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Physica D 86 (1995) 220-237

PHYSICA D

Quantum chaos: unexpected complexity

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Abstract

An overview of the most important, in our opinion, recent results (both numerical as well as theoretical) in the studies of a very controversial topic, the so-called quantum chaos, is presented. We focus on the *dynamical* models and behavior which results, under appropriate conditions, in surprisingly rich statistical properties of the quantum motion, even in case of a discrete energy spectrum of the latter. The results under consideration include the time evolution of an initial narrow wave packet with its diffusion and statistical relaxation to the quantum steady state as well as the statistical properties of the (quasi)energy eigenvalues and eigenfunctions. The guiding line of our discussion is based on the conception of quantum chaos as a *new dynamical phenomenon* in a domain which has been traditionally considered as that of regular motion. A number of open questions and unsolved problems is also presented.

1. Introduction: quantum chaos as a new dynamical phenomenon

The main purpose of this talk is to present a brief overview of most important, in our opinion, recent results, both numerical as well as theoretical, in the studies of a very controversial topic, the so-called *quantum chaos*.

Some researchers see nothing essentially new at all in this phenomenon. Indeed, the problems in this field all belong to the traditional, "old-fashioned" and rather "simple" quantum mechanics of finite-dimensional systems with a given interaction and no quantized fields. Nevertheless, many, including ourselves, consider quantum chaos as a new discovery, in an old field though, of great importance for fundamental physics. To understand this, the phenomenon of quantum chaos should be put into the proper perspective of recent developments in physics. The focus of this perspective is the conception of *dynamical chaos* in classical mechanics (see, e.g., Refs. [1,2]). This conception destroys the deterministic image of classical physics and shows that typically the trajectories of deterministic equations of motion are in a sense random and unpredictable [3]. The mechanism of dynamical chaos lies in the most strong local instability of motion, that is in the exponential separation of almost all close trajectories while the global motion remains bounded. This implies a continuous power spectrum of the motion and correlation decay (not always exponential though) which, in turn, leads to *statistical relaxation*, that is to a non-recurrent (but reversible!) evolution of any nonsingular distribution function (phase space density). Moreover, the relaxation proceeds in both directions of time.

The correlation decay, or *mixing*, is the most important property which provides a meaningful statistical description of a dynamical system in terms of the probability theory. The exponential local instability of motion is certainly sufficient for mixing. However, it

SSDI 0167-2789(95)00103-4

is not clear whether the former is also necessary (apparently not?). In any event, this new mechanism for chaos is qualitatively different from the old one in the thermodynamic limit $N \to \infty$, where N is the number of freedoms [5]. The most striking difference is that the new mechanism provides chaos just in a few freedoms, e.g., for $N \geq 2$ in a conservative system.

Dynamical chaos is one limiting case of the modern general theory of dynamical systems which describes the statistical properties of the deterministic motion (see, e.g., Ref. [5]). No doubt, this theory has been developed on the basis of classical mechanics. Yet, as a general mathematical theory, it does not need to be restricted to classical mechanics only. Particularly, it can be, and indeed was, applied to quantum dynamics with a surprising result. Namely, it had been found from the beginning [6] and was subsequently well confirmed (see, e.g., Ref. [7]) that quantum mechanics does not typically permit the "true" (classical-like) chaos. This is because in quantum mechanics the energy (and frequency) spectrum of any system whose motion is bounded in *phase space* is discrete and its motion is almost periodic. Hence, according to the existing ergodic theory, such a quantum dynamics belongs to the limiting case of regular motion which is opposite to dynamical chaos. The ultimate origin of quantum almost-periodicity is in the discreteness of the phase space itself which is at the basis of quantum physics and directly related to the fundamental uncertainty principle. Yet, another fundamental principle, the correspondence principle, requires the transition to classical mechanics in all cases including the dynamical chaos with its peculiar properties.

Some researchers believe that the only way out of this apparent contradiction is the failure of the correspondence principle. If this were true, quantum chaos would be, indeed, a great discovery. "Unfortunately", there exists a less radical (but also interesting and important) resolution of this difficulty. In our opinion, the fundamental importance of quantum chaos is precisely in that it reconciles the two apparently opposite regimes, regular and chaotic ones, in the general theory of dynamical systems. Namely, as is well established by now [4] quantum chaos possesses all the properties of classical dynamical chaos but only on finite, and different, time scales which, moreover, indefinitely grow towards the classical limit. Thus, quantum chaos is not a particular case of the well understood dynamical chaos but a *new* dynamical phenomenon currently under intensive studies and sharp debates.

Another view of the quantum-classical correspondence for chaotic phenomena emerges from the observation that the border between continuous and discrete spectra is sharp in the limit $|t| \to \infty$ only. On any finite time interval there exists a transition zone which is wider in time the larger is the quasiclassical parameter $q = I/\hbar$, where I is a characteristic value of the action variable of the system. (In what follows we set $\hbar = 1$.) This transition zone on the plane (t, q) is precisely the place where quantum chaos may occur.

It is interesting to notice also that the same mechanism of temporary, transient, chaos works in case of any (e.g., classical) linear waves as well [8], or in linear (many-dimensional) oscillators, or even in the much more general case of completely integrable dynamical systems, both linear and nonlinear (e.g., in a Toda lattice [9]). Thus, quantum chaos, in spite of its generality and importance for fundamental physics, is a particular case of a *new* dynamical phenomenon which we call *pseudochaos* as distinguished from "true" dynamical chaos in the existing ergodic theory.

Another very important example of pseudochaos, from which we have borrowed the term itself, is the computer simulation of dynamical systems [10]. Indeed, in a digital computer any variable is "quantized" (discrete) so that all dynamical trajectories eventually become periodic (this fact is well known in the theory of the so-called pseudorandom number generators).

The absence of the classical-like chaos in quantum dynamics apparently contradicts not only the correspondence principle but also the fundamental statistical nature of quantum mechanics. However, even though the random element in quantum mechanics ("quantum jumps") is unavoidable, it can be singled out indeed and separated from the proper quantum processes. Namely, the fundamental randomness in quantum mechanics is related only to a very specific event – the *quantum measurement* – which, in a sense, is foreign to the proper quantum system itself. This allows us to divide the whole problem of quantum dynamics in two qualitatively different parts:

(i) The proper quantum dynamics as described by a very specific dynamical variable, the wavefunction $\psi(t)$, obeying some deterministic equation, for example the Schrödinger equation. Our discussion below will be limited to this part only.

(ii) The quantum measurement including the registration of the result and, hence, the collapse of the ψ function. This part still remains very vague to the extent that there is no common agreement even on the question whether this is a real physical problem or an ill-posed one so that the Copenhagen interpretation of quantum mechanics answers all the "admissible" questions. In any event, there exists as yet no dynamical description of the quantum measurement including the ψ collapse.

The recent breakthrough in the understanding of quantum chaos has been achieved, particularly, due to the above philosophy of separating out the dynamical part of quantum mechanics. Such a philosophy is accepted, explicitly or more often implicitly, by most researchers in this field. In this approach the initial state $\psi(0)$ of a quantum system is assumed to be the result of some complete quantum measurement. However, the question if any arbitrary $\psi(0)$ could be physically realized remains open (for discussion see, e.g., Ref. [12]). The quantum measurement as far as the result is concerned, is fundamentally a random process. However, there are good reasons to believe that this randomness can be interpreted as a particular manifestation of dynamical chaos [11].

In the presentation below we essentially follow our recent review [7] with the addition of new results, mainly unpublished. In view of the growing interest in the problem of quantum chaos it is practically impossible, within the limits of this talk, to discuss all the numerous papers devoted to this topic with various novel ideas, approaches and methods. Instead, we will focus on the most controversial part of the problem, the quantum dynamics with discrete spectrum. In this class of dynamical systems it is especially clear that quantum chaos is a *new* dynamical phenomenon, not a particular case of the classical-like dynamical chaos. As our numerous discussions reveal, most mathematicians as well as some physicists are still very reluctant to accept such an uncomfortable conception of quantum chaos. So, it appears worthwhile to dwell on this stumbling block again.

Below we shall discuss the quantum dynamics essentially in *momentum* space which is relevant to the applications in nuclear, atomic and molecular physics. There exists a broad class of the so-called *dual* problems related to the conjugated *configurational* (coordinate) space which are most relevant to the solid-state physics. The interrelation between the two had been discovered in Ref. [13], and proved to be very fruitful for both fields of research (for further discussion see, e.g., Ref. [7]).

2. Quantum maps

One class of models extensively used in the studies of quantum dynamics is described by the so-called quantum maps via the unitary operator $\hat{U}_T$ over some time interval T,

$$\overline{\psi(t)} \equiv \psi(t+T) = \hat{U}_T\,\psi(t),$$
$$\hat{U}_T = \exp\left(-i\int^{T} dt\,\hat{\mathcal{H}}\right), \tag{2.1}$$

where $\hat{\mathcal{H}}$ is the Hamiltonian operator. A well-known example is the Kepler map which approximately describes the interaction of a highly excited (Rydberg) Hydrogen atom with a monochromatic electric field (microwave) [14]. The Hamiltonian is given by

$$\mathcal{H} = \tfrac{1}{2}p^2 - 1/r + \varepsilon z\cos(\omega t)\,,$$

and the unitary operator over one orbital period of the electron is given by [14]

$$\hat{U}_T = \exp\left(i\pi\sqrt{\frac{2}{\omega\hat{\nu}}}\right)\exp\left(-ik(\varepsilon)\cos\hat{\phi}\right), \tag{2.2}$$

where $T = 1/n^3$ is the electron period related to the principal quantum number $n \gg 1$ (in atomic units), ε and ω are the strength and frequency of the electric field, respectively, $\nu = -E/\omega = +1/2n^2\omega$ is the action variable (number of photons), and ϕ the conjugate phase ($\hat{\nu} = -i\partial/\partial\phi$). The parameter k depends on the strength of the perturbation and is given by the estimate

$$k(\varepsilon) \sim \begin{cases} \varepsilon/\omega^{5/3}\,, & \varepsilon \lesssim 10\omega/m\,, \\ \exp\left(-\varepsilon m/5\omega\right)/\omega m^2\,, & \varepsilon \gtrsim 10\,\omega/m\,, \end{cases}$$

where m is the magnetic quantum number. The first case ($\varepsilon \lesssim 10\,\omega/m$) describes the regime of diffusive ionization and was extensively studied theoretically [14,87] and experimentally [88]. The second case ($\varepsilon \gtrsim 10\,\omega/m$) describes a recently discovered effect: stabilization of a Rydberg atom in strong fields [15].

The use of maps very much simplifies both analytical and numerical studies of dynamical problems. Yet, in quantum mechanics, two questions still remain open. First, the accuracy of the quantum map; indeed direct quantization of the classical map is usually very simple but generally different from the exact quantum map which should be obtained via integration of the Schrödinger equation. Second, a map (quantum or classical) usually describes the motion in a different (discrete) time τ whose relation to the original (continuous) time t depends on dynamical variables. For example, for the map (2.2),

$$\frac{\partial\tau}{\partial t} = \frac{1}{T} = n^3 = (2\omega\nu)^{-3/2}. \qquad (2.3)$$

For the quantum map it is not clear how to relate, in general, the discrete time $\psi(\nu, \tau)$ to the continuous time $\psi(\nu, t)$. Only for the quantum steady state it is possible to show that the following relation holds for the coarse-grained (phase-averaged) distribution function $f_s = \langle|\psi|^2\rangle$ [17]:

$$f_s(\nu; t) = f_s(\nu; \tau)\,\frac{\partial t}{\partial\tau}\,,$$

where brackets denote here averaging in time.

We would like to especially emphasize that contrary to a widespread misunderstanding, quantum maps describe also conservative systems. In this case they are called quantum *Poincaré maps*. The simplest example is the map which can be obtained from a time-dependent system considered in the so-called extended phase space [1]. A more physical example is the map for the Rydberg hydrogen atom in a static magnetic field which particularly describes long-live states in the continuum [18]. A general method for constructing Poincaré maps in the framework of the period-orbit quantization has been developed in Ref. [19].

One of the most popular models in the studies of classical and quantum dynamical chaos is the so-called *kicked rotator* described by the *standard map*

$$\bar{n} = n + k\cos\phi\,,$$
$$\bar{\phi} = \phi + \bar{n}T\,, \qquad (2.4)$$

where n, ϕ are action–angle variables. Quantization of map (2.4) leads to the unitary operator [6] (cf. Eq. (2.2))

$$\hat{U}_T = \exp\left(-iT\hat{n}^2/2\right)\,\exp\left(-ik\cos\hat{\phi}\right)\,, \qquad (2.5)$$

where $\hat{n} = -i\partial/\partial\phi$. It is important to stress that the standard map approximates more general maps locally in the action. For example, the Kepler map (2.2) is locally approximated by the standard map (2.5) [14] with $T = 6\pi\omega^2/(2\omega\nu)^{5/2}$, $n = \nu - \nu_0$.

The standard map (2.5) is defined on the cylinder ($-\infty < n < +\infty$) where the motion can be unbounded. To describe the bounded motion of a conservative system it is more convenient to make use of another version of the standard map, namely the map on a torus with a *finite* number of states L. In momentum representation $\psi(n, \tau)$ it is described by a finite unitary matrix U_{nm},

$$\overline{\psi(n)} = \sum_{m=-L_1}^{L_1} U_{nm}\,\psi(m)\,,$$

where $L = 2L_1 + 1 \approx 2L_1$, and

$$U_{nm} = \frac{1}{L}\exp\left(iT(n^2+m^2)/4\right) \times \sum_{j=-L_1}^{L_1} \exp[-ik\cos(2\pi j/L) -2\pi i(n-m)j/L]\,, \qquad (2.6)$$

while $T/4\pi = M/2L$ is now rational [20].

There are three quantum parameters in this model, the perturbation k, the period T and the number of states L, but only two classical combinations remain, the perturbation $K = kT$ and the classical size $M = TL/2\pi$, which is the number of resonances over the torus. Notice that the quantum dynamics is generally more rich than the classical one as the former depends on an extra parameter. This is, of course, another representation of Planck's constant which we have set $\hbar = 1$.

The quasiclassical region, where we expect quantum chaos, corresponds to $T \to 0$, $k \to \infty$, $L \to \infty$ while the classical parameters K = const. and M = const.

We would like to mention that in a recent paper [21] the standard map was used to describe the electron motion in a quasi-1D lattice (the model for a thin wire) which is another example of the map's application to the conservative (dual) problem.

3. Two basic time scales in quantum chaos: new facets

3.1. Random time scale and transition to classical chaos

The shortest characteristic time scale t_r of the classically chaotic quantum motion is called *random* time scale, and it is given by the estimate [22]

$$\Lambda t_r \sim \ln q\,, \tag{3.1}$$

where Λ stands for the classical Lyapunov exponent which is the instability rate of the classical motion. Indeed according to the Ehrenfest theorem the motion of a narrow wave packet follows the beam of classical trajectories as long as the packet remains narrow, and hence it is as random as in the classical limit. For the particular case of the standard map, expression (3.1) for the random time scale reduces to

$$\Lambda \tau_r \sim |\ln T|\,, \tag{3.2}$$

where $\Lambda \approx \ln K/2$, $K = kT$. Even though the random time scale is very short, it grows indefinitely as $q \to \infty$. Thus, a temporary, finite-time quantum pseudochaos turns into the classical dynamical chaos in accordance with the correspondence principle. To put it another way, the "true" chaos in the ergodic theory, which is not restricted in time as $t \to \infty$, is a limiting pattern, important in theory but it does not really exist in the quantum world. In a more formal mathematical language it is convenient (and much simpler) to consider the so-called *conditional limit*,

$$t,\, q \to \infty, \quad \tilde{t}(t,q) = \frac{t}{t_r(q)} = \text{const.}\,, \tag{3.3}$$

where the variable $\tilde{t}$ is termed *scaled* time.

Particularly, if we fix time t, then in the limit $q \to \infty$ we obtain the transition to the classical limit in accordance with the correspondence principle while for q fixed and $t \to \infty$ we have the proper quantum evolution in time. For example, the quantum Lyapunov exponent,

$$\Lambda_q(\tilde{t}) \to \begin{cases} \Lambda, & \tilde{t} \ll 1\,, \\ 0, & \tilde{t} \gg 1\,. \end{cases} \tag{3.4}$$

The quantum instability ($\Lambda_q > 0$) was observed in numerical experiments indeed (see Refs. [23,7] and Fig. 1). A new interesting question arises here: what does terminate the instability for $t \gtrsim t_r$? Numerical experiments show that the original wave packet, after a considerable stretching similar to the classical one, is rapidly destroyed. Namely, it gets split into many new small packets. A possible explanation [24] is related to the discreteness of the action variable in quantum mechanics which leads to the "rupture" of a very long stretched packet into many pieces (Fig. 1). Such a mechanism determines a new *destruction time scale* which, for the standard map, is given by the estimate [7]

$$\tau_d \sim \frac{|\ln T|}{2\Lambda}\,. \tag{3.5}$$

This roughly agrees with the results of numerical experiments [23,7]. As expected, $\tau_d \sim \tau_r$ (see Eq. (3.1)).

There is another mechanism which produces a deviation of the quantum packet evolution from the classical motion [24]. We call it *inflation,* because of the increase in time of the phase space area occupied by the Wigner function contrary to the classical phase space density which is conserved (Liouville's theorem). The inflation can be analyzed using the equation

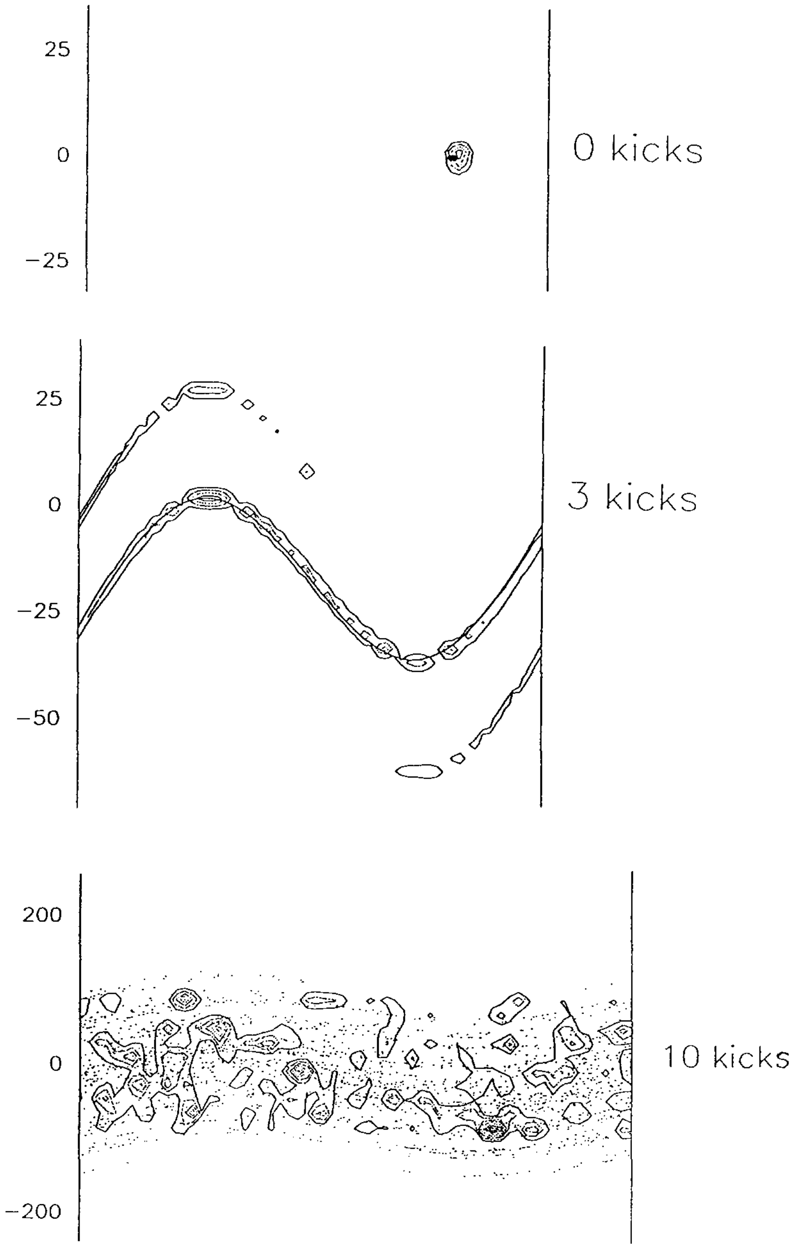

Fig. 1. Chaotic evolution of a wave packet of the kicked rotator in the Husimi representation. The quantum evolution of a minimum uncertainty (coherent-state) wave packet is compared numerically with the classical evolution of an ensemble of 2000 points: $K = 5$, $k = 25$, $T = 0.2$. (a) initial state; (b) after 3 kicks (random time scale); (c) after 10 kicks. Notice the destruction of the wave packet upon spreading over the whole phase interval, and the apparent absence of any substantial squeezing in spite of a large stretching (adapted from Ref. [7]).

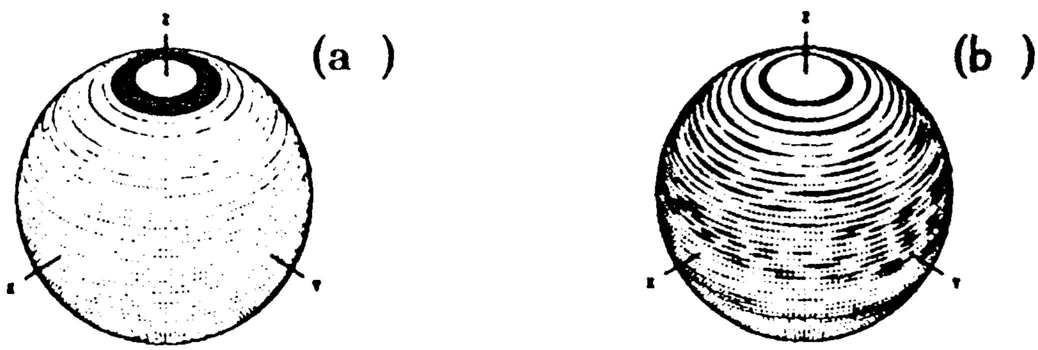

Fig. 2. Squeezing as well as stretching of a wave packet in the Wigner representation [26] (cf. Fig. 1): (a) initial wave packet; (b) stretched/squeezed state.

for the Wigner function W [25]. For the case of the standard map it can be shown [7] that this equation reduces to

$$\frac{dW(n,\phi)}{d\tau} \approx -\frac{1}{24}\frac{\partial^3\mathcal{H}}{\partial\phi^3}\frac{\partial^3 W}{\partial n^3}. \qquad (3.6)$$

This leads to the following estimates [7] for the *inflation time scale* τ_{if} and the *inflation phase size* $(\Delta\phi)_{if}$:

$$\tau_{if} \sim \frac{|\ln(TK^2/\Lambda^2)|}{6\Lambda},$$
$$(\Delta\phi)_{if} \sim \left(\frac{\Lambda}{k}\right)^{1/3} \to 0, \quad k\to\infty. \qquad (3.7)$$

The inflation time is of the order of the destruction time (3.5) and of the random time scale (3.1) as well. This implies a considerable squeezing of a wave packet which, however, is not seen in numerical experiments (Fig. 1). We explain it by the inherent limitation of the resolution of the Husimi probability (distribution function) used in almost all studies of quantum dynamics, including Refs. [23,7]. Indeed, the Husimi probability, being the projection of the Wigner quasiprobability on coherent states, has a technical (artificial) restriction of the resolution in both action *and* phase while the uncertainty principle restricts only the product of both (the area). The only paper we know [26] where the Wigner quasiprobability was used in such studies confirms this explanation. In Fig. 2, reproduced from Ref. [26], a considerable squeezing of the wave packet in the Wigner representation is clearly seen. A controlled resolution by using the distribution functions intermediate between Husimi's and Wigner's ones is described in Ref. [27].

Another interesting and as yet unclear question is: why is the inflation phase size $(\Delta\phi)_{if}$ small and, moreover, why does it indefinitely decrease in the quasiclassical region (3.7)? This theoretical estimate implies a considerable inflation prior to the destruction of the wave packet. It would be interesting to check this prediction numerically.

An important implication of the above picture of packet's time evolution is the rapid and complete destruction of the so-called generalized coherent states [51] in quantum chaos.

3.2. *Relaxation time scale and the periodic-orbit quantization*

The shortest random time scale is very important mainly because it allows the complete transition to classical dynamical chaos. Nevertheless, for statistical mechanics of quantum dynamical systems the most important time scale is another one, much longer, which is called the *relaxation time scale* t_R. It is given by the estimate

$$t_R \sim \rho_0 \le \rho, \qquad (3.8)$$

where ρ is the *full* density of energy (or quasienergy) levels, and ρ_0 is the level density of the *operative eigenstates* only, namely of those states which are actually present in the initial quantum state and, thus, really control the quantum motion. If, for example, the initial state is maximally localized in momentum ($\psi(m,0)=\delta_{mn}$), then (cf. Eq. (6.5))

$$\rho_0 \approx \sum_{m,n} |\varphi_m(n)|^2 \overline{\delta(E-E_m)}\,, \tag{3.9}$$

where $\varphi_m(n)$ and E_m are eigenfunctions and eigenvalues, respectively, and the bar denotes smoothing in energy.

The physical meaning of the estimate (3.8) is very simple, and is related to the fundamental uncertainty principle (see Refs. [10,28,29]). Indeed, for $t \ll t_R$ the discrete spectrum is not resolved, and the quantum motion partially mimics the motion with continuous spectrum. In particular, diffusion and statistical relaxation take place. Notice that typically (see also the estimate below) $t_R \gg t_r$ and therefore local instability is certainly sufficient but not necessary at all for a meaningful statistical description of a dynamical system. Thus quantum relaxation provides an example of a surprising phenomenon of dynamically stable diffusion which was confirmed in numerical experiments [30].

A technical difficulty in evaluating t_R for a particular dynamical problem is that the density ρ_0 depends, in turn, on the dynamics. So, we have to solve a self-consistent problem. For the standard map the answer is known (see Ref. [7]),

$$\tau_R = \rho_0 = 2D_0\,, \tag{3.10}$$

where $D_0 = k^2/2$ is the classical diffusion rate. The quantum diffusion rate depends on the scaled variable $\tilde{\tau} = \tau/2D_0(k)$ and is given by $\tilde{D}_q = D_0/(1+\tilde{\tau})$ [7]. This is an example of scaling in the discrete spectrum which stops eventually the quantum diffusion. Generally

$$\tilde{D}_q \to \begin{cases} D_0\,, & \tilde{\tau} = \tau/\tau_R \ll 1\,, \\ 0\,, & \tilde{\tau} \gg 1\,. \end{cases} \tag{3.11}$$

In closing this section, we would like to mention that one of the recent achievements in the theory of quantum chaos is a new technique of quantization for classically chaotic motion via classical periodic orbits (POQ) (see [19,31,32] and references therein). This technique allows to overcome the most difficult obstacle of the original Gutzwiller theory [34], a wild divergence of the series over the periodic orbits. Namely, it was discovered that the series can be truncated at periodic orbits with period less than $P_{\max}$, where

$$P_{\max} = \pi\rho\,. \tag{3.12}$$

This critical period is also called the Heisenberg time. It is of the order of relaxation time scale (3.8) *if* $\rho \sim \rho_0$. Yet, this is not always the case. For example, in the standard map on the cylinder $\rho=\infty$ (because quasienergies are determined mod $2\pi/T$) while ρ_0 is typically finite (3.10). The latter fact implies statistical relaxation to a non-ergodic state and this results in the *quantum localization*, or quantum suppression of classical diffusion. In our opinion, one of the main open questions in the POQ is precisely whether $P_{\max}$ is related to ρ or to ρ_0. In other words the question: does POQ theory describes the quantum localization too?

Even with the restriction (3.12) the number of periodic orbits to be included in the theory is exponentially large for a chaotic motion ($N_P \sim \exp(\Lambda P_{\max})$). An interesting and important question is: do we really need such a large number of periodic orbits or is it possible to considerably decrease $P_{\max}$ and, hence, N_P? Some interesting remarks on this problem are presented in Refs. [35,36]. It appears that, at least in some examples, only few periodic orbits are sufficient [37].

4. Intermediate asymptotics: diffusion localization as a mesoscopic phenomenon

The quantum localization is a non-universal (but very interesting and important) phenomenon which consists in the formation of non-ergodic or localized states (both a steady state as well as eigenstates) for classically ergodic motion. For the standard map on the torus the *ergodicity parameter* controlling localization can be defined as

$$\lambda = \frac{D_0}{L} \sim \left(\frac{\tau_R}{\tau_e}\right)^{1/2} \sim \frac{k^2}{L} \sim \frac{K}{M}k\,, \tag{4.1}$$

where $\tau_e \sim L^2/D_0$ is a characteristic time of the classical relaxation to the ergodic steady state $|\psi(n)|^2 \approx$

const.

If $\lambda \gg 1$ the final steady state as well as all the eigenfunctions are ergodic (after Shnirelman [38]; see also Refs. [39,40]), that is, the corresponding Wigner functions are close to the classical microcanonical distribution in phase space $\delta(H(n,\phi)-E)$. We call this region, characterized by $\lambda \gg 1$, *quasiclassical asymptotics*. It can be reached, particularly, if the classical parameter K/M is kept fixed while the quantum parameter $k \to \infty$.

However, if $\lambda \ll 1$, all the eigenstates and the steady state are non-ergodic. It means that their structure remains essentially quantum, no matter how large is the quasiclassical parameter $k \to \infty$. We call this region *intermediate asymptotics* or *mesoscopic domain*. Particularly, it corresponds to $K > 1$ fixed, $k \to \infty$ and $M \to \infty$, while $\lambda \ll 1$ remains small. The mesoscopic domain we are speaking about refers to the momentum space. In the dual problems in configurational space the mesoscopic phenomena are currently under intensive studies and may have important applications (e.g. in nanoelectronics; see Ref. [43]).

The size of localized states can be characterized by the so-called *entropy localization length* [20],

$$l_H = e^H, \quad H = -\sum_n |\varphi(n)|^2 \ln |\varphi(n)|^2. \tag{4.2}$$

In the case that the state is exponentially localized, $\varphi(n) \to \exp(-|n|/l)$, $|n| \to \infty$, l_H turns out to be close to the 'asymptotic localization length' l. For example, for the standard map it is known that the eigenstates are exponentially localized with $l \approx \frac{1}{2}D_0$.

Sometimes another measure of localization is used. It is called *participation ratio*,

$$\xi_m = \left(\sum_n |\varphi_m(n)|^4\right)^{-1}, \tag{4.3}$$

which is close but not identical to the previous length l_H.

In terms of the localization length the region of mesoscopic phenomena is defined by the double inequality

$$1 \ll l \ll L.$$

For dual problems in solid state physics the left inequality means that the mesoscopic structure is essentially of a macroscopic size.

An interesting point is that, at least in the standard map, the statistical relaxation to the quantum steady state can be described by a phenomenological diffusion equation (Refs. [7,36]) for the Green function in term of the scaled variables $\tilde{\sigma}$ and $\tilde{n}$,

$$\frac{\partial g(\tilde{n},\tilde{\sigma})}{\partial \tilde{\sigma}} = \frac{1}{4}\frac{\partial}{\partial \tilde{n}}\tilde{D}(\tilde{n})\frac{\partial g}{\partial \tilde{n}} + B(\tilde{n})\frac{\partial g}{\partial \tilde{n}}. \tag{4.4}$$

Here $g(\tilde{n},0) = \delta(\tilde{n}-\tilde{n}_0)$ and

$$\tilde{n} = \frac{n}{2D_0}, \quad \tilde{\sigma} = \ln(1+\tilde{\tau}), \quad \tilde{\tau} = \frac{\tau}{2D_0}.$$

The additional drift term in the diffusion equation (4.4),

$$B(\tilde{n}) = \operatorname{sign}(\tilde{n}-\tilde{n}_0) = \pm 1,$$

describes the so-called quantum coherent backscattering, which is the main cause of localization.

The solution of Eq. (4.4) can be expressed in terms of Error functions [7]. In particular, if we consider the scaled unperturbed energy

$$\tilde{E}(\tilde{\tau}) = \frac{\langle n^2\rangle}{2E_s},$$

where $E_s = D_0^2/4$ is the energy of the quantum steady state, then the relaxation rate is given by

$$\begin{aligned}\tilde{R} &\equiv \frac{d\tilde{E}}{d\tilde{\tau}} \\ &= 2\,e^{-\tilde{\sigma}}\left[\left(\tilde{\sigma}+\tfrac{1}{2}\right)\operatorname{erfc}(\sqrt{\tilde{\sigma}}) - \sqrt{\frac{\tilde{\sigma}}{\pi}}\,e^{-\tilde{\sigma}}\right] \\ &\to \frac{D_0^2}{\sqrt{\pi}\,\tilde{\tau}^2(\ln\tilde{\tau})^{3/2}}, \quad \tilde{\tau}\to\infty.\end{aligned} \tag{4.5}$$

Numerical experiments agree with the analytical result (4.5) only with logarithmic accuracy in τ. More precisely our numerical results [42] (Fig. 3) lead to the asymptotic behaviour $\tilde{R} \sim \left(D_0^2/\sqrt{\pi}\,\tilde{\tau}^2\right)/\sqrt{\ln\tilde{\tau}}$ while in a different approach [41] the expression $\tilde{R} \sim \left(D_0^2/\sqrt{\pi}\tilde{\tau}^2\right)\ln\tilde{\tau}$ is obtained [41]. On the experimental part, the main difficulty lies in the wild fluctuations of the relaxation rate R. To suppress them we made use, besides of the ensemble averaging over about 1000 runs, also of the special time averaging

$$\langle R(\tau)\rangle = \int R(\tau')\, W(\tau' - \tau)\, d\tau' , \tag{4.6}$$

with the weighting function

$$W(s) = A \exp\left(-\frac{p}{1 - (s/F)^2} \right), \quad |s| < F ,$$

where A is the normalizing factor, and p, F parameters.

In closing this section we would like to mention that the quantum diffusion localization occurs not only in time-dependent systems (another common misconception!) but also in conservative systems. The most striking example of the latter is the celebrated Anderson localization in solids (which is, however, a dual problem in our terminology, see Section 1). In momentum space the localization takes place, for example, when the Poincaré map for a conservative system is given by the standard map (see Section 2, and Ref. [56]). It is true that, to the best of our knowledge, nobody has observed as yet localization in conservative models directly described by the Schrödinger equation. In our opinion, this fact is explained by the very special type of models considered. Some indications of localization in Robnik's billiard can be found in Ref. [27] and will be discussed in Section 5.

5. Dynamical fluctuations in quantum chaos

Fluctuations are the most characteristic property of any true statistical process. As we shall see, quantum chaos exhibits a lot of various fluctuations justifying, thus, the term chaos, even in a discrete spectrum. The ultimate origin of these dynamical fluctuations is related to *decoherence*, both in space and time, of a typical quantum state associated with a classically chaotic system. This leads to correlation decay which has been found in many numerical experiments (see, e.g., Ref. [4]). Notice that the very existence of quantum diffusion, observed already in the first numerical experiments [6], implies both correlation decay and decoherence.

Below we consider some particular types of fluctuations studied recently. Among them the energy level fluctuations (distribution) or, to be more precise, the (nearest neighbor) *level spacing distribution*. Interestingly, such fluctuations reveal directly some chaos in the discrete spectrum itself (even for a classically regular motion!).

5.1. Ergodic states

A characteristic statistical property of ergodic eigenstates is the so-called *level repulsion*: the nearest level spacing distribution is described by the Wigner–Dyson formula,

$$p(s) \approx A s^{\beta} \exp(-Bs^2) , \tag{5.1}$$

where the repulsion parameter β takes the value $\beta = 1;2;4 \equiv \beta_e$ only depending on the system symmetry. Quite often this distribution is still called universal (see, e.g., Ref. [44]), in spite of the fact that, as we stressed several times [7], Eq. (5.1) is true in the case of quantum ergodicity only. Even in this case interesting exceptions have been found recently [45] whose mechanism is not completely clear. Instead, we call Eq. (5.1) the *limiting statistics* because it corresponds to the ergodicity parameter $\lambda \to \infty$ (see also Ref. [44]). So far, to our knowledge, the relation between the limiting statistics and limiting ergodicity according to the Shnirelman theorem [38] is not clear.

As was mentioned already above (Section 3.2) the ergodic steady state is close to the classical one up to some quantum fluctuations. For example, in the standard map we would expect the temporal rms energy fluctuations to be of the order

$$\frac{\Delta E}{E_s} \sim \frac{1}{\sqrt{L}} . \tag{5.2}$$

However, to our knowledge, nobody has checked this as yet numerically.

The spatial fluctuations in chaotic eigenfunctions are known to be close to Gaussian (Refs. [46,47]). Interestingly, a clear deviation from the Gaussian distribution was found due to finite dimensions of eigenfunctions in the Hilbert space ($L = 25$) [46]. Besides such irregular (quasirandom) fluctuations, addi-

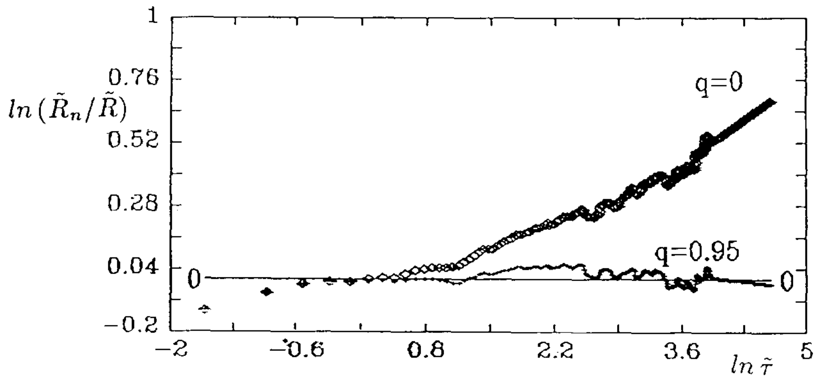

Fig. 3. Time dependence of the discrepancy between the numerically computed ensemble and time-averaged ($p = 0.1$) data $\tilde{R}_n$ and the time-averaged theory $\tilde{R}$ [42]: ($\diamond$) uncorrected theory ($q = 0$); (+) corrected theory ($q = 0.95$), rms deviation 2%.

tional microstructures were observed in chaotic eigenstates. The most studied one is connected to the celebrated Heller scars which appear around some classical (unstable) periodic orbits. However, recent numerical experiments apparently reveal also a different microstructure (Refs. [47,48]) which is currently under study.

5.2. Localized states

Statistical properties of localized states are much more rich and interesting since localization, which is an essential quantum feature, considerably modifies all the statistical properties. Instead of "universal" fluctuations (5.1), the *intermediate statistics* was discovered to hold (Refs. [20,49]) with

$$p(s) \approx A\, s^{\beta} \exp\left[-\tfrac{1}{16}\pi^2 \beta s^2 - \left(B - \tfrac{1}{4}\pi\beta\right) s\right], \quad 0 \le \beta \le 4\,, \tag{5.3}$$

where now the value of the repulsion parameter β (cf. Eq. (5.1)) depends on the degree of localization. In particular it can take the value $\beta = 0$ (Poisson statistics) which corresponds to the classically integrable case (or to extreme localization!). Moreover, the repulsion parameter β was found to be simply related to the localization parameter β_l [50] (see also Ref. [65] below) which gives a measure of the size of the eigenfunctions. Indeed, let us define

$$\beta_l = l_{\langle H \rangle}/l_e\,, \tag{5.4}$$

where $\langle\ \rangle$ indicate averaging over all eigenfunctions, and $l_e \ge l_{\langle H \rangle}$ is the maximal entropy localization length corresponding to ergodic states. Then it has been found that $\beta \approx \beta_e \beta_l$. No explanation of this surprisingly simple relation has been given as yet. Eq. (5.3) represents a "universal" family of distributions parametized by β (or λ).

It is instructive to compare this family with the very popular Brody distribution,

$$p(s) = A\, s^{\beta} \exp\left(-B s^{s+1}\right), \quad 0 \le \beta \le 1\,,$$

which is also a "universal" distribution introduced to fit all empirical data whatever the underlying mechanism for deviations from the limiting statistics. As a consequence the accuracy of the fitting turns out to be rather poor [65].

The dependence $\beta_l(\lambda)$ on the ergodicity parameter λ, or *scaling behaviour*, can be approximated by a simple expression [20],

$$\beta_l(\lambda) \approx \frac{a\lambda}{1 + a\lambda}\,, \quad a \approx 3\,, \quad \lambda \lesssim 10\,. \tag{5.5}$$

Deviations for larger λ remain unclear, and may depend on the boundary conditions and/or $\psi(n)$ symmetry [52] (see also Section 6).

The intermediate statistics (5.3) is due to the quantum localization under the assumption of *complete chaos* in the classical limit. This is why observation of such statistics ($\beta \approx 0.31$) in a billiard with classically divided phase space, containing about $\delta \approx 17\%$ of regular (stable) component of the motion, was rather puzzling [27]. The point is that the presence of a stable component leads to a nonzero level spacing density $p(s) \to p_0 > 0$ as $s \to 0$, contrary to numerical results [27] which confirm Eq. (5.3) down to $s \approx 0.004$. In our opinion, this demonstrates that the effect of quantum localization can be dominant even for $\delta > 0$. To clarify this puzzle we suggest a change of the model in Ref. [27] as follows: the boundary of Robnik's 2D billiard is given by the equation

$$|z + \varepsilon z^m| = 1 , \tag{5.6}$$

where z is the complex coordinate in the billiard plane. For $m = 2$, used in Ref. [27], the requirement for complete classical chaos (large ε) contradicts with that of the diffusive evolution of the particle's velocity direction (small ε) which is necessary for quantum localization (cf. Ref. [8]). To satisfy both conditions we need another parameter, for example $m \gg 1$, so that the billiard boundary becomes slightly wiggly and this leads to diffusion in velocity.

Exponentially localized eigenfunctions $\varphi_m(n)$ show wild fluctuations which are not only very big in size [20] but, moreover, diffusively increase in both directions of n [29],

$$\begin{aligned} &\langle(\eta_{mn} - \langle\eta_{mn}\rangle)^2\rangle = D_\eta\,|\Delta n| , \\ &D_\eta = \frac{1}{l} = -\frac{\langle\eta_{mn}\rangle}{|m-n|} , \end{aligned} \tag{5.7}$$

where $\eta_{mn} \sim \ln|\varphi_m(n)|$. Particularly, such fluctuations result in a surprising increase of the steady-state localization length $l_s \approx 2l$ as compared to that of eigenfunctions (l). Nevertheless, fluctuations of l itself vanish asymptotically as $|m-n| \to \infty$. Namely, the rms dispersion

$$\frac{\Delta l}{l} = \sqrt{\frac{l}{|\Delta n|}} \to 0 , \quad |\Delta n| \to \infty . \tag{5.8}$$

This is not the case for the "global" localization lengths l_H (4.2) or ξ (4.3) which give a measure of the extension of the eigenfunctions. It has been shown that empirical fluctuations of entropy H (and, hence, those of l_H) are described surprisingly well by a simple expression [53]

$$p(H) = \frac{1}{\cosh\left[\pi\left(H - \langle H\rangle\right)\right]} , \qquad \int p\,dH = 1 , \tag{5.9}$$

as shown in Fig. 4. This distribution, which has as yet no explanation leads to the rms dispersion

$$\frac{\Delta l_H}{l_H} \approx 0.66 . \tag{5.10}$$

According to recent preliminary numerical data [54] the fluctuations of the steady-state localization length are qualitatively different,

$$\frac{\Delta\xi_s}{\xi_s} \sim \xi_s^{-0.25} \to 0 , \quad k \to \infty , \tag{5.11}$$

but equally unexplained. Unlike Eq. (5.9) these fluctuations are vanishing in the quasiclassical region. Another interesting feature of fluctuations (5.11) is an "abnormal" (fractal?) exponent ≈ 0.25 instead of 0.5 for the "normal" fluctuations. Possible fractal properties of the quantum steady state are confirmed by the temporal fluctuations of the steady-state energy [55] (cf. Eq. (5.2)),

$$\frac{\Delta E_s}{E_s} \sim l_s^{-0.3} \to 0 , \quad k \to \infty . \tag{5.12}$$

Notice that both exponents, in Eqs. (5.11) and (5.12), are equal within the accuracy of numerical experiments. A naive interpretation of these fluctuations would imply that $\psi_s(n)$ for the chaotic steady state represents a *finite* ensemble of ν statistically independent systems. In this case one expects $\Delta E_s/E_s \sim 1/\sqrt{\nu}$ and comparison with (5.12) leads to

$$\nu \sim l_s^{0.6} . \tag{5.13}$$

6. Statistical models in quantum chaos: random matrices

The region $t \gg t_R$ (if t_R is finite) may be considered as the third, *asymptotic time scale*. Here the temporal fluctuations, in the quantum steady state, remain and are determined by the statistics of eigenfunctions and eigenvalues. These fluctuations are generally very complicated. On the other hand, there exists a well-developed statistical random matrix theory

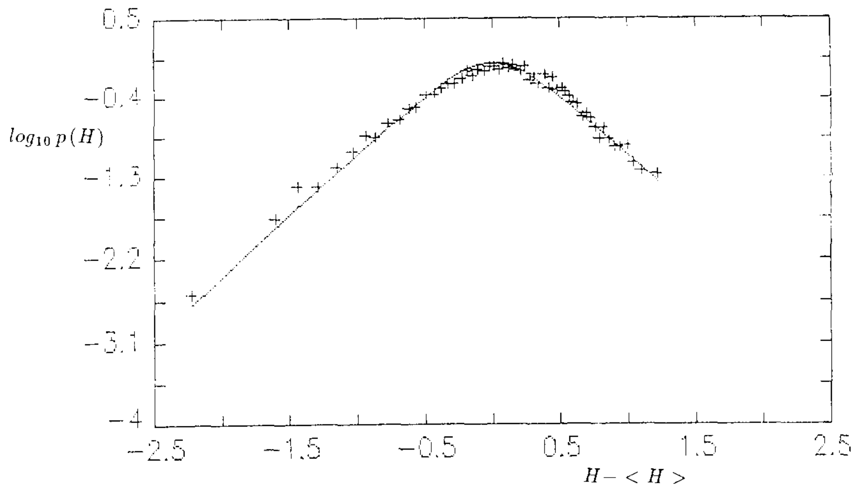

Fig. 4. Fluctuations of entropy $H = \ln l_H$ for strongly localized eigenstates of the kicked rotator on a torus (2.6). The numerical data are locally averaged and the solid line is given by Eq. (5.9).

(RMT) which describes the statistical properties of a typical quantum system with a given symmetry of the Hamiltonian (see, e.g., Ref. [57]). This was actually the first theory aiming at describing statistical properties of quantum systems with discrete spectrum.

At the beginning the object of this theory was assumed to be a very complicated, many-dimensional quantum system as a representative of a certain statistical ensemble. With the understanding of the phenomenon of dynamical chaos it became clear that the number of system freedoms is irrelevant. Instead, the number of quantum states or the quasiclassical parameter q is of importance provided dynamical chaos is present in the classical limit.

Until recently the matrices in this theory were assumed to be homogeneous, that is, all matrix elements have identical statistics. Such a RMT was found to satisfactorily represent the statistical properties of dynamical quantum chaos without localization (for both energies [58] and quasienergies [20]). However, the quantum localization essentially modifies the statistical properties as was discussed above. The statistical counterpart of the theory of quantum localization is not only the old Anderson theory but also a new development in RMT which makes use of the so-called band random matrices (BRM) [59]. These have nonzero random elements within a band of width $2b$ along the main diagonal only. The unitary matrix in the quantized standard map is also of a band structure with $b \approx k$. Here matrix elements are not random (see Eqs. (2.5) and (2.6)). However, they appear to be "pseudorandom" and indeed they lead to statistical properties of eigenvalues and eigenfunctions similar to those of BRM. In particular, a scaling behaviour has been found for both the quantized standard map and the BRM and the appropriate scaling (ergodicity) parameter is [60]

$$\lambda = ab^2/L, \tag{6.1}$$

where $a \sim 1$ is some numerical factor (cf. Eqs. (4.1) and (5.5)), and L stands now for the size of a matrix. Notice that in the dynamical problem discussed above, the ergodicity parameter (4.1) depends on the diffusion rate which includes dynamical correlations. This may explain why the statistics of random matrices and that of the corresponding dynamical models were found to be similar but not identical. For example, no deviations at large λ were found in the former models [61].

In a conservative system the parameter b^2 characterizes the width of the energy shell which is the quantum counterpart of the classical energy surface. Hence, the old RMT describes the locally ergodic quantum structure only, that is for $L \ll b^2$, and $L \ll l$. The global structure is associated instead with band matrices. The former approximation (full random matrices) is very good, for example, in heavy nuclei ($b^2 \sim 10^6$) but not in heavy atoms ($b^2 \sim 10$ only) [67]. The very large width of the energy shells in nuclei has interesting applications in the studies of the weak interaction of elementary particles [69].

An advantage of RMT is that it describes both classes of dual problems. Moreover, the statistical theory proved to be much simpler than the corresponding dynamical one, so that the former is well developed by now [62]. Particularly, the exact dependence $\beta_l(\lambda)$ was found to be very close to the simple relation (5.5) in the whole range ($0 < \lambda < \infty$). Moreover, the distribution of localization lengths ξ was also calculated and turned out to be rather different from that of l_H (5.9) in the dynamical problem. While for the latter both tails of the distribution follow a power-law with different exponents (see Eq. (5.9)), the distribution of ξ in BRM decays exponentially for both $\xi \to \infty$ as well as for $\xi \to 0$ [62].

In all the above papers the BRM were assumed to be *homogeneous*, that is, their statistical properties were the same along the main diagonal. Such BRM do not describe the global structure of conservative quantum systems. For example, their level density ρ grows indefinitely as $L \to \infty$. Moreover L is clearly an irrelevant (technical) parameter for a conservative system with its energy shells of a finite width. Only recently the so-called *inhomogeneous* BRM with increasing diagonal elements were studied in some detail [63],

$$H_{mn} = \frac{m}{\rho}\,\delta_{mn} + v_{mn}\,, \quad v_{mn} = v_{nm}\,. \tag{6.2}$$

Here $\langle v_{mn}\rangle = 0$, $\langle v_{mn}^2\rangle = v^2$ for $|m-n| < b$. Curiously, this had been the first random matrix model introduced and studied many years ago by Wigner [64], precisely to describe conservative quantum systems, (soon completely forgotten also by Wigner himself!), and reconsidered again only recently [63].

In Wigner's BRM the empirical scaling $\beta_l(\lambda)$ was found to be completely different [65] as compared to homogeneous BRM scaling (5.5). Namely,

$$\beta_l \approx 1 - e^{-\lambda}\,, \quad \lambda = a\,b^2/l_e\,, \quad a \sim 1\,, \tag{6.3}$$

where
(i) for "wide" eigenfunctions [65],

$$l_e = c\rho v\sqrt{b} \gtrsim b\,, \quad c \approx 5.29\,, \quad a \approx 1.14\,, \tag{6.4}$$

with the semicircle local spectral density [64]

$$\rho_1(E;n) = \sum_m |\varphi_m(n)|^2\,\delta(E - E_m) \to \frac{\sqrt{8\,b\,v^2 - (E-E_n)^2}}{4\pi\,b\,v^2}\,, \tag{6.5}$$

where the arrow indicates smoothing (and normalization) in energy for sufficiently large ρ (cf. Eq. (3.9));
(ii) for "narrow" eigenstates [66],

$$l_e = c(\rho v)^2 \lesssim b\,, \quad c \approx 42.45\,, \quad a \approx 0.25\,, \tag{6.6}$$

with the Breit–Wigner density [64]

$$\rho_2(E;n) = \frac{\Gamma/2\pi}{(E-E_n)^2 + \Gamma^2/4}\,, \quad \Gamma = 2\pi\,\rho\,v^2\,, \quad |E - E_n| < b/\rho\,. \tag{6.7}$$

In this model the localization length $l_e \geq l$ for ergodic states is always finite and independent of the matrix size $L \gg l_e$ as it should be in conservative systems. We call this *transverse localization* (across the energy shell). The proper, or *longitudinal localization* $l < l_e$ (along the shell) is only possible for the wide eigenstates since for the narrow ones $\lambda \sim b^2/l_e \gtrsim b \gg 1$. The remarkable relation $\beta \approx \beta_l$ has been confirmed for this model [65] to a surprisingly accuracy of about 1% (!?) within the whole available β range ($0.2 < \lambda < 2.5$).

The difficult question of how to relate the BRM parameters to a particular dynamical problem remains open. A straightforward approach – just to calculate the matrix elements H_{mn} for a given Hamiltonian – leads to a curious conclusion, namely that BRM become increasingly sparsed in the quasiclassical region [68]. Moreover, no ergodic states were observed in such a model. Even a simpler question, when one should use homogeneous or inhomogeneous matrices, is still not completely clear, at least, for the problems in momentum space.

7. Conclusion: other developments

Quantum chaos in a discrete spectrum, which has been discussed in the main part of this talk, appears as an interesting new dynamical phenomenon. Yet, it is a particular case of quantum dynamics and, moreover, it is the most difficult one for theoretical analysis. Much simpler is the consideration of unbounded motion with continuous spectrum. This is apparently because the spectral density, in the latter case, is much more smooth (as compared to a sum of δ-functions for the discrete spectrum (6.5)). As a result even the dynamical theory is much more developed in continuous spectrum (see, e.g., Ref. [70]). Notice, however, that continuous spectrum (hence, unbounded motion) is only a necessary condition for the "true" (classical-like, asymptotic in time) chaos in a quantum system but far from being a sufficient one (see Refs. [29,86]). Typically, quantum chaos remains a pseudochaos even in continuous spectrum. In this connection an interesting question which is currently under study concerns the algorithmic complexity of quantum dynamics [71].

Below we briefly mention some interesting and important developments in the studies of unbounded quantum motion (and of related topics).

Perhaps the most active field in this direction is in solid-state physics, apparently motivated by a rich variety of important applications (see, e.g., Refs. [72,73,62]). However, most results here were obtained for dual statistical problems. In the dynamical approach the so-called Harper model and kicked Harper model seem to have attracted the most attention (see, e.g., review [74] and references therein). A wide range of statistical phenomena were observed and studied in some detail using this model, including anomalous diffusion ($\langle(\Delta n)^2\rangle \sim t^\gamma,\ \gamma \neq 1$). Similar phenomena have been found recently also in the kicked rotator model [7] but only for very special values of parameter T.

Another new field of research is the study of decaying systems with energy levels with finite width (see [75] and references therein). Strictly speaking, all energy levels but the ground state have always some finite width ΔE due to interaction with fields. Yet, the statistical properties crucially depend on the *discreteness parameter*

$$d = \rho_0 \Delta E. \tag{7.1}$$

In case of $d \ll 1$, which we call *quasidiscrete spectrum*, the above results for the true discrete spectrum still persist. However, for $d \gtrsim 1$ the statistics of quantum chaos is substantially modified. This is very important, particularly in applications to heavy nuclei [76].

Another interesting problem concerns nonlinear "quantum" equations. In particular this problem allows us to compare the brand new phenomenon of quantum chaos with the old mechanism for statistical laws for large N (the so-called thermodynamic limit), which is a standard approach in traditional statistical physics, both classical and quantum. For infinitely dimensional quantum systems true chaos is also possible, like the chaos in the classical limit. When we speak about the absence of true chaos in quantum mechanics, we mean finite, and even few-dimensional, systems.

It turns out that both mechanisms are very similar; as for any finite N in the latter or q in the former, the dynamics is formally regular and in particular is characterized by a discrete spectrum. The main difference is in the nature of the large parameter, N or q. The similarity comes from the fact that if any of these parameters is large, the motion is controlled by a large number of frequencies which makes it very complicated. The study in quantum chaos helps to better understand the old mechanism for chaos in many-dimensional systems; in particular (for the latter), we conjecture the existence of characteristic time scales similar to those in quantum systems.

The direct relation between these two seemingly different mechanisms of chaos can be traced back in nonlinear quantum models. One interesting example is the nonlinear Schrödinger equation (NSE) [77] (for another example see Ref. [78]). From a physical point of view it describes the motion of a quantum system interacting with many other freedoms whose state is expressed via the ψ function of the system itself (the so-called mean field approximation). This approximation becomes exact in the limit $N \to \infty$ which is a particular case of the thermodynamic limit. Therefore, the mechanism for chaos in this system is the old one. On the other hand, NSE has generally exponentially unstable solutions, hence, the mechanism of chaos here is the new one. Thus, for this particular model both mechanisms describe the same physical process. We would like to emphasize that the true chaos present in these apparently few-dimensional quantum models actually refers to infinite-dimensional systems.

A group of related problems concerns the so-called *quasidegeneracy*, that is, exponentially small level spacings ($s \to 0$ as $q \to \infty$) which result from many effects, particularly from various tunneling processes. In Ref. [79] the tunneling was studied numerically between two domains of regular motion through a chaotic region which was found to accelerate the tunneling.

For example in the case of the kicked rotator in presence of strong localization, most quasienergy levels form narrow doublets because of the parity conservation and tunneling between two localized states centered at the angular momenta $\pm m$ ($m \gg l \ll L$) [80]. The splitting is expected to be similar to that for the Mott states (see, e.g., Ref. [41]),

$$\delta E \sim \exp\left(-C\, m/l\right), \quad l \approx \tfrac{1}{2} D_{cl}, \quad C \sim 1\,. \tag{7.2}$$

This degeneracy must produce a singularity (clustering) in the level spacing distribution

$$p(s) \to l/C\, s, \quad s \to 0, \tag{7.3}$$

if computed for the levels of both parities altogether. For the Mott states $p(s) \to$ const. $\neq 0$ for $s \to 0$ because the degeneracy is accidental [81]. We conjecture that the doublets persist even for violated parity conservation if the system remains time reversible. If true it would be a generalization of the puzzling Shnirelman theorem [82] (see also Ref. [39]) on the multiplicity of quantum spectra in classically nearly integrable (KAM) systems. The mechanism of the latter multiplicity is still unclear. Also the band width [20] and energy growth rate [7] in quantum resonance in kicked rotator seem to be of a similar nature.

In the final conclusion we would like to make a few comments on the problem of quantum measurement. The studies in quantum chaos suggest that it may have a close relation to this problem [11]. First the measurement device is by purpose a macroscopic system for which the classical description is a very good approximation. In such a system the true chaos with exponential instability is quite possible. The chaos in the measurement classical device is not only possible but unavoidable since the measurement system has to be, by purpose again, a highly unstable system. Indeed, a microscopic intervention produces here the macroscopic effect. The importance of chaos for the quantum measurement is in that it destroys the coherence of the initial pure quantum state to be measured converting it into an incoherent mixture. In the existing theories of the quantum measurement this is described as the effect of the external noise [83]. The chaos theory allows to get rid of the unsatisfactory effect of noise and to develop a purely dynamical theory for the loss of quantum coherence (see also Ref. [84]). But this is not yet the whole story. Indeed, besides the loss of coherence the most important effect of the quantum measurement is the redistribution of probabilities $|\psi|^2$ according to the result of the measurement, the famous ψ-collapse, which remains to be explained. Recently, some attempts, which are still to be understood and evaluated, were made to resolve this latter problem [85]. So far we would like simply to mention that these attempts are trying to make use of the nonlinear "semiquantum" equations briefly discussed above in this section.

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