies of quantum chaos is in extensive numerical (computer) simulations of quantum dynamics, or numerical experiments as we use to say. But not only that. First of all, numerical experiments are impossible without a theory, if only semiqualitative, and without even rough estimates to guide the study. Mathematicians may consider such physical theories as a collection of hypotheses to prove or disprove them. What is even more important, in my opinion, that those theories require, and are based upon, a set of new notions and concepts which may be also useful in a future rigorous mathematical treatment.

I would like to mention that with all their obvious drawbacks and limitations the numerical experiments have very important advantage (as compared to the laboratory experiments), namely, they provide the complete information about the system under study. In quantum mechanics this advantage becomes crucial because in the laboratory one cannot observe (measure) the quantum system without a radical change of dynamics.

We call numerical experiments the *third way of cognition* in addition to traditional theoretical analysis, and to the main source of the knowledge and the Supreme Judge in science, the Experiment.

Laboratory experiments are vitally important for the progress in science not simply to prove or disprove some theories but to eventually discover, on a very rare occasion though, new fundamental laws of nature which are taken for granted in numerical experiments and theoretical analysis.

As an illustration of dynamical chaos, both classical and quantal, I will make use of the following "simple" model. In the classical limit it is described by the so-called *standard map*:  $(n, \theta) \rightarrow (\bar{n}, \bar{\theta})$  where

$$\bar{n} = n + k \cdot \sin \theta; \qquad \bar{\theta} = \theta + T \cdot \bar{n}$$
 (1.4)

Here  $n, \theta$  are the action-angle dynamical variables; k, T stand for the strength and period of perturbation. Notice that in full dimensions parameter T is actually  $\omega T/n_0$  where  $\omega$  is the perturbation frequency, and  $n_0$  stands for some characteristic action. The phase space of this model is an infinite cylinder which can be also "rolled up" into a torus of circumference

$$C = \frac{2\pi m}{T} \tag{1.5}$$

with an integer m to avoid discontinuities. Notice that map (1.4) is periodic not only in  $\theta$  but also in n with period  $2\pi/T$ . The latter is a nongeneric symmetry of this model. In the studies of general chaotic properties it is a disadvantage. Nevertheless, the model is very popular, apparently because of its formal and technical symplicity combined with the actual richness of behavior. It can be interpreted as a mechanical system—the rotator driven by a series of short impulses, hence the nickname—"kicked rotator".

The quantized standard map was first introduced and studied in Ref. [15]. It is described also by a map:  $\psi \to \overline{\psi}$  where

$$\bar{\psi} = \hat{R}_T \hat{F}_k \psi \tag{1.6}$$

and where

$$\hat{F}_k = \exp(-ik \cdot \cos \hat{\theta}), \qquad \hat{R}_T = \exp\left(-i\frac{T\hat{n}^2}{2}\right)$$
 (1.7)

are the operators of a "kick" and of a free rotation, respectively. Momentum operator is given by the usual expression:  $\hat{n} = -i\partial/\partial\theta$ .

Sometime it is more convenient to use the symmetric map

$$\bar{\psi} = \hat{R}_{T/2} \hat{F}_k \hat{R}_{T/2} \psi \tag{1.8}$$

which differs from Eq. (1.6) by the time shift T/2, and which is, moreover, time reversible. In the most interesting case of a strong perturbation  $(k \gg 1)$  the operator  $\hat{F}_k$  couples approximately 2k unperturbed states. Also, parameter T can be considered as an effective "Planck's constant" [103].

Notice that in classical limit the motion of model (1.4) depends on a single parameter K = kT but after quantization the two parameters, k and T, can not be combined any longer.

Even though the standard map is primarily a simple mathematical model it can serve also to approximately describe some real physical systems or, better to say, some more realistic models of physical systems. One interesting example is the peculiar diffusive photoeffect in Rydberg (highly excited) atoms (see, e. g., Refs [14, 16, 104] for review).

The simplest 1D model is described by the Hamiltonian (in atomic units):

$$H = -\frac{1}{2n^2} + \varepsilon \cdot z(n,\theta) \cos \omega t \tag{1.9}$$

where z stands for the coordinate along the linearly polarized electric field of strength  $\varepsilon$  and frequency  $\omega$ .

Another approach to this problem is constructing a map over a Kepler period of the electron [17]:  $(N_{\Phi}, \phi) \rightarrow (\bar{N}_{\Phi}, \bar{\phi})$  where

$$\bar{N}_{\Phi} = N_{\Phi} + k \cdot \sin \phi; \qquad \bar{\phi} = \phi + \frac{\pi}{(2\omega)^{1/2}} (-\bar{N}_{\Phi})^{-3/2}$$
(1.10)

Here,  $N_{\Phi} = E/\omega = -1/2\omega n^2$ , and perturbation parameter

$$k \approx 2.6 \frac{\varepsilon}{\omega^{5/3}} \tag{1.11}$$

if the field frequency exceeds that of the electron:  $\omega n^3 \gtrsim 1$ .

Linearizing the second Eq. (1.10) in  $N_{\Phi}$  reduces the Kepler map to the standard map with the same k, and parameter

$$T = 6\pi\omega^2 n^5 \tag{1.12}$$

Thus, the standard map describes the dynamics locally in momentum. In this particular model momentum  $N_{\Phi}$  is proportional to energy as the conjugate phase  $\phi = \omega t$  is proportional to time.

In quantum mechanics, instead of solving Schrödinger's equation with Hamiltonian (1.9) one can directly quantize a simple Kepler map (1.10) to arrive at a quantum map (1.6) with the same perturbation operator  $\hat{F}_k$  (1.7) but with a different rotation operator

$$\hat{R}_{\nu} = \exp(-2i\pi\nu(-2\omega\hat{N}_{\Phi})^{-1/2})$$
(1.13)

Here parameter  $\nu = 1$  (one Kepler's period) for quantum map (1.6), and  $\nu = 1/2$  for symmetric map (1.8).

Notice that in Kepler map's description a new time  $(\tau)$  is discrete (the number of map's iterations), and moreover, its relation to the continuous time t in Hamiltonian (1.9) depends on dynamical variable n or  $N_{\Phi}$ :

$$\frac{dt}{d\tau} = 2\pi n^3 = 2\pi (-2\omega N_{\Phi})^{-3/2}$$
(1.14)

In quantum mechanics such a change of time variable constitutes the serious problem: how to relate the two solutions,  $\psi(t)$  and  $\psi(\tau)$ ? For further discussion of this problem see Ref. [14]. Besides, map's solution  $\psi(N, \tau)$  does not provide the complete quantum description but only some averaged one over the groups of unperturbed states [17].

These difficulties are of a general nature in attempts to make use of the Poincaré map for conservative quantum systems. The straightforward approach would be, first, to solve the Schrödinger equation, and then to construct the quantum map out of  $\psi(t)$ . Usually, this is a very difficult way. Much simpler one is, first, to derive the classical Poincaré map, and then to quantize it. However, generally the second way provides only an approximate solution for the original system. The question is how to reconcile the both approaches?

Another physical problem—the Rydberg atom in constant and uniform magnetic field, I will refer to below, is described by the Hamiltonian (for review see Ref. [18]):

$$H = \frac{p_z^2 + p_\rho^2}{2} - \frac{1}{r} + \frac{\omega L_z}{2} + \frac{\omega^2 \rho^2}{8}$$
(1.15)

Here  $r^2 = \rho^2 + z^2 = x^2 + y^2 + z^2$ ;  $\omega$  is the Larmor frequency in the magnetic field along z axis, and  $L_z$  stands for the component of angular momentum (in atomic units). Unlike the previous model the latter one is conservative (energy preserving). It is simpler for theoretical studies and, hence, more

popular among mathematicians. Physicists prefer time-dependent systems or, to be more precise, the models described by maps which greatly facilitate numerical experiments.

An important class of conservative models are billiards, both classical and quantal [19–21, 9, 105]. Especially popular is the billiard model called "stadium" [20]. Interestingly, instead of a quantum  $\psi$  wave one may consider classical linear waves, e. g., electromagnetic, sound, elastic etc. In the latter case the billiard is called "cavity". Of course, this problem has been studied since long ago, yet only recently it was related to the brand-new phenomenon of "quantum" chaos [22, 23] (see also Refs.[105, 106].

Quantum (wave) billiards are the limiting (and a simpler) case of the general dynamics of linear waves in dispersive media. It seems that the case of a spatially random medium does attract the most attention in this field. A striking example is the celebrated phenomenon of the Anderson localization. True, this is a statistical rather than dynamical problem. On the other hand, one may consider the random potential as a typical one, and the averaged solution as the representation of typical properties in such systems. Instead, in the spirit of the dynamical chaos, one can extend the problem in question onto a class of regular (but not periodic) potentials.

Recently, a deep analogy has been discovered between this rather old problem of wave dynamics in configurational space (in a medium) and of the dynamics in momentum space, particularly, the excitation of a quantum system by driving perturbation [24, 25]. Remarkably, that while the latter problem is described by a time-dependent Hamiltonian the former is a conservative system. This interesting and instructive similarity is discussed in Ref. [26].

# 2. Asymptotic statistical properties of classical dynamical chaos

To understand the phenomenon of quantum chaos it should be put into the proper perspective of recent developments in physics. The central focus of this perspective is the conception of classical dynamical chaos which has destroyed the deterministic image of the classical physics. What is the dynamical chaos? Which should be its meaningful definition?

This is one of the most controversial questions even in classical mechanics. There are two main approaches to the problem. The first one is essentially mathematical [4, 7]. The terms dynamical chaos and randomness are abandoned from rigorous statements, and left for informal explanations only,



Figure 2: A fractal nonergodic motion component for the standard map, K = 1.13 (a); almost ergodic motion, K = 5 (b). Each hatched region is occupied by a single trajectory (after Ref.[14]).

usually in quotes, even in Ref. [27] where a version of the rigorous definition of dynamical randomness (chaos) was actually given. This is not the case in Chaitin's papers (see, e. g., Ref. [28]) but his approach is somewhat separated from the rest of ergodic theory, and is related to a new, *algorithmic theory* of dynamical systems started in the sixties by Kolmogorov (see Refs [27, 28]).

In the mathematical approach to the definition of dynamical chaos a hierarchy of statistical characteristics, such as ergodicity, mixing, K, Markov and Bernoulli properties etc, is introduced. In this hierarchy each property supposed to imply all the preceeding ones (see Fig. 1). However, the latter is not the case in the very important and fairly typical situation when the motion is restricted to a *chaotic component* usually of a very complicated (fractal) structure which occupies only a part of the energy surface in a conservative system or even a submanifold of lesser dimensions (see, e. g., Ref. [29]).

In Fig. 2a an example of the fractal chaotic component for the standard map is shown [14]. The motion is not ergodic as a chaotic trajectory covers about a half of the phase plane only (cf. Fig. 2b for a bigger perturbation K with only tiny islets of stability filled up by regular trajectories). For still bigger K the motion looks like completely ergodic. However, this has not been as yet rigorously proved. Numerical experiments are also not a reliable proof, at least not the direct one, because in computer representation any quantity is discrete. An indirect indication is the dependence of measured chaotic area  $\mu_c$  on the spatial resolution (discreteness)  $\Delta$ . Numerically [30]



Figure 3: Normalized distribution function  $f_n(E)$  in the standard map for various time intervals. The straight line is theoretical dependence  $f_n = \exp(-E)$ ;  $E = (\Delta n)^2 / \tau k^2$ ; statistical errors are shown in a few cases (after Ref.[6]).

$$\mu_c(\Delta) \approx \mu_c(0) + \alpha \Delta^{\beta} \tag{2.1}$$

with nonzero  $\mu(0)$  and fractal exponent  $\beta \approx 0.5$ .

Being nonergodic the motion in the hatched domain in Fig. 2*a* is nonintegrable as the trajectory fills up a finite area of  $\mu(0) \neq 0$ . Hence, no motion integrals exist in this region. From the physical viewpoint there is a good reason to term such a motion chaotic. Anyway, the ergodicity, being the weakest statistical property, is neither necessary nor sufficient for the meaningful statistical description.

In this respect the most important property is mixing that is the correlation decay in time. It implies statistical independence of different parts of a trajectory as the separation in time between them becomes large enough. The statistical independence is the crucial property for the probability theory to be really applicable [31]. Particularly, the central limit theorem predicts Gaussian fluctuations which is, indeed, in a good agreement with the numerical data for the standard map (Fig. 3).

At average, the motion is described by the diffusion equation (also a



Figure 4: Classical (circles) and quantum (crosses) diffusion in the standard map; solid line is a simple theory;  $D_0 = k^2/2$  (after Refs.[32, 33]).

typical statistical law) with the rate [32]

$$D \equiv \frac{\langle (\Delta n)^2 \rangle}{\tau} = \frac{k^2}{2} \kappa(K)$$
 (2.2)

where function  $\kappa(K)$  accounts for short-time correlations [33] (see Fig. 4).

The property of mixing is equivalent to continuous power spectrum of the motion which is the Fourier transform of the correlation function. This is just sufficient to provide the meaningful statistical description with its most important process of *relaxation* for an arbitrary initial distribution function  $f(n,0) \rightarrow f_s(n)$  to some unique steady state. In case of the standard map on a torus, for example, the latter is ergodic

$$f_s(n) = f_e(n) = \frac{1}{C}$$
 (2.3)

if  $K \gg 1$  is big enough. The relaxation is asymptotically exponential [14]

$$f(n,\tau) - \frac{1}{C} \approx \exp\left(-\frac{2\pi^2 |\tau| D}{C^2}\right) \cdot \cos\left(\frac{2\pi n}{C}\right), \quad |\tau| \to \infty$$
 (2.4)

with characteristic relaxation time



Figure 5: Statistics of Poincaré recurrences in discrete spectrum (regular motion):  $N_{\omega} = 5, < \tau >= 6.3$  (squares);  $N_{\omega} = 10, < \tau >= 10$  (triangles);  $N_{\omega} = 100, < \tau >= 5.4$  (crosses);  $\omega_1 = 1$ .

$$\tau_e = \frac{C^2}{2\pi^2 D} \tag{2.5}$$

Notice that both diffusion and statistical relaxation proceed in two directions of time. The theory of dynamical chaos does not need the popular but superficial conception of "time arrow". True, the corresponding diffusion equation

$$\frac{\partial f(n,\tau)}{\partial \tau} = \frac{1}{2} \frac{\partial}{\partial n} D \frac{\partial f}{\partial n}$$
(2.6)

is irreversible in time. However, this is simply because the distribution function  $f(n, \tau)$  is a coarse-grained phase density, averaged over phase  $\theta$ . The finegrained (exact) phase density  $f(n, \theta, \tau)$  obeys the Liouville equation which is time-reversible as are the motion equations. Being time-reversible the statistical relaxation is nonrecurrent that is even the exact phase density  $f(n, \theta, \tau)$ would never come back to the initial  $f(n, \theta, 0)$ . Unlike this almost all trajectories are recurrent, according to the Poincaré theorem, independent of the type of motion (regular or chaotic). The difference is in the distribution of recurrence times: in discrete spectrum this time is strictly bounded from above while for chaotic motion an arbitrary long recurrence time can occur with some probability.

In Fig. 5 an example of the statistics for Poincare's recurrences is shown in regular motion with  $N_{\omega}$  incommensurable frequencies randomly distributed within the interval  $(0, \omega_1)$ . Numerically [34], the upper bound is approximately

$$\frac{\tau_{max}}{\langle \tau \rangle} \approx 0.8 N_{\omega} = 0.8 \omega_1 \rho_{\omega} \tag{2.7}$$

where  $\langle \tau \rangle \approx 6/\omega_1$  is the mean recurrence time under given conditions (particularly, for a given set of frequencies), and  $\rho_{\omega} = N_{\omega}/\omega_1$  is the density of frequencies. The latter quantity is going to play the central role in the problem of quantum chaos.

For  $\tau < \tau_{max}$  the distribution function is close to exponential

$$P(\tau) = e^{-\tau/\langle \tau \rangle} \tag{2.8}$$

as predicted by the probability theory for a random process with continuous spectrum which is the limit for  $N_{\omega} \to \infty$ . Actually, in the above example, the spectrum is discrete but this apparently crucial property turns out to only restrict the random behavior to a finite time interval proportional to the frequency density.

Typically, chaotic motion possesses much stronger statistical properties than mixing. Here we come to the second approach to the definition of dynamical chaos which is essentially physical (see, e. g., Refs [5, 6]). In this approach the conception of random trajectories in a dynamical system is introduced from the beginning, and it is related to the strong (exponential) *local instability* of motion. This is characterized by a positive Lyapunov's exponent  $\Lambda$  or, more generally, by the Kolmogorov—Sinai (KS) dynamical entropy [4].

The main difficulty here is in that the instability itself is not sufficient for chaotic motion. One additional condition is boundedness of the motion to exclude, for example, the hyperbolic motion which hardly can be termed chaotic. Further, the separated unstable periodic trajectories must be also excluded, possibly, by the requirement of some minimal dimensions of a chaotic component. To the best of my knowledge, the complete set of conditions for an arbitrary motion component to be considered chaotic has not been found as yet, and it constitutes a difficult problem. Nevertheless, such a difinition of classical dynamical chaos is commonly accepted in the physical literature.

The crucial quantity  $\Lambda$  characterizes linearized equantions of motion. For example, in the standard map these are

$$\bar{\eta} = \eta + k \cdot \cos \theta(\tau) \cdot \xi; \qquad \bar{\xi} = \xi + T \cdot \bar{\eta}$$
(2.9)

where new dynamical variables,  $\xi = d\theta$  and  $\eta = dn$ , form the additional tangent space. Lyapunov's exponent is defined by the limit

$$\Lambda = \lim_{|\tau| \to \infty} \frac{1}{|\tau|} \ln v(\tau) > 0 \tag{2.10}$$

where  $v^2 = \xi^2 + \eta^2$ , and v(0) = 1 is assumed. The last inequality in (2.10) means exponential instability of motion. The instability is time reversible as well as  $\Lambda$ . Actually, there are two (for a 2D map)  $\Lambda$  of opposite signs  $(\Lambda_1 + \Lambda_2 \equiv 0)$ . The latter condition is equivalent to the area preservation in Hamiltonian systems.

In the standard map [6]

$$\Lambda \approx \begin{cases} 0.07K & K \ll 1\\ \ln \frac{K}{2} & K \gg 1 \end{cases}$$
(2.11)

The first expression holds, of course, within the chaotic component of motion only which decomposes, for  $K \ll 1$ , into infinitely many exponentially narrow chaotic layers  $(\Delta n \lesssim \exp(-\pi^2/\sqrt{K}))$ .

Remarkably, the main condition for chaos  $(\Lambda > 0)$  is related to *linear* equations (2.9) with time-dependent coefficients though. As this dependence  $(\theta(\tau))$  is very complicated for a chaotic motion, the mathematical analysis of those linear equations is almost as difficult as that of the original nonlinear ones. However, numerically the criterion  $\Lambda > 0$  is much simpler than, say, the spectrum or correlation function as the former requires much shorter computation time because the instability is fast. Actually, one needs to discern between the exponential and linear instabilities. The latter is always present in nonlinear oscillations due to the dependence of motion frequencies on initial conditions (see Eq. (1.2) and Refs [35, 36]).

According to the algorithmic theory of dynamical systems the information J(t) associated with the chaotic trajectory of length t is asymptotically

$$\frac{J(t)}{|t|} \to \Lambda; \quad t \to \pm \infty \tag{2.12}$$

that is just proportional to the rate of exponential instability. This is the most important implication of the Alekseev—Brudno theorem (see Ref. [27]). It means that for each new time interval one needs a new information which cannot be extracted from the measurement, to arbitrarily high but finite accuracy  $\nu > 0$ , of any preceeding section of the trajectory (even the infinite one!).

Obviously, over some finite time interval the prediction of a chaotic trajectory is possible depending on the *randomness parameter* [37]

$$r = \frac{\Lambda |t|}{|\ln \nu|} \tag{2.13}$$

Prediction is restricted to a finite domain of temporary determinism  $(r \leq 1)$  which goes over, as r increases, to the infinite region of asymptotic randomness  $(r \gg 1)$ . Notice that the average information per unit time (2.12) does not depend on the measurement accuracy  $\nu > 0$ .

For the regular motion with discrete spectrum the specific information decreases with time

$$\frac{J(t)}{|t|} \to \frac{\ln|t|}{|t|} \tag{2.14}$$

and the prediction is asymptotically possible contrary to the conclusion in Ref. [35].

Another way to understand the requirement of exponential instability for chaos is to consider the so-called symbolic dynamics (see, e.g., Ref.[27]) which is a mathematical description of the trajectory recording to a finite accuracy. For a map the total number of symbolic trajectories  $M^{\tau} = \exp(\tau \ln M)$  grows exponentially in time where the total number of symbols  $M \sim 1/\nu$  determines the measurement accuracy. If the motion instability is also exponential, then all the symbolic trajectories are realized for a sufficiently large map's period  $T \gtrsim (\ln M)/\Lambda$ .

The ultimate origin of the complexity (particularly, unpredictability) of a chaotic trajectory lies in the continuity of the phase space in classical mechanics. This is no longer true in quantum mechanics which leads to the most important peculiarity of the quantum chaos.

On the first glance the important condition for chaos  $\Lambda > 0$  is not invariant with respect to the change of time. To avoid this difficulty the instability should be considered not in time but rather in the oscillation phase, e. g.  $\theta$  for the standard map, or per map's iteration like in Eq. (2.11). In other words, the appropriate quantity is a dimensionless entropy, e.g.,  $\Lambda \to \Lambda / < \omega >$ where  $< \omega >$  is some average frequency of the motion.

To summarize, the physical definition reads: the dynamical chaos is exponentially unstable motion bounded, at least, in some variables.

Remarkably, the instability is determined from the linear equations, the role of nonlinearity being to bound the unstable motion. On the other hand, any motion can be described equivalently by the linear Liouville equation for the fine-grained distribution function or phase space density. Being a stronger statistical property the exponential instability implies the continuous spectrum and, hence, the correlation decay. Yet, the latter is not always expotential but may be instead a power-law one (see, e. g., Ref. [29]).

The role of exponential instability in the statistical description of dynamical systems is not completely clear, it seems to be only sufficient but not a necessary condition. Nevertheless, the conception of random trajectories of a purely dynamical system is of the fundamental importance as it destroys the mysterious image of the random and opens the way for quantitative studies in this large part of natural phenomena. Indeed, the theory of dynamical chaos shows that the random processes are not controlled by some qualitatively different laws, to account for by means of some additional statistical hypotheses, but constitute a very specific, even though typical, part of general dynamics. An interesting question if there are "more random", or "true random", processes remains, as yet, open.

### 3. The correspondence principle and quantum chaos

Absence of the claassical-like chaos in quantum mechanics apparently contradicts not only with the correspondence principle, as mentioned above, but also with the fudamental statistical nature of quantum mechanics. However, even though the random element in quantum mechanics ("quantum jumps") is inavoidable, indeed, it can be singled out and separated from the proper quantum processes. Namely, the fundamental randomness in quantum mechanics is related only to a very specific event – the quantum measurement – which, in a sense, is foreign to the proper quantum system itself.

This allows to divide the whole problem of quantum dynamics into two qualitatively different parts: (i) the proper quantum dynamics as described by the wave function  $\psi(t)$ , and (ii) the quantum measurement including the registration of the result, and hence the  $\psi$  collapse.

Below I am going to discuss the first part only, and to consider  $\psi$  as a specific dynamical variable ignoring the common term for  $\psi$ , the probability amplitude. Variable  $\psi$  obeys some purely dynamical equation of motion, e.g., the Schrödinger equation. This part of the problem is essentially mathematical, and it naturally belongs to the general theory of dynamical systems.

As to the second part of the problem – the quantum measurement – this is a hard nut for physicists. Currently, there is no common opinion even on the question whether this is a real physical problem or an ill-posed one so that the Copenhagen interpretation of (or convention in) quantum mechanics answers all the admissible questions. In any event, there exists as yet no dynamical description of the quantum measurement including the  $\psi$  collapse. An interesting recent discussion of this question in the light of quantum cosmology can be found in Ref.[38]. In my opinion, one could find more "earthy" problems as well. Below I comment about the quantum measurement on a few occasions, but I will not discuss it in any detail as

this certainly goes beyond the frame of my lectures here.

Recent breakthrough in the understanding of quantum chaos has been achieved, particularly, due to the above philosophy of separating the dynamical part of quantum mechanics accepted, explicitly or more often implicitly, by most researchers in this field.

Currently, there are several approaches to the definition of quantum chaos. The first natural move was to extend onto the quantum mechanics the classical definition of dynamical chaos as exponentially unstable motion. One of a few physicists who still adheres to this philosophy is Ford [39]. He insists that the quantum chaos is deterministic randomness in quantum mechanics over and above that contained in wavefunction or the expansion postulate. The latter refers to the quantum measurement as mentioned above. Some mathematicians implicitly accepted the same definition, and "succesfully" constructed the quantum analogue to the classical KS-entropy (see, e.g., second Ref.[39]).

For bounded in phase space quantum systems the quantum KS-entropy is identically zero because of discrete spectrum, and the classical-like chaos is impossible. Is it possible for unbounded quantum motion? The answer is *yes* as was found recently but the examples of such a chaos are rather exotic. The first one was briefly mentioned in Ref.[40]. We consider here another example following the second Ref.[41] (for a more physical example see Ref.[42] while some general consideration are presented in Ref.[43], and a rigorous mathematical treatment is given, e.g., in second Ref.[39]).

Consider the flow on an N-dimensional torus specified by the equation

$$\dot{\theta}_i = \nu_i(\theta) \tag{3.1}$$

If  $N \ge 3$  the classical chaos is possible with positive Lyapunov exponents that is the solution of the linearized equations is exponentially unstable. Consider now the Hamiltonian system

$$H(n,\theta) = \sum_{k} n_k \nu_k(\theta)$$
(3.2)

linear in momenta  $n_k$  canonically conjugated to coordinates  $\theta_k$ . Then, the equations for  $n_k$  coincide (in reverse time) with the linearized equations (3.1). Hence, as soon as system (3.1) is chaotic the momenta of system (3.2) grow exponentially fast.

It is easily verified that the density  $f(\theta, t) = |\psi(\theta, t)|^2$  of quantized system (3.2) obeys exactly the same (continuity) equation

$$\frac{\partial f}{\partial t} + \sum_{k} \frac{\partial (f\nu_k)}{\partial \theta_k} = 0$$
(3.3)

as for classical system (3.1) with the same (particularly, chaotic) solution. The peculiarity of this and similar examples is in that to achieve the true chaos not only the quantum motion must be unbounded and, hence, of a continuous spectrum but the momenta have to grow exponentially in time.

This is why most physicists reject the above definition of quantum chaos and adhere to another one which reads (see, e.g., Ref.[11]): quantum chaos is the quantum dynamics of classically chaotic systems whatever it could happen to be, I would add.

Logically, this is most simple and clear definition. Yet, it is completely inadequate and even helpless, in my opinion, just because that chaos my turn out to be a perfectly regular motion, much surpassing that in the classical limit. The point is that the discreteness of quantum spectrum supresses any transitions for a sufficiently weak perturbation, no matter what is the corresponding classical motion [44]. For example, in the standard map this occurs if the perturbation parameter  $k \leq 1$  independent of classical parameter K = kT which controls the transition to chaos. This specific quantum stability is also called *perturbative localization*, or *transition localization*.

For this reason Berry proposed [45] to use the term "quantum chaology" which essentially means studying the absence of chaos in quantum mechanics.

My position is somewhere in between. I would like to define the quantum chaos in such a way to include some essential part of the classical chaos. It would be natural to include the mixing property which provides the meaningful statistical description of quantum dynamics. The difficulty is in that the discrete spectrum prohibits even the mixing in the sense of the ergodic theory. Yet, it turns out that the *finite-time analogues* of all the asymptotic properties in the ergodic theory, mixing including, can be formulated as we shall see below (cf. Fig. 5 as an example). For this reason, I currently adhere to the following definition: the quantum chaos is finite-time statistical relaxation in discrete spectrum.

A drawback of this definition is that such a chaos occurs also in the classical systems of linear waves as already mentioned. The term quantum chaos (in this definition) is, nevertheless, meaningful, in my opinion. Unlike classical linear waves, which are no doubt a limiting approximation to generally nonlinear waves, the linear quantum mechanics is as yet the fundamental and universal theory.

In such interpretation the classical-like asymptotic (infinite-time) chaos remains as an important limiting pattern to compare with the true quantum dynamics.

# 4. The uncertainty principle and the time scales of quantum dynamics

The main difficulty in the problem of quantum chaos is in that one needs to reconcile the quantum discrete spectrum, which apparently prohibits any dynamical chaos, with the correspondence principle, which does require some chaos, at least, sufficiently far in the quasiclassical region. But this is also the principal importance of the phenomenon of quantum chaos which reveals the deep interrelation between the two opposites – order and chaos – in the theory of dynamical systems (see Fig.1). To put it another way, the quantum chaos, properly interpreted, unveils a very complicated and reach nature of what has been, and still is, considered as a dull order, the almost periodic motion of discrete spectrum.

The other side of this difficulty is discreteness of the phase space in quantum mechanics, the size of an elementary cell being  $\sim \hbar = 1$ .

We resolved the above difficulty by introducing the characteristic time scales of the quantum motion on which the latter is close to the classical chaotic dynamics [41]. Actually, the first of those time scales had been discovered and explained by Berman and Zaslavsky already in 1978 [46], and was subsequently confirmed in many numerical experiments (see, e.g., Refs. [47]). We call it *random time scale* for the reasons given below. This scale is characterized, generally, by the estimate

$$t_r \sim \frac{\ln q}{\Lambda} \tag{4.1}$$

where q is some big quantum (quasiclassical) parameter, and  $\Lambda$  stands for the Lyapunov exponent.

In the standard map  $\Lambda \approx \ln(K/2)$  (see Eq.(2.11))and there are two quantum parameters: k and 1/T. The transition to the classical limit corresponds to  $k \to \infty$ ,  $T \to 0$  while the classical parameter K = kT = const. It may seem strange that perturbation period  $T \to 0$  in the classical limit. This is because in full dimensions  $T \sim 1/n_0$  (see Eq.(1.4) and below), and characteristic action  $n_0 \to \infty$ . General estimate (4.1) takes now the form [41]

$$\tau_r \sim \frac{|\ln T|}{\ln(K/2)} \tag{4.2}$$

This corresponds to the optimal (least spreading) configuration of the initial  $\psi(0)$ , a coherent state.

The physical meaning of this time scale is in the fast (exponential) spreading of the initially narrow wave packet. Thus, the exponential instability is present in quantum mechanics as well but only on a very short time interval (4.1,2).

This can be explained in two ways. On the one hand, the initial wave packet can not be less, in size, than a quantum phase-space cell. On the other hand, in Hamiltonian systems, the local instability leads not only to the expansion in a certain direction but also to the contraction in some other direction which rapidly brings the initial wave packet to the size of the quantum cell.

Accoding to the Ehrenfest theorem a wave packet follows the beam of classical trajectories but only as long as it remains narrow, that is only on time scale (4.1). Nevertheless, characteristic time interval  $\tau_r$  grows indefinitely in quasiclassical region, as  $T \to 0$ , in accordance with the correspondence principle. However, the transition to the classical chaos is (conceptually) difficult as it includes two limits  $(T \to 0 (q \to \infty) \text{ and } t \to \infty)$  which do not commute (see Fig.1). This is a typical situation in the quasiclassical region as was stressed, particularly, by Berry [10].

Substituting  $t_r$  (4.1) for t into Eq.(2.13) we arrive at the quantum randomness parameter

$$r_q \sim \frac{\ln q}{|\ln \nu|} \gtrsim 1 \tag{4.3}$$

The latter inequality is the condition for the motion of a narrow wave packet to be random. It is equivalent to

$$q\nu \gtrsim 1$$
 (4.4)

Again, the transition to the classical chaos includes two noncommuting limits:  $q \to \infty, \nu \to 0.$ 

The first time scale (4.1) is rather short, and the important question is: what happens next? Numerical experiments revealed [15,41] that some classical-like chaos persists on a much longer time scale  $t_R$ , generally, of the order

$$\ln t_R \sim \ln q \tag{4.5}$$

which means some power-law dependence  $t_R \sim q^{\alpha}$  (see Fig.8 below).

For the quantized standard map

$$\tau_R \sim k^2 \tag{4.6}$$

On this time scale the diffusion in n proceeds and, moreover, closely follows classical diffusion in all details, again in agreement with the correspondence



Figure 6: Quantum diffusion in the standard map:  $K = 10; k = 6.56; T = 1.52; E = \langle n^2 \rangle / 2$  is the energy. Solid line - a single run; dashed and dotted lines - different averages over  $10^4$  runs; straight line - classical diffusion (after Ref.[48]).

principle. Subsequently, these numerical results were confirmed both numerically (see, e.g., Ref.[48]) as well as analitically [49]. In the Fig.6 the data from Ref.[48] are reproduced which demonstrate a classical-like behavior up to  $\tau \sim 40$  for k = 6.56. The dependence of the initial rate of quantum diffusion on classical parameter K, shown in Fig.4, is in a good agreement with the classical dependence even for those K values where a simple theory fails. We call  $t_R$  the diffusion or (statistical) relaxation time scale.

This similarity to the classical chaos is, however, only partial. Unlike the classical one the quantum diffusion was found to be perfectly stable dynamically. This was proved in striking numerical experiments with the time reversal [50]. In a classical chaotic systems the diffusion is immediately recovered due to numerical "errors" (not random !) amplified by the local instability. On the contrary, the quantum "antidiffusion" proceeds untill the system passes, to a high accuracy, the initial state, and only than the normal diffusion is restored. An example of the time reversal in classical and quantum standard map is shown in Fig.7 [50]. The stability of quantum chaos on relaxation time scale is comprehensible as the random time scale (4.1) is



Figure 7: The effect of time reversal at  $\tau = 150$  in classical (1) and quantum (2) chaos for the standard map with k = 20; T = 0.25. The straight lines show the same classical diffusion in different scales. The accuracy of the quantum reversal in E at  $\tau = 300$  is better than  $10^{-10}$  (!) (after Ref.[50]).

much shorter. Yet, the accuracy of the reversal is surprising. Apparently, this is explained by a relatively large size of the quantum wave packet as compared to the unavoidable rounding-off errors. In the standard map, for example, the size of the optimal, least-spreading, wave packet  $\Delta \theta \sim \sqrt{T}$  [41]. On the other hand, any quantity in the computer must exceed the error  $\delta < T$ , hence  $(\delta \theta)^2 / \delta^2 \gtrsim (T/\delta) \delta^{-1} \gg 1$ .

Beyond the relaxation time scale, that is for  $t \gg t_R$ , the quantum diffusion stops, and a certain steady state is formed which may or may not be close to the classical statistical equilibrium as will be discussed in detail below. For  $k \to \infty$  (kT = const) the time scale  $t_R \to \infty$ , again in accordance with the correspondence principle, but this quasiclassical transition is also characterized by the same double limit as for  $t_r$  above.

Thus, various properties of the classical dynamical chaos are also present in quantum dynamics but only temporarily, within finite and different time scales  $t_r$  or  $t_R$ . This is the crucial distinction of the quantum ergodic theory from the classical one which is asymptotic in t. It seems that any substantial progress in the mathematical theory requires a generalization of the existing ergodic theory to a finite time. Perhaps, it is better to say that a new nonasymptotic (finite-time) ergodic theory needs to be created.

Why the existing ergodic theory is asymptotic? I suspect that the main reason is technical rather than physical or mathematical. Namely, the asymptotic analysis is, typically, much simpler. Remember, for example, the conception of continuous phase space in classical mechanics. One particular difficulty in a finite-time ergodic theory is the important distinction between discrete and continuous spectrum of the motion which is unambiguous only asymptotically in time.

The conception of a finite-time chaos in discrete spectrum appears unusual and even strange, indeed. Yet, in my opinion, it has no intrinsic defects or contradictions. Moreover, such a notion already exists in the rigorous algorithmic theory of dynamical systems. For a physicist, the decisive argument is that the finite-time chaos perfectly fits a broad class of quantum processes and, moreover, provides an arbitrarily close approach to the classical chaos in accordance with the fundamental correspondence principle. Also, notice that in numerical experiments on the digital computer the finite-time pseudochaos is only possible as any quantity in the computer is discrete. In computer representation any dynamical system is "superquantized" in a sense (for discussion see, e.g., Ref.[41]).

This philosophy, which has not yet many adherents, resolves also the double limit ambiguity discussed above. From the physical viewpoint there is no reason to take the limit  $t \to \infty$  at all. Instead, the time should be fixed for any particular problem, the regime of quantum motion depending on the quasiclassical parameter q as outlined in Fig.8. In this picture the asymptotic classical chaos is but a *limiting pattern* to compare with the true (quantum) dynamics.

The real quantum chaos, nevertheless, is called sometimes *pseudochaos* or transient chaos to distinguish an "ugly" reality from the perfect ideal.

Of the two characteristic time scales of quantum motion discussed above the relaxation time scale  $t_R$  is most important simply because it is much longer than the other one,  $t_r$ . Peculiarity of quantum statistical relaxation is in that it proceeds in spite of the discrete energy spectrum. As is well known, the latter is always the case for the quantum motion bounded in phase space. The crucial property is a finite number of quantum states on the energy surface or, better to say, within an *energy shell*. In this case [41]

$$t_R \lesssim \rho$$
 (4.7)

where  $\rho$  is the finite energy level density ( $\hbar = 1$ ) (cf. Eq.(2.7)).

The physical meaning of this estimate is very simple and is related to the



Figure 8: Classically chaotic quantum motion: 1 - random time scale  $t_r \sim \ln q$ ; 2 - relaxation time scale  $t_R \sim q^{\alpha}$ ;  $q \gg 1$ , the quasiclassical parameter.

fundamental uncertainty principle<sup>1</sup>. For sufficiently short time the discrete spectrum is not resolved, and a classical-like diffusion is possible, at most up to  $t \sim \rho$ . The same is true for the standard map on a torus which has also a finite number (C) of now quasienergy states. Since quasienergy is determined  $mod(2\pi/T)$  the level density is

$$\rho = \frac{TC}{2\pi} = m \gtrsim t_R \tag{4.8}$$

Notice that  $\rho$  is classical parameter as is m because while  $C \to \infty$  parameter  $T \to 0$  in the classical limit.

The situation is much less clear for the standard map on a cylinder where the motion can be unbounded in n. In some special cases the quasienergy spectrum is, indeed, continuous, yet this does not mean chaotic motion but rather the peculiar quantum resonance. A more complicated case of continuous spectrum will be discussed below. On the other hand, all the numerical evidence indicates that typically the quasienergy spectrum is discrete in spite of infinite number of levels. Formally, the level density  $\rho$  is then also infinite. Yet, relaxation time scale  $t_R$  is finite. The point is that the quantum motion does not depend on all quasienergy eigenstates but only on those which are actually present in the initial quantum state  $\psi(0)$  and, thus, control the motion. We call them operative eigenstates (for given initial conditions). If their density is  $\rho_0 \leq \rho$  a better estimate for  $t_R$  is (cf.Eq.(4.7)):

$$t_R \sim \rho_0 \tag{4.9}$$

<sup>&</sup>lt;sup>1</sup>In a different way this first principle was used in Ref.[51] to explain the Anderson localization in a random potential.

For  $\rho_0$  to be finite all eigenfunctions have to be localized that is to decrease sufficiently fast in n. To the best of my knowledge there are as yet no rigorous results on the eigenfunction localization and/or the spectrum even for such a simple model as the standard map.

If the localization length is l, the density  $\rho_0 \sim Tl/2\pi$  (for sufficiently localized initial state), and  $\tau_R = t_R/T \sim l$ . Actually, Eq.(4.9) is an implicit relation because  $\rho_0$  depends, in turn, on dynamics. Consider, first, the unbounded standard map where the rate of classical diffusion has the form (2.2), and

$$D \sim k^2 \tag{4.10}$$

for  $K \gg 1$  (complete classical chaos). Suppose, further, that the width (in n) of the initial state  $\Delta n_0 = l_0 \ll l$ . Then the final width due to a diffusion during time  $\tau_R$  is  $\Delta n_f \sim (\tau_R D)^{1/2} \sim l$ . Since  $\tau_R \sim l$ , we arrive at the remarkable estimate

$$\tau_R \sim l \sim D \tag{4.11}$$

which relates essentially quantum characteristics  $(\tau_R, l)$  with the classical quantity D.

Thus, the quantum diffusion in the unbounded standard map is always localized, and a certain steady state is formed which has no counterpart in classical mechanincs.

For the bounded standard map the situation is qualitatively different depending on a new parameter

$$\lambda = \frac{l}{C} \sim \frac{D}{C} \tag{4.12}$$

which we term the ergodicity parameter. Indeed, the quantum localization occurs for  $\lambda \ll 1$  only. In the opposite limiting case  $\lambda \gg 1$   $(D \gg C)$  the relaxation time scale, being finite, is nevertheless long enough for the relaxation to the ergodic steady state to be accomplished. In this case the final steady state is close to that in classical mechanics. The same is true for conservative systems of two freedoms like billiards or cavities. In terms of the relaxation times  $\lambda^2 \sim \tau_R/\tau_e$  (see Eqs.(2.5) and (4.11)).

## 5. Finite-time statistical relaxation in discrete spectrum

We turn now to a more accurate description of the quantum relaxation in

the standard map. First, what are the quasienergy eigenfunctions? We shall discuss this in detail below. So far it is sufficient to know that the quantum localization is approximately exponential with eigenfunctions

$$\varphi_m(n) \approx \frac{1}{\sqrt{l}} \exp\left(-\frac{|m-n|}{l}\right)$$
 (5.1)

and the steady state

$$g_s(n) = \overline{|\psi_s(n)|^2} \approx \frac{1}{l_s} \exp\left(-\frac{2|n|}{l_s}\right)$$
(5.2)

Here the bar means averaging in time, and the initial conditions are  $g(n,0) = \delta(n)$  so that g is actually the Green function. Generalization to arbitrary conditions is obvious.

Using these definitions the more accurate relations were found numerically for the standard map (see, e.g., second Ref.[41]):

$$l_s \approx 2l \approx D \tag{5.3}$$

Surprisingly, the localization lengths for eigenfunctions and for the steady state are rather different. This is due to big fluctuations around the simple exponential dependence. Generally, relations (5.3) depend also on system's symmetry [107,108].

The first attempt to describe the quantum relaxation in standard map was undertaken in Ref.[52]. The idea was very simple: the diffusion rate should be proportional to the number of quasienergy levels which are not yet resolved in time  $\tau$ . This number decreases, for  $\tau \geq \tau_R$ , as  $\tau^{-1}$ , hence

$$D(\tau) \sim D(0) \frac{\tau_R}{\tau} \tag{5.4}$$

where D(0) is the classical diffusion rate. This result was corrected in Ref.[53] where, in a more sophisticated way, the so-called level repulsion was taken into account to give for the rate of energy variation

$$\frac{dE(\tau)}{d\tau} \equiv \dot{E}(\tau) \sim \dot{E}(0) \left(\frac{\tau_R}{\tau}\right)^{1+\beta}$$
(5.5)

where  $\beta$  is the repulsion parameter. Preliminary fitting of Eq.(5.5) to some numerical data looked as an agreement with  $\beta \approx 0.3$ .

However, recent extensive numerical similations [48] revealed a different dependence for  $\tau \gg \tau_R$  (in our notations)

$$\dot{E}(\tau) \approx c \dot{E}(0) \left(\frac{\tau_R}{\tau}\right)^2 \ln \frac{\tau}{\tau_R}$$
 (5.6)

supported by a different theory. Numerically (my fit)

$$\tau_R \approx 2l_s; \quad c \approx 0.2 \tag{5.7}$$

in apparent contradiction with Eq.(5.5).

Still another phenomenological theory was proposed in Ref.[14] and developed in Ref.[54]. It is based on the general diffusion equation (see, e.g., Ref.[5]):

$$\frac{\partial}{\partial \tau}g(n,\tau) = \frac{1}{2}\frac{\partial^2}{\partial n^2}Dg - \frac{\partial}{\partial n}Ag$$
(5.8)

The second term describes a "drift"

$$A \equiv \frac{\langle \Delta n \rangle}{\tau} = \frac{dD}{dn} + B \tag{5.9}$$

Introducing this relation into Eq.(5.8), we obtain

$$\frac{\partial g}{\partial \tau} = \frac{1}{2} \frac{\partial}{\partial n} D \frac{\partial g}{\partial n} - \frac{\partial}{\partial n} Bg$$
(5.10)

In our problem the last term represents the so-called "backscattering", or reflection of  $\psi$  wave propagating in *n*. Negligible in the beginning the backscattering eventually suppresses the diffusion and leads to the formation of steady state (5.2).

From Eq.(5.10) the general expression for steady state  $g_s(n)$  is

$$\ln g_s = 2 \int \frac{B(n) \, dn}{D(n)} \tag{5.11}$$

For homogeneous diffusion  $(D = const) g_s$  is given by Eq.(5.2) with  $l_s = D$ , hence

$$B = -\frac{n}{\mid n \mid} \tag{5.12}$$

The analysis of quantum relaxation can be performed using the two first moments of  $g(n,\tau)$ :  $m_1 = \langle n \rangle$  (n > 0) and  $m_2 = \langle n^2 \rangle = 2E$ . Notice that for initial  $g(n,0) = \delta(n)$  the solution is symmetric with respect to n = 0, and we can consider n > 0 only. The equation for the moments are derived from Eq.(5.10)

$$\dot{m_1} = \frac{1}{2}Dg(0,\tau) + B;$$
  $\dot{m_2} = D + 2m_1B$  (5.13)

Here B = -1 but we keep it for further analysis. The second equation shows that one should distinguish the rate of energy variation from the diffusion rate just because of the backscattering.

The quantity  $g(0,\tau)$  in the first equation, called *staying probability* is of independent interest as a characteristic of the relaxation process.

In our case Eqs.(5.13) describe the evolution of initially spreading Gaussian distribution into the final exponential steady state (5.2). Accidentally, the ratio of moments

$$\frac{m_1^2}{m_2} \equiv \frac{\gamma^2}{2} \approx 0.5 \tag{5.14}$$

remains almost constant which allows for a simple solution

$$-t = \xi + \ln(1 - \xi); \qquad \xi(0) = 0 \qquad (5.15)$$

Here the new variable and time are

$$\xi = \frac{2\gamma}{D}\sqrt{E}; \qquad t = \frac{\gamma^2}{D}\tau \qquad (5.16)$$

Initially, as  $\tau \to 0$ , Eq.(5.15) describes the classical diffusion ( $\xi^2 \approx 2t, E \approx D\tau/2$ ) independent of  $\gamma$ . For constant D and B the relaxation  $\gamma \to 1$  is exponential

$$\xi \approx 1 - e^{-t-1}; \qquad t \to \infty \tag{5.17}$$

To explain the power law relaxation observed numerically in Refs.[48,52,53] one needs to take account of the explicit time dependence for both  $D(\tau) = Ds(\tau)$  and  $B(\tau) = -s(\tau)$ . Notice that their ratio must be independent of time, at least asymptotically, to provide the exponential steady state (see Eq.(5.11)).

The solution for the moments (see Eq.(5.13)) can be obtained by a change of time

$$\tau \to \tau' = \int s(\tau) \, d\tau$$
 (5.18)

Then, Eq.(5.15) shows that a power-law tail is only possible for  $s(\tau) \sim \tau^{-1}$ . This is in accord with the first simple estimate (5.4). Assuming

$$\tau' = \tau_R \ln\left(1 + \frac{\tau}{\tau_R}\right) \tag{5.19}$$

we arrive at the following implicit dependence  $E(\tau)$ :

$$e^{\xi}(1-\xi)\left(1+\frac{\tau}{\tau_R}\right)^p = 1; \qquad p = \frac{\gamma^2 \tau_R}{D} = 1$$
 (5.20)

The value of exponent p is obtained from asymptotic relation  $(\tau \to \infty)$ :  $\xi(\tau) \sim \tau^{-p} \sim s(\tau) \sim \tau^{-1}$ . Hence, the relaxation time scale is

$$\tau_R = \frac{D}{\gamma^2} \approx 2D \tag{5.21}$$

The value of  $\gamma^2 \approx 0.5$  was derived from the best numerical data available [48]. It is only a half of the theoretical value (5.14). Besides, Eq.(5.20) does not contain the logarithmic dependence like Eq.(5.6) [48]. The latter seems to agree better with the numerical data for large  $\tau$ . The origin of this discrepancies will be discussed below.

From the first Eq.(5.13) we can derive also implicit dependence of the staying probability on time:

$$g(0,\tau) - \frac{2}{D} = \frac{e^{-\xi(\tau)}}{(\tau_R + \tau)\xi(\tau)} \to \frac{1}{e\tau}$$
 (5.22)

which is in agreement with numerical data in Ref.[55].

Recently, an exact solution of diffusion equation (5.10) with B = -1 (n > 0) has been found [109] in the form

$$D \cdot f(z,s) = \frac{1}{\sqrt{\pi s}} \exp\left(-\frac{(z+s)^2}{s}\right) + e^{-4z} \cdot \operatorname{erfc}\left(\frac{z-s}{\sqrt{s}}\right)$$
(5.23)

where the function

$$erfc(u) = \frac{2}{\sqrt{\pi}} \int_{u}^{\infty} e^{-v^2} dv$$

and z = n/2D;  $s = \tau/2D$ . The dependence  $E(\tau)$  can be found from the equation (see Eqs.(5.13), D, B = const):

$$\ddot{m_2} = 2 - D \cdot g(0,\tau) \tag{5.24}$$

Asymptotically, as  $\tau \to \infty$ 

$$D \cdot g(0,\tau) \to 2 + \frac{D}{\sqrt{\pi}\tau [\ln(\frac{\tau}{2D})]^{3/2}}; \qquad \dot{E}(\tau) \to \frac{4D^4}{\sqrt{\pi}\tau^2 [\ln(\frac{\tau}{2D})]^{3/2}}$$
(5.25)

where the decrease in time of D and B is taken into account as before, via the change of time variable (5.19).

Comparison of Eqs.(5.25), (5.20) and (5.6) shows that the accuracy of the "exact" solution (5.23) is logarithmic only. This is because of the original simplifying assumption of a purely exponential steady state (5.2). The exact distribution is not known but, most likely, it contains some power-law factor [41].

In many-dimensional systems or for a quasiperiodic driving perturbation the diffusion localization is typically absent besides some special cases (see, e.g., Ref.[56]). For three freedoms or two driving frequencies the localization persists but its length is exponentially large. However, the perturbative localization, mentioned above, occurs in all cases of discrete spectrum.

On the other hand, even in the lowest dimensions under consideration the so-called delocalization is possible if the motion is allowed to be unbounded. Consider, for example, the standard map on a cylinder with the perturbation k(n) depending on momentum:

$$D(n) = D_0 n^{2\alpha} \tag{5.26}$$

with some constant  $\alpha$ . To solve this problem it is essential to assume that the backscattering remains unchanged, that is B = -1 as before, since it does not depend on system's parameters. Then, using Eq.(5.11), we obtain the steady state distribution in the form

$$-\ln g_s(n) = \begin{cases} \frac{2n^{1-2\alpha}}{(1-2\alpha)D_0} & \alpha \neq \frac{1}{2} \\ \frac{2}{D_0}\ln n & \alpha = \frac{1}{2} \end{cases}$$
(5.27)

In agreement with previous results [32] the critical value of the parameter is  $\alpha_c = 1/2$ . For  $\alpha < \alpha_c$  the localization remains exponential while for  $\alpha > \alpha_c$  delocalization occurs because  $g_s(n) \to const \neq 0$  as  $n \to \infty$ . In the critical case the steady state distribution is a power law:

$$g_s \sim n^{-2/D_0}$$
 (5.28)

and the localization takes place for sufficiently small  $D_0 < 2$  only, when  $g_s(n)$  is normalizable. Notice that for the localization of energy, that is for the mean energy  $\langle E \rangle = \langle n^2 \rangle / 2$  to be finite in the steady state, a more strong condition is required, namely

$$D_0 < \frac{2}{3}$$
 (5.29)

This result was recently confirmed numerically in Ref.[57].

In spite of all this theoretical developments no rigorous treatment of the quantum relaxation exists so far.

The analogy with disordered solids mentioned by the end of Lecture 1, being very fruitful, is nevertheless restricted since it concerns the correspondence between eigenstates only. The properties of motion in the two problems, both dynamical and even statistical, are generally different. For example, the ratio of the localization lengths for eigenfunctions and for the steady state is different:  $l_s \approx 2l$  in momentum space, and  $l_s \approx 4l$  in disordered solids (see, e.g., Ref.[59]).

The most striking difference is in the absence of the diffusion stage of motion in 1D solids [110]. This is because the level density of the operative eigenfunctions

$$\rho \sim \frac{ldp}{dE} \sim \frac{l}{u} \tag{5.30}$$

which is the localization (relaxation) time scale (4.9), is always of the order of the time interval for a free spreading of the initial wave packet at a characteristic velocity u. In other words, the localization length l is of the order of the free path for backscattering. On the contrary, in momentum space, for instance, in the standard map each scattering (one map's iteration) couples  $\sim k$  unperturbed states, so that  $\sim k^2 \gg 1$  scatterings are required to reach the localization  $l \sim k^2$ .

Another (qualitative) explanation of this surprising difference is in that the density of quasienergy levels for driven systems is always higher as compared to that of energy levels. The same is true for a conservative system of two freedoms as compared with the one-freedom motion in solids. Thus, the Anderson localization is the spreading, rather than diffusion, localization.

Interestingly, the asymptotic relaxation  $(t \to \infty)$  in solids [110] is the same as in the momentum space (5.6). Yet, the decay of the staying probability is different ([110], cf. Eq.(5.22))

$$g_s \sim t^{-3} \tag{5.31}$$

Nevertheless, the analogy in question remains very fruitful and extensively used in the studies of quantum chaos (see, e.g., Ref.[55]).

#### 6. The quantum steady state

The quantum diffusion localization generally results in the formation of a peculiar steady state which has no classical counterpart. The statistical relaxation to this steady state is also surprising because the motion spectrum is discrete.

The ultimate origin of this steady state is in localization of all the eigenfunctions. In a homogeneous systems like the standard map on a cylinder the localization is asymptotically exponential because the equation for eigenfunctions is linear whose behavior is described by the Lyapunov exponents in n. This is the most powerful method, borrowed from the solid-state physics, to numerically calculate localization length l [58]. However, a simple exponential dependence (5.1) is only the average behavior superimposed by big fluctuations

$$\varphi(n) \approx \frac{1}{\sqrt{l}} \exp\left(-\frac{|n|}{l} + \xi_n\right)$$
 (6.1)

By definition  $\langle \xi_n \rangle = 0$  while the dispersion is not only big but grows with  $|\Delta n|$  as [25]

$$\langle (\xi_n - \xi_m)^2 \rangle = D_{\xi} \mid n - m \mid; \qquad D_{\xi} \approx \frac{1}{l} \approx \frac{2}{D}$$
 (6.2)

Nevertheless, the accuracy of numerical determination of l can be fairly high:

$$\frac{\Delta l}{l} \approx \left(\frac{l}{n}\right)^{1/2} \tag{6.3}$$

for sufficiently large n. Fluctuations  $\xi_n$  have a big impact on the steady state as was already mentioned above. Namely, they double the localization length (5.3). This is essentially numerical result, no accurate theory still exists [32]. Also, it is not clear if the steady state is purely exponential asymptotically or there is a power-law factor like in solids [59].

Initial part  $(|n| \sim l)$  of the distribution for both eigenstates as well as the steady state must deviate from a simple exponential dependence. Again, big fluctuations impede the direct numerical measurement. Instead, two integral characteristics were studied. One is the average energy in the steady state. For exponential localization (5.2)

$$E_s = \frac{\langle n^2 \rangle}{2} = \frac{l_s^2}{4} = \frac{D^2}{4}$$
(6.4)

and it is in agreement with numerical results within a factor of 2.

Another integral quantity – the entropy H – was introduced in Ref.[60] (see also Refs.[13, 103]) as a different measure of quantum localization. The entropy localization length, which is also called the Shannon width, is defined as

$$l_H = e^H;$$
  $H = -\sum_n |\varphi(n)|^2 \ln |\varphi(n)|^2$  (6.5)

For exponentially localized eigenfunctions

$$l_{\bar{H}} = el = \frac{eD}{2} \approx 1.4D \tag{6.6}$$

where  $l_{\bar{H}} = \exp(\bar{H})$ , and  $\bar{H}$  is the average over all eigenfunctions. Numerically,  $l_{\bar{H}} \approx D$  that is less, partly due to fluctuations which decrease entropy and  $l_{\bar{H}}$  by a factor of 2. Again, deviations from exponential dependence are apparently present but not very big.

Just because  $l_H$  essentially depends on the main part of the distribution its fluctuations are much bigger as compared to those for l (6.3). Namely [61]:

$$\frac{\Delta l_H}{l_H} \approx 0.5 \tag{6.7}$$

Fluctuations of entropy H were numerically found [61] to be described quite well by a simple expression  $(l_{\bar{H}} \gg 1)$ :

$$\frac{dp}{dH} = \frac{a}{\pi \cosh[a(H - \bar{H})]} \tag{6.8}$$

with  $a \approx 3$ . So far there is no idea as to the explanation of this distribution.

There is another class of localized eigenfunctions which we call *Mott's* states. They were conjectured by Mott [62] in the context of the Anderson localization and further studied in Refs. [55,48,63,64]. Mott's state is also called the *double-hump* state for its shape of two exponential peaks separated by distance L (in n). These states exist in pairs of the symmetric and antisymmetric superpositions of the two peaks. The mechanism of their formation can be qualitatively explained as follows. The exponential localization is the effect of resonant backscattering, that is the backscattering on a resonant harmonic of random (or sufficiently irregular) potential. Hence, the exponentially localized states are in a sense the unperturbed ones. The perturbation (nonresonant potential) mixes them. For close unperturbed states this increases still more the fluctuations. However, for distant states a new, double-hump, structure is formed. The principal parameter is the overlapping integral

$$v = \int_{-\infty}^{\infty} dn \,\varphi_1^0(n) \,\varphi_2^0(n) \approx e^{-L/l} \left(\frac{L}{l} + 1\right) \tag{6.9}$$

which determines the energy splitting in the pair:  $\Delta \epsilon \sim v$ .

We studied numerically [65] the structure of Mott's states in the standard map assuming two versions of dependence  $\Delta \varepsilon(L)$ :

$$l_m \omega = A e^{-L/l_m} \tag{6.10a}$$

$$l_m \omega = A \left( 1 + \frac{L}{l_m} \right) e^{-L/l_m} \tag{6.10b}$$

where  $\omega = T\Delta\varepsilon/2\pi$ , and A is a constant. The first dependence is usually accepted in literature, the second one is suggested by parameter (6.9). Our

preliminary results seem to better confirm the second law with fitting parameters

$$A \approx 0.05; \qquad l_m \approx D_n \approx l_s \approx 2l$$
 (6.11)

The fitting to the first dependence gives a close  $l_m$  but larger  $A \approx 0.15$ .

In disordered solids the structure of Mott's states was directly calculated in Ref.[63] via the correlation functions. The result is of the form of Eq.(6.10a) with  $A \approx 5$ , and  $l_m = l = l_s/4$  (cf.Eq.(6.11)).

The importance of Mott's states, for which they actually were sought, is a large matrix element

$$n_{12} = \int dn \, n\varphi_1(n) \, \varphi_2(n) \approx \frac{L}{2} \tag{6.12}$$

The latter expression holds for  $L \gg l_m$ . The additional logarithmic dependence in the long-time relaxation (5.6) is explained just by the effect of Mott's states in the low-frequency part of the spectrum [48].

The probability for a given unperturbed (exponential) eigenstate to form the Mott pair with  $L > L_1$  can be estimated as

$$p_{1} = 2\alpha \int_{L_{1}}^{\infty} \omega(L) \, dL = 2\alpha A e^{-L_{1}/l_{m}} \tag{6.13a}$$

$$p_1 = 2\alpha A \left( 2 + \frac{L_1}{l_m} \right) e^{-L_1/l_m}$$
(6.13b)

for two dependences  $\omega(L)$  in Eq.(6.10), respectively. In both cases  $\alpha \approx 1.5$  according to our numerical experiments. The total probability  $p_1 \ll 1$ , and this explains why multi-hump states are very rare. We have found a few states which could be interpreted as distorted three-hump eigenfunctions.

In disordered solids  $p_1 > 1$  but this is not necessarily a contradiction because Eqs.(6.10,13) are asymptotic. Nevertheless, it would be interesting to analyze the structure of Mott's states in more detail.

The time-averaged density  $g_s(n)$  (5.2) determines a certain invariant measure of the quantum motion which is qualitatively different from the classical measures (microcanonical, Gibbs' etc). One important distinction is in that the former depends on initial conditions as the quantum steady state results from the localization of a spreading initial state. Moreover, if the width of initial state exceeds the localization length this dependence becomes even more complicated.

Another difference is in that the relaxation of initial state into the steady state is never as full as in the classical mechanics. For example, average quantities like energy  $E_s = \langle n^2 \rangle /2$  (6.4) oscillate, and can even come back, close to the initial value  $E_0$  since the motion spectrum is discrete.

Does it make any physical sense to speak about statistical relaxation in discrete spectrum? In my opinion, it does. First, such Poincare's recurrences are extremely rare, and their time scale has nothing to do with characteristic relaxation time scale  $\tau_R$  (4.6). Second, which is even more important, those recurrences are but large fluctuations characteristic for any statistical systems.

The same occurs in classical mechanics – for trajectories, and this is the difference. In fact, the quantum density  $g(n, \tau)$  plays an intermediate role between the classical density (which would never come back for chaotic motion) and the classical chaotic trajectory with its Poincare's recurrences. Namely, the quantum density which actually describes a single quantum system represents, nevertheless, a finite statistical ensemble of  $M \sim l_s$  systems. Hence, finite fluctuations in the quantum steady state. For example, the energy fluctuations

$$\frac{\Delta E_s}{E_s} \approx \frac{1.5}{\sqrt{l_s}} \sim \frac{1}{\sqrt{M}} \tag{6.14}$$

in a reasonable agreement with numerical experiments (see, e.g., Ref. [48,52] and Fig.6. Numerical factor in Eq. (6.14) is taken from our recent computer simulations [101].

One can say also that the mixing, which is responsible for relaxation, is terminated by localization, so that the quantum mixing is only partial or a *finite-time mixing*. Such a partial relaxation with persistent fluctuations is clearly seen, for example, in Fig.6. Notice, that a big fluctuation in this run, which is a partial recurrence towards the initial state  $E_0 = 0$ , is approximately symmetric with respect to the minimum of E. Moreover, the growth of the fluctuation follows the "antidiffusion" law (cf. Fig.7,  $\tau > 150$ ) while its decay is the "normal" diffusion (cf. initial part of dependence  $E(\tau)$ in Fig.7). This is another manifistation of time-reversibility in the dynamical chaos.

The smooth (up to fluctuations) steady state (5.2) is formed only if localization length  $l_s \gg 2\pi/T$ , the period of standard map in n. In the opposite limit  $l_s \ll 2\pi/T$  the quantum measure  $g_s(n)$  reveals the classical resonance structure [32]. Since quantum diffusion requires both K > 1 (classical border) and k > 1 (quantum border) this regime is only possible near K = 1where the diffusion in the chaotic component is very slow:

$$l_s \approx D \approx 0.3 \, (\Delta K)^3 \, k^2 \sim \frac{k^2}{\tau_{cr}^3} \tag{6.15}$$

and where the resonance structure is critical with characteristic time scale  $\tau_{cr}$  [32,66].



Figure 9: The quantum steady state in the standard map: a - homogeneous localization, K = 5, k = 10, T = 0.5, the straight line:  $-\ln \bar{f}_N = x = 2n/l_s; \bar{f}_N = \bar{f}(n)2l_s/(1+x); b$  - inhomogeneous localization,  $K = 1.5, k = 10, T = 0.15, l_s \approx 2, < l_s > \approx 7$ , the straight line:  $-\ln \bar{f}(n) = 2n/< l_s > (after Ref.[32]).$ 

The border between the two regimes is approximately at

$$l_s T \approx \frac{l_s}{k} \approx 1$$
 (6.16)

At the border,  $\tau_{cr} \sim k^{1/3}$  as was recently confirmed in Ref.[67].

For  $l_sT \ll 1$  the localization length  $< l_s >$  averaged over the resonance structure is

$$< l_s > \approx \frac{k}{\sqrt{3}}$$
 (6.17)

and the interpolation between the two regimes is approximately described by the expression [14]

$$< l_s >= \frac{l_s}{2} + \left(\frac{l_s^2}{4} + \frac{k^2}{3}\right)^{1/2}$$
 (6.18)

Two examples of the quantum steady state are shown in Fig.9 for homogeneous (a) and inhomogeneous (b) localization, respectively.

The nature of a new time scale is a controversial question. In my understanding it characterizes the phase motion in  $\theta$  rather than the excitation in *n* assumed in Ref.[26]. Indeed, the localization length for  $l_sT \gtrsim 1$  is only  $\sim k$ , hence, the relaxation time scale  $\tau_R \sim 1$ , and does not depend on *k* at all. Since  $\tau_{cr}$  (in my interpretation) is also of the order of local instability rise time the ratio of the two time scales  $\tau_R/\tau_r \sim 1/\tau_{cr} \sim k^{-1/3} \ll 1$  in the critical structure is opposite as compared to the usual  $\tau_R \gg \tau_r$ .

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The quantum steady state is only possible in discrete spectrum. The conditions for the latter in an unbounded quantum map remain unknown. For the standard map on cylinder the spectrum is continuous for the rational values of parameter  $T/4\pi = p/q$  due to periodicity of this map in n. This results in an additional motion integral which can be termed quasicoordinate by analogy with the quasimomentum in spatially periodic potential [14]. By the same analogy the momentum n grows linearly in time, hence the term quantum resonance [15,68]. The mechanism of this resonance is especially clear in case q = 1 when rotation operator  $\hat{R} = \hat{I}$  becomes identity (see Eq.(1.7)).

In Ref.[69] the continuous spectrum was proved to exist also for very special Liouville's (transcendental)  $T/4\pi$  (see below) but if this condition is only a technical limitation remained unclear. This constitutes a very subtle mathematical problem. We shall try to discuss it using semiempirical theory of the quantum resonance [14] which leads to the expression

$$< n^2 > \approx D\tau^2 \exp\left(-\frac{q}{\pi D}\right)$$
 (6.19)

This is the asymptotic energy growth in quantum resonance with denominator  $q \gtrsim D$ . A detuning  $\varepsilon(q) = |T/4\pi - p/q|$  would stop the growth in time  $\tau(\varepsilon)$  which we assume to satisfy the condition (see Eq.(1.7))

$$\varepsilon\tau < n^2 >= \nu \sim 1/2\pi \tag{6.20}$$

According to a few numerical results in Ref.[102]  $\nu \approx 0.02$ .

Consider now irrational

$$\frac{T}{4\pi} = \frac{1}{m_1 + \frac{1}{m_2 + \dots}} \equiv (m_1, \dots, m_i, \dots)$$
(6.21)  
$$\frac{p_i}{q_i} = (m_1, \dots, m_i) \to \frac{T}{4\pi}; \qquad q_{i+1} = m_{i+1}q_i + q_{i-1}$$

where  $p_i/q_i$  are the convergents of  $T/4\pi$ . Comparing Eqs.(6.19-6.21) we can formulate the following conjecture: there exist infinitely many irrational values of  $T/4\pi$  which provide unbounded energy growth and, hence, a continuous spectrum; moreover,  $T/4\pi$  can be adjusted in such a way to achieve any desired growth rate.

Take growth law in the form

$$\langle n^2 \rangle = G \tau^{\gamma} \tag{6.22}$$

Substituting this into Eqs.(6.19, 6.20) and excluding  $\tau$ , we arrive at the relation

$$\varepsilon(q_i) = \frac{\nu}{G} \left(\frac{D}{G}\right)^{\frac{1+\gamma}{2-\gamma}} \exp\left(-\frac{q_i}{\pi D} \cdot \frac{1+\gamma}{2-\gamma}\right) \approx \frac{c}{q_i q_{i+1}}$$
(6.23)

where the latter expression  $(c \sim 1)$  follows from the continuous fraction representation (6.21) of  $T/4\pi$ . This relation determines a map for the construction of desired  $T/4\pi$ :

$$m_{i+1} \approx \frac{q_{i+1}}{q_i} \approx \frac{cG}{\nu q_i^2} \left(\frac{G}{D}\right)^{\frac{1+\gamma}{2-\gamma}} \exp\left(\frac{q_i}{\pi D} \cdot \frac{1+\gamma}{2-\gamma}\right)$$
(6.24)

Successive convergents determine the quantum resonances which operate in turn, each one on its own time scale

$$\tau_i = \left(\frac{\nu}{cG} q_i q_{i+1}\right)^{\frac{1}{\gamma+1}} = \left(\frac{G}{D}\right)^{\frac{1}{2-\gamma}} \exp\left(\frac{q_i}{\pi D(2-\gamma)}\right) \tag{6.25}$$

Since these time scales rapidly increase the diffusion is inhomogeneous in time, its local rate  $\Gamma \equiv d < n^2 > /d\tau$  oscillating from about zero up to

$$\Gamma_{max}(\tau_i) = 2D\left(\frac{G}{D}\right)^{\frac{1}{2-\gamma}} \exp\left(\frac{q_i}{\pi D} \cdot \frac{\gamma - 1}{2-\gamma}\right)$$
(6.26)

The ratio

$$\frac{\Gamma_{max}(\tau_i)}{<\Gamma>} = \frac{2}{\gamma} \ge 1; \qquad <\Gamma> = \gamma G \tau^{\gamma-1}$$
(6.27)

where  $< \Gamma >$  is the mean rate from Eq.(6.22).

For maximal  $\gamma = 2$  a single resonance operates according to Eq.(6.19). In the whole interval  $0 < \gamma \leq 2$  the motion is unbounded, and the spectrum is (singular) continuous with a fractal structure in agreement with the rigorous results in Ref.[69]. Irrationals which are approximated by rationals to exponential accuracy, like those satisfying Eq.(6.23), are called transcendental numbers. A new conjecture is that even among those  $T/4\pi$  values there are (infinitely many) such ones which provide the diffusion localization. They correspond, particularly, to  $\gamma = 0$  with any finite G. A more general condition is that asymptotically

$$m_{i+1} < \exp(aq_i); \qquad a < \frac{1}{2\pi D}$$
 (6.28)

For a particular value of  $T/4\pi$  satisfying this condition the energy  $E_s = G/2$  of the quantum steady state is determined by maximal  $G_i$  found from Eq.(6.23)

$$G_i^{3/2} = \frac{\nu}{c} D^{1/2} m_{i+1} q_i^2 \exp\left(-\frac{q_i}{2\pi D}\right)$$
(6.29)

If this  $G_{max} < D^2/2$  (6.4) the resonances are irrelevant, and the usual exponential steady state is formed described by Eqs.(5.2) and(6.4). This is just the case for a typical irrational  $T/4\pi$  when  $G_{max} \sim \sqrt{D} \ll D$  if quasiclassical parameter  $k \gg 1$  is big enough.

The change of time is a serious problem in quantum mechanics as explained above. For the steady state this problem can be solved [14] as follows. The steady state distribution is proportional to invariant measure and, hence, to (sojourn) time t. Whence, upon a change of time  $t \to \tilde{t}$ 

$$\frac{\widetilde{g_s}(n)}{g_s(n)} = \frac{d\widetilde{t}}{dt} \tag{6.30}$$

the steady state distribution does change as well even though  $g_s$  does not depend on time ! Now we can change momentum n in such a way to provide  $\tilde{\tilde{g}}_s(\tilde{n}) = g_s(n)$ . We have

$$\frac{d\tilde{n}}{dn} = \frac{d\tilde{t}}{dt} \tag{6.31}$$

Partcularly, if  $\tilde{t} = \tau$ , the map time (the number of map's iterations)

$$\frac{\widetilde{g}_s}{g_s} = \frac{1}{T} = \frac{d\widetilde{n}}{dn} \tag{6.32}$$

where  $T = 2\pi/\Omega(n)$  is map's period. If, moreover, *n* is action, the map momentum  $\tilde{n} = E/2\pi$  is proportional to the energy (cf. Kepler map (1.10)).

### 7. Asymptotic statistical properties of quantum chaos

The well developed random matrix theory (RMT) (see, e.g., Refs.[70, 111, 112]) is a statistical theory which describes average properties of a "typical" quantum system. At the beginning, the object of this theory was assumed to be a very complicated, partcularly, many-dimensional quantum system as the representative of a certain statistical ensemble. With understanding the phenomenon of dynamical chaos it became clear that the number of system's freedoms is irrelevant. Instead, the number of quatum states, or the quasiclassical parameter, is of importance.

Until recently the ergodicity of eigenfunctions, that is the absence of any operators commuting with the Hamiltonian, was assumed. Of course, that is not always the case (for a very interesting and instructive review of first attempts to prove the quantum ergodicity, see Ref.[71]). One of a few rigorous results in quantum chaos is an old theorem due to Shnirelman (announced in Ref.[72] with a full proof published only now [73]). Loosely speaking the theorem states that the classical ergodicity implies the ergodicity of most quantum eigenfunctions sufficiently far in the quasiclassical region that is for sufficiently large quantum parameters. The quantum ergodicity was further discussed in Refs.[75] and well confirmed in numerical experiments with quantum billiards [21].

Shnirelman's definition of quantum ergodicity is of an integral type

$$\int dp \, dq \, W_n(p,q) \, f(p,q) \to \int dp \, dq \, g_\mu(p,q) \, f(p,q) \tag{7.1}$$
$$n \to \infty$$

for any sufficiently smooth function f of the phase space. Here  $W_n$  are Wigner eigenfunctions, and

$$g_{\mu} = \delta(H(p,q) - E) \frac{dE}{dp \, dq} \tag{7.2}$$

is microcanonical (ergodic) measure. The quantity  $\rho(E) = dpdq/dE$  is the classical counterpart of the mean level density.

To understand the quantum limitations of ergodicity and the importance of the quasiclassical asymptotics  $(n \to \infty)$  we consider as an example the Rydberg atom in magnetic field (see Eq.(1.15)).

In Ref.[76] the eigenfunctions of this model were found, for chaotic motion in the classical limit, in the form

$$\psi_i = c \sum_m \frac{\varphi_m}{\sqrt{\Omega(m)}} \tag{7.3}$$

Here c is normalizing constant,  $\varphi_m$  are some unperturbed eigenfunctions with a fixed quantum number m, and

$$\Omega(m) = 2^{3/2} \left( \omega \left( m + \frac{1}{2} \right) - E \right)^{3/2} \approx \frac{1}{n^3}$$
(7.4)

is the electron longitudinal frequency depending on quantum numbers n, m. In the classical limit the ergodic measure is

$$g_{\mu}(m) = \int dn \frac{\delta(H(m,n) - E)}{\rho(E)} \approx \frac{1}{\rho\Omega}$$
(7.5)

where  $\Omega = \partial H / \partial n$ .

In quantum mechanics this measure is discrete, and to satisfy ergodicity (7.1) the change in  $\Omega$  must be small, hence,  $\omega \to 0$ , and  $m \sim |E| / \omega \to \infty$ . On the other hand, classical ergodicity (chaos) takes place under condition [18]

$$\varepsilon = \frac{|E|}{\omega^{2/3}} \lesssim 1 \tag{7.6}$$

Therefore, the condition for quatum ergodicity is

$$\omega \ll \varepsilon^3 \tag{7.7}$$

The RMT operates with finite matrices  $N_m \times N_m$  so that expansion similar to Eq.(7.3)

$$\psi_i = \sum_j a_{ij} \varphi_j \tag{7.8}$$

is always finite, the ergodicity meaning that

$$< |a_{ij}|^2 > = \frac{1}{N_m} \tag{7.9}$$

In other words, all probabilities  $|a_{ij}|^2$  are equal at average. This is not the case in a physical system whose energy shell, corresponding to the classical energy surface, is bounded. Hence, the conventional RMT is a *local theory* applicable far within a quantum energy shell. We will come back to this important question below.

Statistical properties of quasienergy eigenstates (for driven systems) were first studied in Refs.[77,78] (see also Ref.[13]) using, as a model, the standard map on a torus. Owing to condition (1.5) the parameter  $T/4\pi = m/2C$  is rational. But for a finite system, with C states, the spectrum is discrete, of course, so that no delicate problems, discussed above, arise. This model represents the quantum dynamics within the energy shell of a two-freedom conservative system.

The ergodicity depends on the parameter

$$\lambda = \frac{l}{C} = \frac{D}{2C} \tag{7.10}$$

and corresponds to large values of the latter. In the quasiclassical region  $\lambda \sim \frac{K}{m}k \to \infty$  (K = kT and  $m = CT/2\pi$  remain constant). Thus, sufficiently high quantum states are ergodic in accordance with the Shnirelman theorem.

The structure of ergodic eigenstates well agrees with the prediction of RMT, namely, the fluctuations are nearly Gaussian with the probability density

$$p(a) = \frac{\Gamma(N_m/2)}{\sqrt{\pi} \,\Gamma\left(\frac{N_m-1}{2}\right)} (1-a^2)^{\frac{N_m-3}{2}} \approx \left(\frac{N_m}{2\pi}\right)^{1/2} \exp\left(-\frac{N_m a^2}{2}\right)$$
(7.11)



Figure 10: Fluctuations in ergodic eigenfunctions for the standard map on a torus:  $K \approx 20, k = 20, T/4\pi = 4/51$ ; I - RMT, II - Gaussian approximation (after Ref.[78]).

Here a, assumed to be real, stand for amplitudes in expansion (7.8). Interestingly, a slight difference between the two distributions was clearly observed in Ref.[78] for  $N_m = 25$  using the  $\chi^2$  criterion (Fig.10).

Big spatial fluctuations in a chaotic eigenstate are not completely random but reveal the structure of classical periodic trajectories. This interesting phenomenon had been discovered by Heller in numerical experiments with the quantum stadium billiard [79], and was subsequently confirmed by many others (see, e.g., Ref.[80]), particularly, in quantum maps. The microstructure was observed so far as some enhancements along classical periodic trajectories in both configurational and phase spaces. Such enhancements were termed "scars" by Heller.

A general theory of scars in conservative system with arbitrary number of freedoms N was developed by Berry [81,10] (see also Ref.[82]). He made use of the Wigner function W which is the quantum counterpart of the classical fine-grained phase space density. Notice that W is generally not positively definite.

Within a scar W forms complicated diffraction fringes, rapidly oscillating and rather extended along the energy surface. The relative width of the central fringe contracts with the quantum number n as  $\sim n^{-1/2}$ . In this sense the scars have essentially quantum structure which vanishes in quasiclassical region. Yet, this transition to the classical limit is not a trivial one as the fringe amplitude does not depend on n. To get rid of scars one needs a coarsegrained (averaged) density W, which is called also the Husimi distribution, and which is positively definite. Then average density of a scar vanishes  $\sim n^{-(N-1)}$ .

As the scars are maximally localized (essentially within one quantum cell of the phase space) they do not violate Shnirelman's integral ergodicity (7.1). However, it is not completely clear why they are not seen in the fluctuations of eigenfunctions (Fig.10).

According to Berry's theory the Wigner chaotic eigenfuntion can be approximately represented as a sum over classical periodic trajectories:

$$W(x) \approx \frac{dE}{dx} \delta(E - H(x)) \times \left[1 + \nu \sum_{s} \exp\left(-\frac{N-1}{2} \Lambda_{s} T_{s}\right) \cos(S_{s} + \gamma_{s}) \delta(X_{s})\right]$$
(7.12)

Here x = (p,q) is a point in 2N-dimensional phase space while X = (P,Q)describe 2(N-1)-dimensional Poincaré section transverse to a periodic trajectory at X = 0. The periodic trajectory is characterised by action S and quasiclassical phase as well as by instability rate  $\Lambda$  and period T. Each term in sum (7.12) represents a scar which, by the way, can be of any sign, that is it may produce both a bump or a dip in phase density W. Explicit expression for  $\delta(X)$  is given in Refs.[81,10], and  $\nu$  is some numerical factor.

A difficult mathematical problem in this theory is apparent strong divergence of series (7.12) since the number of periodic trajectories with  $T_s < T$ grows as  $\exp((N-1)\Lambda T)$  (see, e.g., Ref.[83]). One way to approach this problem is as follows [54]. Let us try to consider Eq.(7.12) as an expansion in the basis of certain "coherent" states, the "scars"

$$W_s = \frac{1}{T_s} \delta(X_s) \delta(E - H(x)); \qquad \int W_s dx = 1$$
(7.13)

which are localized on periodic trajectories. A peculiar property of such coherent states is in that they are stationary that is they don't move in phase space, nor they are spreading. The mechanism of localization is essentially the same as for the diffusion discussed above but now it concerns the exponential spreading of a narrow wave packet prior to diffusion. The difference is in the level density which, for a scar, is  $\rho_s \sim T_s$ . Hence, the time scale for the localization of instability is  $T_s$ , and this is a simple explanation of the exponential factor in Eq.(7.12).

Oscillating  $\delta(X)$  tails of unknown length overlap to produce somehow the average ergodic (microcanonical) distribution  $\sim \delta(E - H(x))$  (see Eq.(7.12)) as well as the Gaussian fluctuations discussed above. The total number of separated scars is  $\sim n^{N-1}$ . Since the number of periodic trajectories grows

as  $\exp((N-1)\Lambda T)$  the longest period  $T_m$  of the basis scars is given by the estimate

$$\Lambda T_m \sim \ln n \tag{7.14}$$

and it coincides with the random time scale (4.1)  $(q \sim n)$ . This is the time interval for a wave packet spreading over the whole energy surface. The scars with longer periods  $T_s \geq T_m$  are not separated from each other, that is even their central fringes do essentially overlap, hence they are crucially modified. As a crude approximation one can simply drop these higher terms which makes series (7.12) trivially convergent. It is not excluded that this approach could provide some physical justification for a formal procedure of smoothing  $\delta(E - H)$  [10]. It is essential that under a natural assumption of random phases in Eq.(7.12) [26] the divergence is only logarithmic in T and, hence, insensitive to the exact truncation border.

In a recent theory of the series over classical periodic orbits [105] the truncation border was found to be at  $T_{max} \sim t_R$ , the relaxation time scale. In this theory the series represent quantum eigenvalues. A natural conjecture is that the truncation  $T_{max}$  in both cases is of the order of the corresponding localization time scale.

Another characteristic statistical property of chaotic eigenstates is the distribution of their eigenvalues, the energies. Particularly, the spacings s between neigbouring levels are distributed, according to RMT, as

$$p(s) \approx A \, s^{\beta} \, e^{-Bs^2} \tag{7.15}$$

where A, B are obtained from normalization and condition  $\langle s \rangle = 1$ .

In the old RMT the level repulsion parameter  $\beta$  could take 3 values only  $(\beta = 1; 2; 4)$  depending on system's symmetry. In Refs.[77,13] this property was confirmed for ergodic quasienergy eigenstates as well.

A new problem is the impact of localization on the statistical properties of chaotic eigenstates. It was first adressed in Ref.[60] for the quantized standard map on a torus to discover a new class of spacing statistics which is now called the *Izrailev distribution*:

$$p(s) \approx As^{\beta} \exp\left(-\frac{\pi^2}{16}\beta s^2 - \left(B - \frac{\pi\beta}{4}\right)s\right)$$
 (7.16)

where now  $\beta$  is a continuous parameter in the whole interval (0,4). This semiempirical relation was found using Dyson's model of charged bars on a ring. In this model the parameter  $\beta$ , which is the inverse bar temperature, can take any value. Yet, for the level repulsion of ergodic eigenstates only 3 values, given above, make sense. Izrailev has found that the intrermediate values describe localized eigenstates. The Izrailev distribution is also called *intermediate statistics* as contrasted to the *limiting statistics* (7.15) for ergodic states. This intermediate statistics should be distinguished from another one proposed in Ref.[84] to account for the lack of ergodicity in the classical limit. Earlier a few cases of big deviations of unknown nature from the limiting statistics (mainly in heavy nuclei) were described by a purely empirical *Brody's distribution*  $(0 \le \beta \le 1)$ :

$$p(s) = A s^{\beta} e^{-Bs^{1+\beta}}$$
(7.17)

The next important step would be to relate parameter  $\beta$  in Eq.(7.16) to the localization length l or rather to the ergodicity parameter  $\lambda = l/C$ . Instead, Izrailev introduced a new ergodicity parameter

$$\beta_{\bar{H}} = \exp(\bar{H} - H_e) \approx \frac{2l_{\bar{H}}}{C} \tag{7.18}$$

Here  $H, l_H$  are the average entropy of eigenstates and corresponding length, respectively (see Eq.(6.5)); and  $H_e \approx \ln(C/2)$  is the entropy of an ergodic state which is less than maximal  $(\ln C)$  owing to fluctuations (7.11). Surprisingly, the new parameter  $\beta_H \approx \beta$  proved to be very close to the repulsion parameter  $\beta$  of intermediate statistics (7.16). Why this relation is so simple remains an open question.

Particularly, in case of strong localization ( $\beta_H \ll 1$ ) the spacing distribution (7.16) approaches the Poisson law

$$p(s) = e^{-s} (7.19)$$

which originally was associated with the completely integrable systems and regular dynamics. Also, this limiting case shows that Eq.(7.16) is an approximation because clearly  $p(0) \neq 0$  for sufficiently small  $\beta_{\bar{H}}$ . At most, the residual level repulsion could be exponentially small.

In any event, this limit explains the absence of repulsion for Anderson localization in infinite disordered solids. Yet, in a finite sample the repulsion must appear which is also an interesting mathematical problem.

Notice, that Poisson distribution holds only for all levels. For the operative eigestates, which determine the quantum dynamics, the repulsion reappears again. This is another difficult problem.

The level repulsion does not change the relaxation time scale (4.9) but can modify the relaxation tail (see, e.g, Eq.(5.5) and Ref.[55]). In this context an interesting question concerns the repulsion among specific Mott's states (6.10). For each pair of such states the repulsion is very strong in the sense that their spacing is bounded from below by ovelapping untegral (6.9). On the other hand, the total number of Mott's pairs increases as the spacing  $(\omega)$ decreases owing to the growth of state's size L. Both effects seem to cancel, and the integral repulsion vanishes. Indeed, from Eqs.(6.13a) and (6.10a) (both versions (a) and (b) are asymptotically equivalent), we have

$$p_1 = 2\alpha l_m \omega \tag{7.20}$$

This is in apparent disagreement with numerical results in Ref.[55] where the level attraction was inferred from the asymptotic behavior of the staying probability (5.22). However, this conclusion is very sensitive to the exact relaxation law. On the other hand, our result is in agreement with another relaxation (5.6) observed in Ref.[48]. To conclude, this question certainly requires further studies.

Empirical dependence  $\beta_{\bar{H}}(\lambda)$  was found in Refs.[60,85,86]. Parameter  $\beta_{\bar{H}}$  was defined by Eq.(7.18) with the entropy averaged over all eigenstates. The dependence can be approximately described by two expressions

$$\beta_{\bar{H}} \approx \begin{cases} \frac{4\lambda}{1+4\lambda} & \lambda \le 0.5\\ 1 - \frac{1}{4\sqrt{\lambda}} & \lambda \ge 0.1 \end{cases}$$
(7.21)

Besides the limit  $\lambda \to 0$  there is no explanation of this dependence so far, nor even the physical mechanism underlying Eq.(7.21) has been identified.

For example, we could use a simple Eq.(5.1) for localized eigenstates. On a torus it becomes

$$\varphi_m(n) \approx \left(\frac{2\lambda}{1+\lambda\sinh(1/\lambda)}\right)^{1/2} \cosh\left(\frac{|m-n|}{l}\right)$$
 (7.22)

with the Izrailev ergodicity parameter

$$\beta_{\bar{H}} \approx \begin{cases} 2e\lambda \left(1 + \frac{e^{-1/\lambda}}{\lambda}\right) & \lambda \ll 1\\ 1 - \frac{1}{360\lambda^4} & \lambda \gg 1 \end{cases}$$
(7.23)

that is quite different from Eq.(7.21). Thus, the real dependence  $\beta_H(\lambda)$  is related to deviations from simple eigenfunction shape (5.1).

Remarkably, dependence (7.21) has the nature of a scaling in the sense that  $\beta_{\bar{H}}$  and  $\beta \approx \beta_{\bar{H}}$  depend on the ratio  $\lambda = l/C = D/2C$  only, whatever the underlying mechanism could be.

The importance of this scaling is in that both quantities,  $\beta$  and  $\lambda$ , are invariant with respect to the rotation of the basis in Hilbert space whereas the intermediate quantities,  $\beta_H$  and H, are not.

The standard map on a torus can be considered also as a model for the so-called *longitudinal localization* in conservative systems that is one along the energy shell which destroys the ergodicity.

The ststistical counterpart of the theory of quantum localization is not only old Anderson's theory but also a new development in RMT which makes use of the so-called band random matrices (BRM, see, e.g., Refs. [87]). These have nonzero random elements within a band of width 2b along the main diagonal only. They are defined in a certain physically significant basis, and also are not invariant under basis rotation.

The unitary matrix in quantized standard map (1.6) is also of a band structure with  $b \approx k$  but nonrandom elements. This similarity suggests that appropriate scaling parameter would be [88]

$$\lambda_r = r \frac{b^2}{N_m} \tag{7.24}$$

where  $N_m$  is matrix size, and r some numerical factor. All matrix elements are assumed to have the same statistical properties. Indeed, the scaling  $\beta_H(\lambda_r)$  is similar but not identical to that for the dynamical problem (7.21). In fact, the first dependence is the same for  $r \approx 1.5$  and it persists even farther, up to  $\lambda_r \approx 3$ . The second region ( $\beta_H \approx 1$ ) is apparently different but it has not yet been studied in detail. Notice, that the origin of the difference can be attributed not so much to the distinction between random and regular matrix elements as to the different boundary conditions for a square matrix and a torus.

For  $\lambda \ll 1$  the matrix of eigenfunctions  $a_{ij}$  (7.8) is also a band matrix with  $a_{ij}$  smoothly decreasing off the diagonal but with a much larger effective width  $(\sim b^2)$ .

In a conservative system the BRM may represent both longitudinal as well as tranverse localization. The latter is related to a finite width of the energy shell. The relative width depends on a particular dynamics, and vanishes in the classical limit when shell becomes a surface. Transverse localization is a universal phenomenon independent of motion's ergodic properties. The type of localization depends on the structure of matrix elements. If their distribution along the main diagonal is homogeneous the longitudinal localization only is represented, generally with the intermediate statistics as described above. To account for a finite energy shell the diagonal (unperturbed) matrix elements have to grow, at average, along the diagonal. If, moreover, the eigenfunctions are ergodic the limiting statistics (7.15) persists in spite of localization as was found empirically in heavy nuclei and atoms as well as in simple dynamical models like billiards [21, 70, 111]. However, distant correlations among many levels may change on the scale of energy shell's width  $N_E$ . Generally, the old RMT describes the local quantum structure only, that is for  $N_m \leq N_E$ , even for ergodic eigenstates. The global structure is associated with band matrices. The former approximation is very good, for example, in heavy nuclei  $(N_E \sim 10^6)$  but not in heavy atoms  $(N_E \sim 10 \text{ only})$  [89].

A new type of statistical properties for the quantum chaos has been introduced recently in Ref.[90]. It is the statistics of bands (or gaps) in the fractal spectrum of a particle in the quasiperiodic critical potential. For a particular model the band "attraction" (or clustering) was found with the parameter  $\beta \approx -3/2$  (cf.Eq.(7.15)) in the limit of small gaps. The attraction parameter characterizes also the fractal dimensions of the spectrum  $d_f = -\beta - 1 = 1/2$ in this model. Apparently, the same statistics can be applied to the nonresonant unbounded motion in the standard map (see Eqs.(6.22)-(6.24)).

Also, I would like just to mention (and to attract attention to) a very interesting and less known theorem due to Shnirelman [91] (for the proof see Ref.[73]). It is related to the KAM integrability which is intermediate between the complete integrability with independent levels (see Eq.(7.19)) and the quantum chaos with level repulsion (7.16). The KAM structure is highly intricate as its chaotic part, being of exponentially small measure, is everywhere dense.

In quantum mechanics the beautiful Shnirelman theorem, which even does not need translation, asserts:

$$\forall N \exists C_N > 0, \quad \forall n > 1 \quad \min(\lambda_{n+1} - \lambda_n, \lambda_n - \lambda_{n-1}) < C_N n^{-N}$$
(7.25)

where  $\lambda_n^2$  are the energy eigenvalues. Thus, asymptotically as  $n \to \infty$ , a half of level spacings is exponentially small. A striking difference from both the complete integrability and quantum chaos !

# 8. Conclusion: the quantum chaos and traditional statistical mechanics

The dunamical chaos in classical mechanics seems to be a fundamentally new mechanism underlying statistical laws in physics as compared to the traditional ("old") statistical mechanics (TSM). It is indeed ! The only problem with this mechanism is in that the classical chaos does not exist, strictly speaking, as our world is quantal. Now, in quantum mechanics the chaos is waning and becoming a sort of pseudochaos which only mimics some properties of the "true" chaos and, moreover, on finite time scales only. Besides, it turns out that such a quantum chaos is rather similar in mechanism to TSM [92, 93].

Let us consider these complicated relations in some detail. The paradigm of TSM is the many-dimensional *linear* oscillator which can be described by the matrix of coefficients in its quadratic Hamiltonian. This is a completely integrable system with purely discrete spectrum. But the same is true for a broad class of quantum systems as described by Hamiltonian or unitary matrices. In both cases the main *dynamical* problem is to diagonalize the matrix, and to find its eigenvalues and eigenvectors. The principal difference is in the nature of matrix's size. In TSM it is the number of freedoms Nwhile the quantum counterpart is that of states n.

If any of these parameters is big the statistical description becomes meaningful. In TSM it is achieved, in the formal theory, by taking the thermodynamic limit  $N \to \infty$ . Then the spectrum becomes continuous if the eigenfunctions are delocalized. This is indeed the case, under certain conditions, and not only for the simple linear oscillator but also for a broad class of completely integrable systems (see, e.g., [4] and references therein). Moreover, in the thermodynamic limit the completely integrable system (for any finite N) becomes a K system with positive (nonzero) KS entropy. This is a very strong statistical property.

In quantum mechanics we have the classical limit  $n \to \infty$  with its new dynamical chaos. Yet, the main problem in quantum chaos is *finite* (no matter how large) n. This semiclassical region is characteristic for the quantum chaos. Actually, the same problem exists in TSM as well. What would be the impact of finite N on the statistical properties here? From the studies of quantum chaos we know that one still can speak about statistical relaxation in spite of the discrete spectrum. A striking example of such a process was observed in old numerical experiments [94] with the completely integrable Toda lattice of 5 freedoms only! The transition to pseudochaos and statistical relaxation in this simple model is shown in Fig.11.

Thus, a new phenomenon - quantum chaos - turns out to be the old TSM of completely integrable systems, both classical and quantal, under  $N \to \infty$ . Moreover, the quantum chaos provides a new insight into the old theory as to the impact of a finite N on the statistical properties. Such a reconcilation of the two apparently unrelated theories seems to be very satisfactory from the physical point of view.

The interrelation between the two mechanisms of chaos becomes especially close in a particular class of models described by the Nonlinear Schrödinger equation (NSE). The simplest NSE is known to be completely integrable



Figure 11: Classical pseudochaos in the Toda lattice: the time dependence of harmonic normal mode energies  $E_k(t)$ , k = 1 - 5, is shown; the total energy E = 1.32 (a) and 132 (b) (after Ref.[94]).

but some additional perturbation, either driving or conservative, can produce already the true (asymptotic) chaos in this quantum system [96]. Depending on the physical nature of the nonlinearity  $(|\psi|^2)$  it can be interpreted as the classical freedoms of motion. An example is the interaction of some quantum system with even a single mode of the electromagnetic field with infinitely many quanta. Such a system is called *semiclassical* or *partly classical* one. For any finite number of quanta the NSE is an approximation, the so-called mean-field approximation. Physically, the chaotic solutions are just the result of this approximation and hence an artifact. Yet, in the classical limit for that single freedom only the NSE becomes exact which demonstrates that even a single classical freedom is sufficient for the true chaos (an instructive analysis of such a model is presented in Ref.[97]).

On the other hand the above limit can also be interpreted as an infinite quantum system in which the old mechanism is operative. This is especially clear for the model discussed in second Ref.[96]. The interaction of many electrons is described here in the mean-field approximation by the NSE for

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one of them. A remarkable peculiarity of this example is in that the old mechanism for chaos is explicitly reduced to the new one, the dynamical chaos. This is also some explanation how the exponential local instability arises, in the thermodynamic limit, within a completely integrable system.

In the very conclusion I would like to make a few comments on the problem of quantum measurement. The studies in quantum chaos suggest that the latter may have close relation to this problem [95]. First, the measurement device is by purpose a macroscopic system for which the classical description is a very good approximation. Together with the measured quantum microsystem it forms a semiclassical object in which the true chaos is already possible as discussed above. Further, the chaos in the measurement device is not only possible but inavoidable because the latter has to be, by purpose again, a highly unstable system. Indeed, a microscopic interaction produces, in the process of measurement, the macroscopic effect.

The importance of chaos in the quantum measurement is in that it destroys coherence of the initially pure quantum state converting it into the incoherent mixture. In the existing theories this is described as the effect of some external noise. In the standard map, for example, such a process was studied in Ref.[98]. Typically, a sufficiently weak noise does not affect the classical-like diffusion on relaxation time scale  $t_R$  (4.9). Yet, even arbitrarily weak noise destroys coherent localization and provides finite and permanent diffusion rate  $D_N$  where

$$\frac{D_N}{D} \sim \begin{cases} D\tilde{D} & D\tilde{D} \le 1\\ 1 & D\tilde{D} \ge 1 \end{cases}$$
(8.1)

Here  $\tilde{D}$  is the diffusion rate under noise only. A sufficiently strong noise restores the permanent classical diffusion (for  $1/D \leq \tilde{D} \leq D$ ). Notice that the critical noise level  $\tilde{D}_{cr} \sim 1/D \rightarrow 0$  in the classical limit as  $D \rightarrow \infty$ .

A more interesting effect recently under intensive studies (see, e.g., [99] and references therein) is in that the noise of a special type substantially inhibits the quantum transitions preserving the initial state. This effect is similar to the impact of quantum measurements but unlike the latter admits the dynamical description (cf. Ref. [100] on the quantum Zeno effect).

The chaos theory allows to get rid of the unsatisfactory inclusion of external noise, and to develope a purely dynamical theory for the loss of quantum coherence. Particularly, the special type of noise in the latter example is related to the specific construction of the device for the measurement of a given quantity.

This is almost dynamical theory of quantum measurement except one, perhaps most difficult and important, link in the chain - the probability redistribution according to the result of the particular measurement (for discussion see Ref.[14]). The main difficulty here is in that a certain modification of the quantum mechanics appears to be inavoidable, and not simply the studies of solutions to the known fundamental equations.

This seems to be a very intriguing problem but it certainly goes far beyond the scope of my lectures and, perhaps, of the whole physics.

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#### References

- 1. P.Peebles, The Large-Scale Structure of the Universe, Princeton Univ. Press, 1980.
- 2. P.Glansdorff and I.Prigogine, Thermodynamic Theory of Structure, Stability and Fluctuations, Wiley, 1972.
- 3. H.Haken, Advanced Synergetics, Springer, 1983.
- 4. I.Kornfeld, S.Fomin and Ya.Sinai, Ergodic Theory, Springer, 1982.
- 5. A.Lichtenberg, M.Lieberman, Regular and Stochastic Motion, Springer, 1983; G.M.Zaslavsky, Chaos in Dynamic Systems, Harwood, 1985.
- 6. B.V.Chirikov, Phys.Reports 52 (1979) 263.
- 7. V.I.Arnold and A.Avez, Ergodic Problems of Classical Mechanics, Benjamin, 1968.
- 8. B.V.Chirikov and V.V.Vecheslavov, in: *Analysis etc*, Eds. P.Rabinowitz and E.Zehnder, Academic Press, 1990, p. 219.
- 9. Proc. Les Houches Summer School on Chaos and Quantum Physics, Elsevier, 1991.
- 10. M.Berry, Some Quantum-to-Classical Asymptotics, Ref.[9].
- 11. B.Eckhardt, Phys.Reports 163 (1988) 205.
- 12. G.Casati and L.Molinari, Suppl.Prog.Theor.Phys. 98 (1989) 287.
- 13. F.M.Izrailev, Phys.Reports 196 (1990) 299.
- 14. B.V.Chirikov, Time-Dependent Quantum Systems, Ref.[9].
- 15. G.Casati et al, Lecture Notes in Physics 93 (1979) 334.
- 16. G.Casati et al, Phys.Reports 154 (1987) 77.
- 17. G.Casati et al, Phys.Rev. A 36 (1987) 3501.

- 18. H.Friedrich and D.Wintgen, Phys.Reports 183 (1989) 37.
- 19. Ya.G.Sinai, Usp.Mat.Nauk 25 #2 (1970) 141.
- 20. L.A.Bunimovich, Zh.Eksp.Teor.Fiz. 89 (1985) 1452.
- 21. O.Bohigas and M.Giannoni, Lecture Notes in Physics 209 (1984) 1.
- P.Šeba, Phys.Rev.Lett. 64 (1990) 1855; J.Krug, ibid 59 (1987) 2133;
   R.Prange and S.Fishman, ibid 63 (1989) 704.
- 23. B.V.Chirikov, Linear Chaos, preprint INP 90-116, Novosibirsk, 1990.
- 24. S.Fishman et al, Phys.Rev. A 29 (1984) 1639.
- 25. D.L.Shepelyansky, Physica D 28 (1987) 103.
- 26. R.Prange, these proceedings.
- 27. V.M.Alekseev and M.V.Yakobson, Phys.Reports 75 (1981) 287.
- G.Chaitin, Information, Randomness and Incompleteness, World Scientific, 1990.
- 29. B.V.Chirikov, Chaos, Solitons and Fractals 1 (1991) 79.
- 30. D.Umberger and D.Farmer, Phys.Rev.Lett. 55 (1985) 661.
- 31. M.Kac, Statistical Independence in Probability, Analysis and Number Theory, Math. Association of America, 1959.
- 32. B.V.Chirikov and D.L.Shepelyansky, Radiofizika 29 (1986) 1041.
- 33. A.Rechester et al, Phys.Rev. A 23 (1981) 2664.
- 34. V.V.Vecheslavov, private communication, 1991.
- 35. M.Born, Z.Phys. 153 (1958) 372.
- 36. G.Casati et al, Phys.Lett. A 77 (1980) 91.
- 37. B.V.Chirikov, Intrinsic Stochasticity, in: Proc. Int. Conf. on Plasma Physics, Lausanne, 1984, Vol. 2, p. 761.
- M.Gell-Mann and J.Hartle, Quantum Mechanics in the Light of Quantum Cosmology, in: Proc. 3rd Int. Symposium on the Foundations of Quantum Mechanics in the Light of New Technology, Tokyo, 1989.
   J.Ford, Quantum Chaos: Is there any?, in: Directions in Chaos, Ed. Hao
- J.Ford, Quantum Chaos: Is there any?, in: Directions in Chaos, Ed. Hao Bai-Lin, World Scientific, 1987; F.Benatti et al, Lett.Math.Phys. 21 (1991) 157.
- 40. A.N.Gorban and V.A.Okhonin, Universality Domain for the Statistics of Energy Spectrum, preprint KIP-29, Krasnoyarsk, 1983.
- B.V.Chirikov, F.M.Izrailev and D.L.Shepelyansky, Sov.Sci.Rev. C 2 (1981) 209; Physica D 33 (1988) 77.
- 42. S.Weigert, Z.Phys. B 80 (1990) 3.
- 43. M.Berry, True Quantum Chaos? An Instructive Example, Proc. Yukawa Symposium, 1990.
- 44. E.V.Shuryak, Zh.Eksp.Teor.Fiz. 71 (1976) 2039.
- 45. M.Berry, Proc.Roy.Soc.Lond. A 413 (1987) 183.
- 46. G.P.Berman and G.M.Zaslavsky, Physica A 91 (1978) 450.
- 47. M.Toda and K.Ikeda, Phys.Lett. A 124 (1987) 165; A.Bishop et al, Phys.Rev. B 39 (1989) 12423.
- 48. D.Cohen, Phys.Rev. A 44 (1991) 2292.

- D.L.Shepelyansky, Quasiclassical Approximation for Stochastic Quantum Systems, preprint INP 80-132, Novosibirsk, 1980; V.V.Sokolov, Teor.Mat.Fiz. 61 (1984) 128.
- 50. D.L.Shepelyansky, Physica D 8 (1983) 208; G.Casati et al, Phys.Rev.Lett. 56 (1986) 2437.
- 51. P.Allen, J.Phys. C 13 (1980) L166.
- 52. B.V.Chirikov, Usp.Fiz.Nauk 139 (1983) 360.
- 53. G.P.Berman and F.M.Izrailev, On the Dynamics of Correlation Functions and Diffusion Limitation in the Region of Quantum Chaos, preprint KIP-497, Krasnoyarsk, 1988; Correlations and Diffusion Suppression in Quantum Chaos, preprint KIP-663, Krasnoyarsk, 1990.
- 54. B.V.Chirikov, CHAOS 1 (1991) 95.
- 55. T.Dittrich and U.Smilansky, Nonlinearity 4 (1991) 59; 85.
- 56. G.Casati et al, Phys.Rev.Lett. 59 (1987) 2927.
- 57. F.Benvenuto et al, Z.Phys. B 84 (1991) 159.
- 58. D.L.Shepelyansky, Phys.Rev.Lett. 56 (1986) 677.
- 59. I.M.Lifshits, S.A.Gredeskul and L.A.Pastur, Introduction to the Theory of Disordered Systems, Wiley, 1988.
- 60. F.M.Izrailev, Phys.Lett. A 134 (1988) 13; J.Phys. A 22 (1989) 865.
- 61. B.V.Chirikov and F.M.Izrailev, Statistics of Chaotic Quantum States: Localization and Entropy (in preparation).
- 62. N.Mott, Phil.Mag. 22 (1970) 7.
- 63. L.P.Gorkov et al, Zh.Eksp.Teor.Fiz. 84 (1983) 1440; 85 (1983) 1470.
- 64. R.Blümel et al, J.Chem.Phys. 84 (1986) 2604.
- 65. G.Casati, B.V.Chirikov and F.M.Izrailev, The Structure of Mott's States (in preparation).
- 66. S.Fishman et al, Phys.Rev.Lett. 53 (1984) 1212; Phys.Rev. A 36 (1987) 289.
- 67. J.Jensen and O.Niu, Phys.Rev. A 42 (1990) 2513.
- 68. F.M.Izrailev and D.L.Shepelyansky, Teor.Mat.Fiz. 43 (1980) 417.
- 69. G.Casati and I.Guarneri, Comm.Math.Phys. 95 (1984) 121.
- 70. T.Brody et al, Rev.Mod.Phys. 53 (1981) 385; F.Haake, Quantum Signatures of Chaos, Springer, 1990.
- 71. P.Pechukas, J.Phys.Chem. 88 (1984) 4823.
- 72. A.I.Shnirelman, Usp.Mat.Nauk 29 #6 (1974) 181.
- 73. A.I.Shnirelman, On the Asymptotic Properties of Eigenfunctions in the Regions of Chaotic Motion, addendum in Ref.[74].
- V.F.Lasutkin, The KAM Theory and Asymptotics of Spectrum of Elliptic Operators, Springer, 1991.
- M.Berry, J.Phys. A 10 (1977) 2083; A.Voros, Lecture Notes in Physics 93 (1979) 326.
- 76. M.Yu.Kuchiev and O.P.Sushkov, Phys.Lett. A 158 (1991) 69.
- 77. F.M.Izrailev, Phys.Rev.Lett. 56 (1986) 541.
- 78. F.M.Izrailev, Phys.Lett. A 125 (1987) 250.
- 79. E.Heller, Phys.Rev.Lett. 53 (1984) 1515.

- R.Jensen, Phys.Rev.Lett. 63 (1989) 2771; R.Scharf et al, Phys.Rev. A 43 (1991) 3183.
- 81. M.Berry, Proe.Roy.Soc.Lond. A 423 (1989) 219.
- 82. E.B.Bogomolny, Physica D 31 (1988) 169.
- 83. Ya.G.Sinai, Introduction to Ergodic Theory, Princeton, 1976.
- 84. M.Berry and M.Robnik, J.Phys. A 17 (1984) 2413.
- 85. S.Fishman et al, Phys.Rev. A 39 (1989) 1628.
- 86. G.Casati et al, Phys.Rev.Lett. 63 (1990) 5.
- T.Seligman et al, Phys.Rev.Lett. 53 (1985) 215; M.Feingold et al, Phys.Rev. A 39 (1989) 6507; M.Wilkinson et al, J.Phys. A 24 (1991) 175.
- 175. 88. G.Casati et al, Phys.Rev.Lett. 64 (1990) 1851; J.Phys. A 24 (1991) 4755.
- 89. B.V.Chirikov, Phys.Lett. A 108 (1985) 68.
- 90. T.Geisel et al, Phys.Rev.Lett. 66 (1991) 1651.
- 91. A.I.Shnirelman, Usp.Mat.Nauk 30 #4 (1975) 265.
- 92. B.V.Chirikov, Foundations of Physics 16 (1986) 39.
- 93. B.V.Chirikov, Dynamical Mechanisms of Noise, preprint INP 87-123, Novosibirsk, 1987.
- 94. J.Ford et al, Prog. Theor. Phys. 50 (1973) 1547.
- 95. B.V.Chirikov, Wiss.Z. der Humboldt-Univ. 24 (1975) 215.
- 96. F.Benvenuto et al, Phys.Rev. A 44 (1991) R3423; G.Jona-Lasinio et al, Quantum Chaos without Classical Counterpart, 1991 (unpublished).
- 97. R.Fox and J.Eidson, Phys.Rev. A 36 (1987) 4321.
- 98. E.Ott et all, Phys.Rev.Lett. 53 (1984) 2187.
- 99. I.B.Khriplovich and V.V.Sokolov, Physica A 141 (1987) 73.
- 100. T.Petrosky et al, Physica A 170 (1991) 306.
- 101. G.Casati et al, The Quantum Steady State: Temporal Fluctuations in Discrete Spectrum (in preparation).
- 102. G.Casati et al, Phys.Rev. A 34 (1986) 1413.
- 103. R.Graham, these proceedings.
- 104. P.Koch, these proceedings.
- 105. U.Smilansky, these proceedings.
- 106. O.Agam et al, Experimental Realization of "Quantum Chaos" in Dielectric Waveguides, Technion-Phy-91, 1991 (unpublished).
- 107. U.Smilansky, private communication, 1992.
- 108. J.Pichard et al, Phys.Rev.Lett. 65 (1990) 1812.
- 109. G.Casati, private communication, 1992.
- 110. E.P.Nakhmedov et al, Zh.Eksp.Teor.Fiz. 92 (1987) 2133.
- 111. O.Bohigas, these proceedings.
- 112. H.Weidenmüller, these proceedings.