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BUDKER'S PROBLEM OF PARTICLE CONFINEMENT AND WHAT HAS COME ABOUT OF IT

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ABSTRACT

One of numerous Budker's ideas was born out of his deep insight into the old problem of adiabatic invariance, the relatively simple but crucial condition for realization of his proposal which set out a new direction in the controlled nuclear fusion research termed nowadays the open systems. The studies of this apparently very particular application problem had quickly developed, nevertheless, into the fundamental research of a brand-new, at that time, and very surprising phenomenon, the so-called dynamical chaos, which has grown by now in a whole large field in physics and mathematics with many applications. A brief history of this research is traced back to Budker's original intuition. The principal results in the chaos theory, gained since then, are discussed using, as a simple example, the first Budker's trap with "magnetic plugs".

1. A little of the history

In 1954 A.M. Budker, then a young theoretician in the now Kurchatov Institute, proposed a new approach to realization of the controlled nuclear fusion termed today the open systems[1]. He himself baptized this invention "the particle trap with magnetic plugs". In such a device the particle confinement was relied upon the adiabatic invariance of particle's orbital magnetic moment proportional to its action. Since this invariance is approximate only the immediate question was if such a trap would confine at least a single particle for the sufficiently long time required for the fusion reactions. Budker's intuition suggested that a purely theoretical solution was hardly possible here, and he turned to experiment. He invented a nice way to check the confinement by making use of the decay electrons of tritium inside the magnetic trap. This experiment has been quickly realized by his pupil Rodionov[2], and it confirmed fairly good confinement properties of Budker's trap. Since the single particle stability was certainly not the main difficulty in the controlled nuclear fusion the former minor application problem seemed to have been exhausted. Apparently, such was the conclusion of the American physicists in a similar research highly classified before 1958 [3,4]. But this was not the case in Budker's group!

His always interesting, deep, and exciting discussions, or rather disputes, acted as the powerful attraction to a purely physical problem of the adiabatic invariance.

This is a rather general problem of primary importance in both classical and quantum physics. In the former the action-angle variables are the most natural quantities for describing the oscillatory dynamics while in the latter just the actions are quantized and assume discrete values only the adiabaticity meaning suppression of the quantum transitions. Besides, Rodionov's experiment revealed that the electron life time in the magnetic trap was big but finite indicating some puzzling slow motion instability.

All this excited our curiosity, and the studies into the adiabatic noninvariance continued somewhat contrary to Budker's plans. He was hesitating as, on the one hand, he would like to concentrate all the efforts on the central problem of nuclear fusion but, on the other hand, he well understood the scientific importance of the adiabaticity problem and, moreover, was himself very curious about it. Eventually, he blessed the new research to which he himself had given, unintentionally though, a powerful initial momentum.

At the beginning two Budker's insights proved to be very important in guiding us in this new field of research.

The first one was the idea of some accumulation of small changes in the action variables under a periodic adiabatic (low-frequency) perturbation. Eventually, this idea has been developed into the conception of the nonlinear resonance as the central construction in the theory of motion stability in Hamiltonian systems.

The second, also a vague, idea was born out of Budker's consideration and criticism of a then new proposal of the strong focusing particle accelerators. The idea was in that the nonlinearity of a particle oscillation, i.e. a dependence of the oscillation frequencies on the actions, may cause a loss of the oscillation phase "memory" resulting eventually in a diffusion-like behavior. Apparently, it was the first idea of chaos in nonlinear oscillations. Curiously, these Budker's doubts about the strong focusing, also classified at that time and never published afterwards, received some confirmation in one of the first numerical experiments (computer simulations) on the particle dynamics in a simple model of the strong focusing accelerator [5]. The simulation was done in the course of design of the first CERN Proton Synchrotron, and has led to severe tolerances on the nonlinearity. Since particle's oscillation in the magnetic trap was strongly and unavoidably nonlinear (see Eq.(3) below) this Budker's idea turned out to be of a primary importance. It completed the resonance picture of nonlinear oscillations and allowed for the first estimates of the particle stability in the magnetic trap [6] (for a recent review see Ref.[7]).

2. A nonperturbative problem

Consider, as a simple example, the first Budker's trap with axisymmetric magnetic field

$$B(s) = B_0 \left(1 + \frac{s^2}{L^2}\right) \tag{1}$$

where s is coordinate along a magnetic line, and subzero refers to the midplane of the trap ($s = 0$). The unperturbed Hamiltonian for the longitudinal particle bouncing has the form

$$H_0(p, s) = \frac{p^2}{2} + \mu\,\omega(s) = \mu\,\omega_0 + J\,\Omega(\mu) \tag{2}$$

where J, $\Omega = \sqrt{2\mu\omega_0}/L$ are the longitudinal action and frequency, respectively; ω stands for Larmor's frequency; $\mu = E \cdot Sin^2\theta/\omega = const$ in this approximation; $E = v^2/2$, and we use the atomic units $e = m = \hbar = 1$. Below we will consider small pitch-angles $\theta \ll 1$ only, hence $\mu \approx \mu_0\theta_0^2$, $\mu_0 = E/\omega_0$ which is also the big quantum parameter of a particle in magnetic field.

The last expression (2) shows that particle's oscillation in this trap is harmonic (by the shape), yet strongly nonlinear in the sense of action-dependent frequencies

$$\Omega(\mu) = \frac{\sqrt{2\,\mu\,\omega_0}}{L}; \qquad <\omega> = \omega_0 + \frac{J}{L}\sqrt{\frac{\omega_0}{2\,\mu}} \tag{3}$$

where $<\omega>$ is the mean Larmor frequency over the longitudinal bouncing which determines the principal resonances of an electron in the trap

$$\frac{<\omega>}{2\Omega} = l \sim \lambda \tag{4}$$

Here l is any positive integer, and we introduced the adiabaticity parameter $\lambda \gg 1$ which is convenient to define by the relation

$$\lambda = \frac{2\,L}{3\,\rho_m} = \frac{2\,L\,\omega_0}{3\,v} \tag{5}$$

with the maximal Larmor radius ρ_m.

Resonances (4) cause the exchange of energy between the two freedoms of the motion, and particularly, a change in μ (and in J) at each transit of trap's midplane:

$$\Delta\mu = 2\sqrt{\mu}\,A \cdot \sin\phi_0 = -\frac{2\,\mu\,\Omega}{2\,\mu\,\omega_0 + J\,\Omega} \cdot \Delta J \tag{6}$$

where the amplitude

$$A = 0.83\, r_0 \sqrt{\omega_0}\, \exp(-\lambda) \tag{7}$$

ϕ_0 is Larmor's phase, and r_0 the distance to the trap axis.

As expected the nonadiabatic effect is exponentially small in parameter λ (for an analytic function $\omega(s)$). Particularly, it implies that the whole problem cannot be solved by means of the standard perturbation theory as an asymptotic expansion in the small perturbation parameter $1/\lambda$ because the nonadiabatic effects lie beyond any order of such an expansion. Instead, the problem may be divided into two parts. First, one needs to exactly evaluate the nonadiabatic effect in a single transit (6) which is possible to do for sufficiently simple models like one in question. This is an auxilliary problem. The main part is to describe accumulation of the nonadiabatic changes over many bouncings. This can be realized by the construction of a canonical (Hamiltonian) map over a bouncing half-period [7]: $\mu,\ \phi \to \overline{\mu},\ \overline{\phi}$ where

$$\begin{array}{lll} \overline{\mu} = \mu + 2\sqrt{\overline{\mu}}\,A\cdot\sin\phi & \to & \mu + 2\sqrt{\mu_l}\,A\cdot\sin\phi \\ \overline{\phi} = \phi + G(\overline{\mu}) + \frac{A}{\sqrt{\overline{\mu}}}\cdot\cos\phi & \to & \phi + G'_\mu(\mu_l)\cdot(\overline{\mu}-\mu_l) \end{array} \tag{8}$$

Here $\phi \equiv \phi_0$, and μ_l is a resonant value of μ (see Eq.(4)). The latter approximate map, linearized in μ, is also canonical, and provides the local description of the motion which depends on a single parameter

$$K = |2\sqrt{\mu}\,A\cdot G'_\mu| = 5.6\,\frac{r_0}{L}\cdot\frac{\lambda^2 e^{-\lambda}}{\theta_0^4} < 1 \tag{9}$$

The latter inequality is the condition for stability that is for eternal particle's confinement. Thus, the accumulating nonadiabatic effects widen the loss cone in the trap from some θ_a (adiabatic loss cone, not specified in the model under discussion) up to $\theta_b = \theta_0(K=1)$. The unstable region corresponds to $\theta_0 < \theta_b$, the latter being called the chaos border.

3. Dynamical chaos

A real surprise was the nature of the unstable motion which looked (in numerical experiments) highly irregular as if driven by some misterious random forces even though the motion equations (8) were fairly simple, completely regular, and of the two freedoms only. In any event, the motion admitted only statistical description, e. g., via a diffusion equation [7]

$$\frac{\partial f(\mu,\tau)}{\partial\tau} = \frac{\partial}{\partial\mu}D_\mu(\mu)\frac{\partial f}{\partial\mu} \tag{10}$$

where τ is discrete time (the number of map's iterations), and

$$D_\mu = \frac{\langle(\Delta\mu)^2\rangle}{2} = \mu A^2 = \frac{\mu\,\mu_0}{22}\cdot\frac{\theta_b^8}{\lambda^2} \tag{11}$$

is the diffusion rate.

An appropriate solution to this equation describes exponential relaxation

$$f \to \exp\left(-\frac{\tau}{\tau_L}\right) \tag{12}$$

with mean particle's life time

$$\tau_L = \frac{\mu_b}{A^2}\ln\left(\frac{\mu_b}{\mu_a}\right) \sim \frac{\lambda^2}{\theta_b^6} \tag{13}$$

The mechanism of the dynamical chaos is explained by the most strong (exponential) local instability of motion that is by the fast divergence of close trajectories.

In a more formal language it means the exponential solutions to the linearized motion equations:

$$\delta \sim \exp(\Lambda\,\tau) = \left(\frac{K}{2}\right)^{\tau} \tag{14}$$

where $\delta^2 = (\Delta P)^2 + (\Delta\phi)^2$, and

$$\Lambda = \ln\left(\frac{K}{2}\right), \qquad K > 4 \tag{15}$$

is Lyapunov's exponent for the linearized equations. That strong instability amplifies arbitrarily fine details of the initial conditions (far digits of $P(0)$ and $\phi(0)$) to such an extent that they eventually affect the global behavior of the chaotic system. Thus, the arithmetics of the initial conditions plays a role of some inherent "noise" which makes a chaotic trajectory very complicated and unpredictable. The ultimate origin of the dynamical chaos lies in the continuity of the phase space in classical mechanics.

Another graphic view of the chaos mechanism is the "scattering" of a typical particle's trajectory on the unstable periodic orbits. The number of the latter grows exponentially with the period ($n(T) \to \exp(\Lambda T)$, $T \to \infty$), and they are everywhere dense. That scattering also leads to the exponential instability of trajectories. A nontrivial part of the chaos mechanism is in that the instability must be not slower than exponential, any power-law instability would be insufficient.

In these conditions the trajectory loses any physical meaning, and only statistical description remains applicable. Remarkably, the dynamical equations still can be used in the theory to completely evaluate all the statistical properties of the motion without any *ad hoc* statistical hypotheses like, e.g., random phases.

The diffusion equation of the type (10) holds true for the ergodic variable only that is for one in which the steady-state distribution is a constant [7]. In discrete time τ it is μ, and $f_s(\mu|\tau) = const$. In the continuous time t ($dt/d\tau = \pi/\Omega(\mu)$), for example, the ergodic variable is different, namely, $\nu = \sqrt{\mu}$ with the new diffusion rate

$$D_\nu = \frac{\Omega < (\Delta\nu)^2 >}{\pi} = \frac{\Omega\, A^2}{\pi} = \frac{\nu\omega_0\sqrt{\mu_0}}{104} \cdot \frac{\theta_b^8}{\lambda^3}$$

Particularly, the steady state now is $f_s(\nu|t) = const$ again but is different from that in the discrete time.

The inherent statistical behavior may happen, of course, to be very unusual. An interesting example is the impact of the chaos border at $\theta_0 = \theta_b$. This border generally has a very complicated hierarchical structure, an intricate mixture of regular and chaotic domains on all the spatial scales [8]. Even though that critical structure occupies a very narrow region along the border it completely changes the asymptotic ($|\tau| \to \infty$) statistical properties of the whole chaotic component of the motion. Particularly, the relaxation follows exponential dependence (12) only initially, and then turns to a power law

$$f \to \tau^{-p} \tag{16}$$

where the exponent

$$p = \frac{2N - 1}{2N - 2} \tag{17}$$

depends on the number of freedoms N only. Apparently, such relaxation was actually observed in a laboratory experiment [9] even though the authors interpretered their data in a rather different way (for discussion see Ref.[7]).

Particle's motion in a magnetic field is, of course, not the only example of the dynamical chaos. On the contrary, the chaos is a generic phenomenon in nonlinear mechanics. After the nature and mechanism of the chaos had been understood it began to be found literally everywhere, from the microworld (gauge fields of elementary particles [10], atoms in electromagnetic fields [11]) to celestial mechanics (comet Halley [12] and the Solar system itself [13]), and even to cosmology [14]. One of the most beautiful nonlinear phenomenon, a universal instability of many-dimensional oscillator systems (termed Arnold diffusion) has been discovered and studied in detail (for review see Refs.[15] and [16], some recent developments are presented in Ref.[17]). This diffusion is spreading along the everywhere dense set of very narrow channels - the chaotic layers around separatrices of nonlinear resonances.

Interestingly, the Arnold diffusion, which is a generic phenomenon for the weak perturbation of a completely integrable Hamiltonian system (the so-called nearly integrable system), is essetially a nonadiabatic effect too like a very specific Budker's problem. The point is that a weak perturbation (with small parameter $\varepsilon \to 0$) gives birth to low frequencies ($\sim \sqrt{\varepsilon}$) near each of the resonances. Then, the rest of the perturbation (except a resonance term) acts as a high-frequency perturbation. This is essentially adiabatic situation, called sometime reversed adiabaticity, with a big parameter $\lambda \sim 1/\sqrt{\varepsilon}$.

4. Quantum chaos

One of the most challenging problems in this field is the so-called quantum chaos that is chaotic behavior in quantum mechanics which is commonly accepted as the most deep and universal physical theory. The principal difficulty here is in that the energy (and frequency) spectrum of any quantum motion bounded in phase space is discrete. In the magnetic trap, for example, the energy level density

$$d = \frac{2\lambda\mu_0}{\omega_0} \tag{18}$$

is finite. Hence, the quantum evolution is almost periodic which has been always considered as something opposite to the chaotic motion with a continuous spectrum. Meanwhile, the fundamental correspondence principle requires transition to the classical chaos as some quantum parameter grows indefinitely ($\mu_0 \to \infty$ in our example). Clearly, the quantum chaos cannot be a particular case of the dynamical chaos as it is known in classical mechanics.

One resolution of this apparent contradiction is the conception of some characteristic time scales in quantum motion on which various statistical properties of the classical chaos persist [18]. As the quantum parameter $\mu_0 \to \infty$ the time scales grow indefinitely in accordance with the correspondence principle. The most important one is the relaxation time scale

$$t_R \sim d \tag{19}$$

This has a very simple physical meaning directly related to the uncertainty principle. Indeed, for $t \lesssim t_R$ the discrete spectrum is not yet resolved. A more detailed analysis reveals that scale t_R is even less by a factor of θ_b which is the fraction of the energy surface occupied by a chaotic motion. Finally, in discrete time,

$$\tau_R \sim t_R \theta_b \Omega \sim \mu_0 \theta_b^2 \tag{20}$$

Beyond this time scale the classical diffusion is suppressed by the quantum interference effects. The degree of suppression depends on the ratio (see Eq.(13))

$$R = \frac{\tau_R}{\tau_L} \sim \frac{\mu_0 \theta_b^8}{\lambda^2} \sim 1 \tag{21}$$

The latter estimate determines the quantum localization border. If $R \ll 1$ the particle diffusion is suppressed, and the loss cone shrinks back from θ_b to θ_a. For small θ_b (large λ, see Eq.(9)) the μ_0 threshold is fairly high ($\mu_0 \approx 10^8\, E[eV]/B[G]$ for electrons).

A more accurate theory of the quantum localization can be constructed on the basis of diffusion equation (10) by introducing an additional (dynamical) term (usually called the drift)

$$\frac{\partial g}{\partial \tau} = \frac{\partial}{\partial \mu} D_\mu \frac{\partial g}{\partial \mu} - \frac{\partial g}{\partial \mu} \tag{22}$$

which describes the quantum coherent backscattering in a rather universal way [19]. Here $g(\mu, \tau)$ is a Greene function with the initial condition $g(\mu, 0) = \delta(\mu - \mu_b)$. The quasistationary solution to this equation, as $\mu \to 0$, is

$$g \to \mu^{1/A^2} \tag{23}$$

diffusion suppression corresponding to $A^2 \lesssim 1$ which coincides with Eq.(21).

The global structure of quantum motion on the plane (μ_0, θ_b) consists of 4 main regions:

$$\mu_0 \lesssim \theta_b^{-2} \tag{24.a}$$

the whole chaotic cone ($\theta_0 \lesssim \theta_b$) is too small ($\mu_b \lesssim 1$) to host even a single quantum state;

$$\theta_b^{-2} \lesssim \mu_0 \lesssim \lambda \theta_b^{-5} \tag{24.b}$$

there are many states within the chaotic cone but no transitions among them because the quantum perturbation parameter $|\Delta\mu| \sim \sqrt{\mu} A \sim \mu_0 \theta_b^5/\lambda \lesssim 1$ (see Eqs.(8) and (11)) is too small;

$$\lambda \theta_b^{-5} \lesssim \mu_0 \lesssim \lambda^2 \theta_b^{-8} \tag{24.c}$$

there are many transitions but no diffusion due to the quantum localization ($R \lesssim 1$, see Eq.(21)); and

$$\lambda^2 \theta_b^{-8} \lesssim \mu_0 \qquad (24.d)$$

classical diffusion and relaxation persist in this quantum domain as $R \gtrsim 1$. Although brief the above consideration demonstrates, as I hope, that the quantum chaos is a new phenomenon in dynamical systems which reveals the surprising complexity of the almost periodic motion with discrete spectrum.

5. A final remark: creative chaos

Development of the research of dynamical chaos in our Institute, initiated by an "innocent" Budker's problem of particle's stability in a magnetic trap, gives a beautiful example of the instability and chaos in the science itself.

The chaos is not always that bad after all!

In a broader sense this opens a little the rather surprising facet of the chaos, the creativity. It warns us of the limits of human reasoning, and calls for a free random search in the hope of lucky findings.

6. Acknowledgment

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