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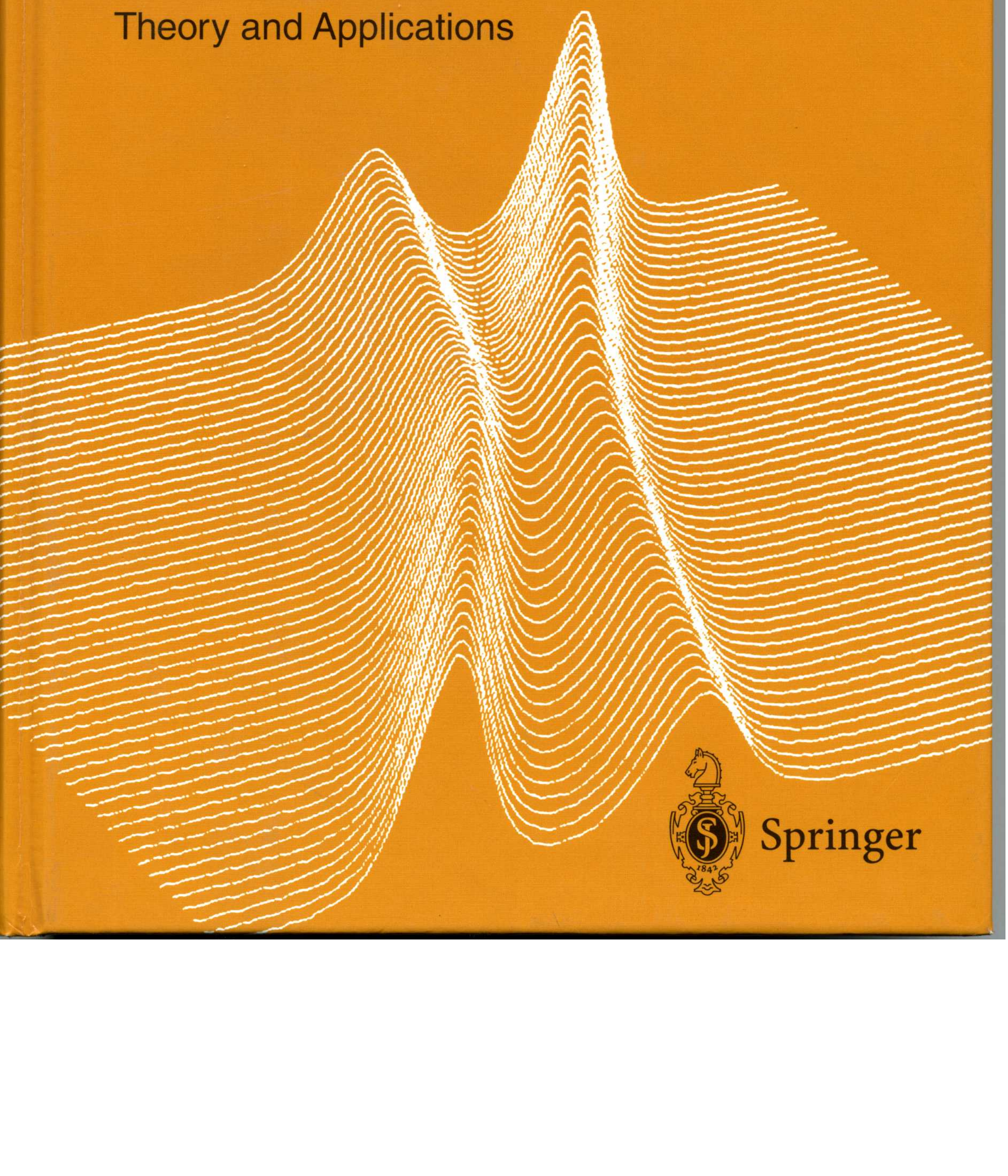

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Quantum Chaos and Ergodic Theory

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Abstract

The conception of quantum chaos is described in some detail. The most striking feature of this novel phenomenon is in that all the properties of classical dynamical chaos are retained but, typically, on finite and different time scales only. The necessary reformulation of the ergodic and algorithmic theories, as parts of the general theory of dynamical systems, is discussed. A number of specific unsolved problems is listed.

1. Introduction

This paper is primarily addressed to mathematicians with the main purpose of explaining new physical ideas in the so-called *quantum chaos* which has recently been attracting ever growing interest of many researchers [1–5, 10].

The breakthrough in understanding of this phenomenon has been achieved, particularly, due to a new philosophy accepted, explicitly or more often implicitly, in most studies of quantum chaos. Namely, the whole physical problem of quantum dynamics was separated into two different parts: (i) the proper quantum motion described by a specific dynamical variable $\phi(t)$ which obeys, e.g., the Schrödinger equation, and (ii) the quantum measurement including ψ collapse which, as yet, has no dynamical description. In this way one can single out the vague problem of the fundamental randomness in quantum mechanics which is related to the second part only, and which in a sense is foreign to the proper quantum system. The remaining first part then fits perfectly the general theory of dynamical systems.

The importance of quantum chaos is not only in that it represents a new unexplored field of nonintegrable quantum dynamics with many applications, but also, and this is most interesting for the fundamental science, in reconciling the two seemingly different dynamical mechanisms for the statistical laws in physics.

Historically, the first mechanism is related to the *thermodynamic limit* $N \to \infty$ in which the completely integrable system becomes chaotic for typical (random) initial conditions (see, e.g.,[6]). A natural question—what happens for large but finite number of freedoms N—has still no rigorous answer but the new phenomenon of quantum chaos, at least, presents an insight into this problem too. We call this mechanism, which is equally applicable in both classical and quantum mechanics, the *traditional statistical mechanics* (TSM).

The second (new) mechanism is based upon the strong (exponential) local instability of motion characterized by positive Lyapunov's exponent $\Lambda > 0$ [6, 7]. It is not at all restricted to large N, and is possible, e.g., for $N > 1$ in a Hamiltonian system. However, this mechanism has been considered, until recently, in the classical mechanics only. We term this the *dynamical chaos* as it does not require any random parameters or any noise in the equations of motion.

The quantum system bounded in phase space has a discrete energy (frequency) spectrum and is similar, in this respect, to the finite-N TSM. Moreover, such quantum systems are even completely integrable in the Hilbert space (see, e.g. [3]). Yet, the fundamental correspondence principle requires the transition to classical mechanics, including dynamical chaos, in the *classical limit* $q \to \infty$, where q is some quasi-classical parameter, e.g., the quantum number n (the action variable, $\hbar = 1$). Again, a natural physical conjecture is that for finite but large q there must be some chaos similar to finite-N TSM. Yet, in a chaotic quantum system the number of degrees of freedom N does not need to be large similarly to the classical chaos. The quantum counterpart of N is q, both quantities determining the number of frequencies which control the motion. Thus, mathematically, the problem of quantum chaos is the same as that for the finite-N TSM.

The main difficulty here (especially for mathematicians) is that the both problems suggest some chaos in the discrete spectrum which is completely contrary to the existing theory of dynamical systems and to the ergodic theory where such a spectrum corresponds to the opposite limit of regular motion.

The ultimate origin of the quantum integrability is discreteness of the phase space (but not, as yet, of the space-time!) or, in the modern mathematical language, the noncommutative geometry of the former.

As an illustration I will make use of the simple model described classically by the *standard map* (SM) [7, 8]:

$$\bar{n} = n + k \sin\theta \; ; \qquad \bar{\theta} = \theta + T\bar{n} \tag{1}$$

with action-angle variables n, θ, and perturbation parameters k, T. The quantized standard map (QSM) is given by [9, 10]

$$\bar{\psi} = \exp(-\mathrm{i}k\cos\bar{\theta}) \exp\left(-\mathrm{i}\frac{T}{2}\hat{n}^2\right) \psi \; , \tag{2}$$

where the momentum operator $\hat{n} = -i\partial/\partial\theta$. To provide the complete boundedness of the motion we consider SM on a torus of circumference (in n)

$$L = \frac{2\pi m}{T} \tag{3}$$

with integer m to avoid discontinuities. The quasi-classical transition corresponds to quantum parameters $k \to \infty$, $T \to 0$, $L \to \infty$ while classical parameters $K = kT = \text{const}$, and $m = LT/2\pi = \text{const}$ remain unchanged.

QSM models the *energy shell* of a conserved system which is the quantum counterpart of the classical energy surface.

In the studies of dynamical systems, both classical and quantal, most problems unreachable for rigorous mathematical analysis are treated "numerically" using the computer as a universal model. With all obvious drawbacks and limitations such "numerical experiments" have very important advantage as compared to the laboratory experiments, namely, they provide the complete information about the system under study. In quantum mechanics this advantage becomes crucial as in the laboratory one cannot observe (measure) the quantum system without a radical change of its dynamics.

2. Definition of Quantum Chaos

The common definition of classical chaos in physical literature is the *strongly unstable motion*, that is one with positive Lyapunov's exponents $\Lambda > 0$. The Alekseev-Brudno theorem then implies that almost all trajectories of such a motion are unpredictable, or random (see [11]). A similar definition of quantum chaos, which still has adherents among both mathematicians as well as a few physicists, fails because, for the bounded systems, the set of such motions is empty due to the discreteness of the phase space and, hence, of the spectrum.

The common definition of quantum chaos is *quantum dynamics of classically chaotic systems* whatever this might happen to be. Logically, this is a simple and clear definition. Yet, in my opinion, it is completely inadequate from the physical viewpoint just because such a chaos may turn out to be a perfectly regular motion as, for example, in case of the *perturbative localization* [12]. In QSM this corresponds to $k \leq 1$ when all quantum transitions are suppressed independent of classical parameter K which controls the chaos.

I would like to define quantum chaos in such a way as to include some essential part of classical chaos. The best definition I have managed to invent so far reads: the *quantum chaos is statistical relaxation in a discrete spectrum.* This definition is certainly in contradiction to the existing ergodic theory as the relaxation (particularly, correlation decay) requires the mixing, hence, a continuous spectrum. In what follows I will try to explain a new, modified, concept of mixing which is necessary to describe the peculiar phenomena of quantum chaos.

3. The Time Scales of Quantum Dynamics

The first numerical experiments with QSM already revealed the quantum diffusion in n close to the classical one under conditions $K \geq 1$ (classical stability border) and $k \geq 1$ (quantum stability border) [9]. Further studies confirmed this conclusion and showed that the former followed the latter in all details but on a *finite time interval* only [10, 13]. The latter fact was the clue to understanding the dynamical mechanism of the diffusion, which is apparently an aperiodic process, in a discrete spectrum. Indeed, the fundamental uncertainty principle implies that the discreteness of the spectrum is not resolved for sufficiently short time intervals. Whence, the estimate for the *diffusion (relaxation) time scale*:

$$t_R \sim \rho_0 \leq \rho \,. \tag{4}$$

Here ρ is the density of (quasi)energy levels, and ρ_0 is the same for the *operative eigenstates* which are actually present in the initial quantum state $\psi(0)$. In QSM the quasi-energies are determined mod $2\pi/T$ and, surprisingly, $\rho = LT/2\pi = m$ is a classical parameter (3). As to ρ_0, it depends on the dynamics and is given by the estimate [10, 13]:

$$\frac{\rho_0}{T} \sim \frac{t_R}{T} \equiv \tau_R \sim D \equiv \frac{\langle(\Delta n)^2\rangle}{\tau} \leq \frac{m}{T} \,. \tag{5}$$

Here τ is discrete map's time (the number of iterations), and D is the classical diffusion rate. This remarkable expression relates an essentially quantum characteristic (τ_R) to the classical one (D). The latter inequality in Eq. (5) follows from that in Eq. (4), and is explained by the boundedness of QSM on a torus.

In the quasi-classical region $\tau_R \sim k^2 \to \infty$ (see Eq.(1)) in accordance with the correspondence principle. Yet, the transition to the classical limit is (conceptually) difficult to understand (and still more to accept) as it involves two limits ($k \to \infty$ and $t \to \infty$) which do not commute. The second limit is related to the existing ergodic theory which is asymptotic in t. Meanwhile the new phenomenon of quantum chaos requires the modification of the theory to a finite time which is a difficult mathematical problem still to be solved. The main difficulty is in that even the distinction between the two opposite limits in the ergodic theory—discrete and continuous spectra—is asymptotic only.

In a relatively new *algorithmic theory* of dynamical systems the finite-time trajectories are also considered but, as yet, with the strongest statistical property—the randomness—only, which is generally unnecessary for a meaningful statistical description.

Besides the relatively long time scale (5) there is another one given by the estimate [14, 10]

$$t_r \sim \frac{\ln q}{\Lambda} \to \frac{T|\ln T|}{\ln(K/2)} \tag{6}$$

where q is some (large) quasi-classical parameter, and where the latter expression holds for QSM. It may be termed the *random time scale* since here the quantum motion of a narrow wave packet is as random as classical trajectories according to the Ehrenfest theorem. This was well confirmed in a number of numerical experiments [15]. The physical meaning of t_r is in the fast spreading of a wave packet due to the strong local instability of classical motion.

Even though the random time scale t_r is very short it grows indefinitely in the quasi-classical region ($q \to \infty$, $T \to 0$), again in agreement with the correspondence principle.

The big ratio t_R/t_r implies another peculiarity of quantum diffusion: it is dynamically stable as was demonstrated in striking numerical experiments [16].

4. The Quantum Steady State

As a result of quantum diffusion and relaxation some steady state is formed whose nature depends on the *ergodicity parameter*

$$\lambda = \frac{l_s}{L} \simeq \frac{D}{L}\,, \tag{7}$$

where l_s is the so-called localization length (see Eq.(10) below). If $\lambda \gg 1$ the quantum steady state is close (on average) to the classical statistical equilibrium which is described by ergodic phase density $g_{\mathrm{cl}}(n) = \mathrm{const}$ (for SM on a torus) where n is continuous variable. In quantum mechanics n is integer, and the quantum phase density $g_q(n, \tau)$ in the steady state fluctuates [17, 5], the ergodicity description can be given by relation

$$g_q(n) = \overline{|\psi_s(n,\tau)|^2} = \frac{1}{L}\,, \tag{8}$$

where the bar denotes time averaging.

According to numerical experiments the ergodicity does not depend on the initial state which implies that all eigenfunctions $\phi_m(n)$ are also ergodic, on average, with Gaussian fluctuations [17, 5]:

$$\langle |\phi_m(n)|^2 \rangle = \frac{1}{L}\,. \tag{9}$$

This is always the case sufficiently far in the quasi-classical region as $\lambda \sim k^2/L \sim Kk/m \to \infty$ with $k \to \infty$ ($K = kT$ and $m = LT/2\pi$ remain constant) in accordance with Shnirelman's theorem [18].

An interesting unsolved problem is the microstructure of ergodic eigenfunctions, particularly, the so-called 'scars' [29] which reveal the set of classical periodic trajectories (see [30] for the theory of scars).

Finite fluctuations (9) show that a single chaotic quantum system, described by $\psi_s(n,\tau)$, represents, in a sense, finite statistical ensemble of $M \sim L$ "particles". The fluctuations can result in partial recurrences toward the initial state but the recurrence time is much longer as compared to the relaxation time scale τ_R and sharply depends on the recurrence domain.

If $\lambda \ll 1$ the quantum steady state is qualitatively different from the classical one. Namely, it is localized in n within the region of size l, around the initial state if the size of the latter $l_0 \ll l_s$. Numerical experiments show that the phase space density, or the *quantum statistical measure*, is approximately exponential [10, 13]

$$g_s(n) \simeq \frac{1}{l_s}\exp\left(-\frac{2|n|}{l_s}\right) \; ; \qquad l_s \simeq D \tag{10}$$

for initial $g(n,0) = \delta(n)$. The quantum ensemble is now characterized by $M \sim l_s \sim k^2$ "particles".

The relaxation to this steady state is called *diffusion localization*, and it is described approximately by the diffusion equation [19, 28]

$$\frac{\partial g}{\partial \tau'} = \frac{1}{2}\frac{\partial}{\partial n}D\frac{\partial g}{\partial n} \pm \frac{\partial g}{\partial n} \tag{11}$$

for initial $g(n,0) = \delta(n)$, where the signs "$\pm$" correspond to $n \neq 1$, and where new time

$$\tau' = \tau_R \ln\left(1 + \frac{\tau}{\tau_R}\right) \tag{12}$$

accounts for the discrete motion spectrum [20]. The last term in Eq. (11) describes "backscattering" of ψ wave propagating in n which eventually results in the diffusion localization. The fitting parameter $\tau_R \sim 2D$ was derived from the best numerical data available (see Ref. [21], where a different theory of diffusion localization was also developed).

5. Concluding Remarks

In conclusion I would like to briefly mention a few important results for unbounded quantum motion. In SM this corresponds to $L \to \infty$. First, there is an interesting analogy between dynamical localization in momentum space and the celebrated Anderson localization in disordered solids which is a statistical theory. It was discovered in [22] and essentially developed in [23]. The analogy is based upon (and restricted by) the equations for eigenfunctions. The most striking (and less known) difference between the two problems is in the absence of a diffusion regime in $1D$ solids [24]. This is because the energy level density of the operative eigenfunctions in solids

$$\rho_0 \sim \frac{ldp}{dE} \sim \frac{l}{u} \sim t_R \tag{13}$$

which is the localization (relaxation) time scale, is always of the order of the time interval for a free spreading of the initial wave packet at characteristic velocity u.

Another similarity between the two problems is in that the Bloch extended states in a periodic potential correspond to a peculiar quantum resonance in QSM for rational $T/4\pi$ [9, 10].

An interesting open question is the dynamics for irrational Liouville's (transcendental) $T/4\pi$.

As was proved in [25] the motion can be unbounded in this case unlike a typical irrational value. The latter is the result of numerical experiments, no rigorous proof of localization for $k \gg 1$ has been found as yet.

In [28] the conjecture is put forward, supported by some semiqualitative considerations, that depending on a particular Liouville's number the broad range of motions is possible, from a purely resonant one ($|n| \sim \tau$) down to complete localization ($|n| \leq l$).

If quantum motion is not only unbounded but its rate in unbounded variables is exponential, then "true" chaos (not restricted to a finite time scale) can occur. A few exotic examples together with considerations from different viewpoints can be found in [10, 26]. However, such chaos does not seem to be a typical quantum dynamics.

The final remark is that the quantum chaos, as defined in Sect. 2, comprises not only quantum systems but also any linear, particularly classical, waves [27]. So, it is essentially the *linear wave chaos.* Moreover, a similar mechanism also works in completely integrable nonlinear systems like the Toda lattice, for example [31]. From a mathematical point of view all these new ideas require reconsideration of the existing ergodic theory. Perhaps it is better to say that a new ergodic theory is wanted which, instead of benefiting from the asymptotic approximation ($|t| \to \infty$ or $N \to \infty$), could analyze the finite-time statistical properties of dynamical systems. In my opinion, this is the most important conclusion emerging from first attempts to comprehend quantum chaos.

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