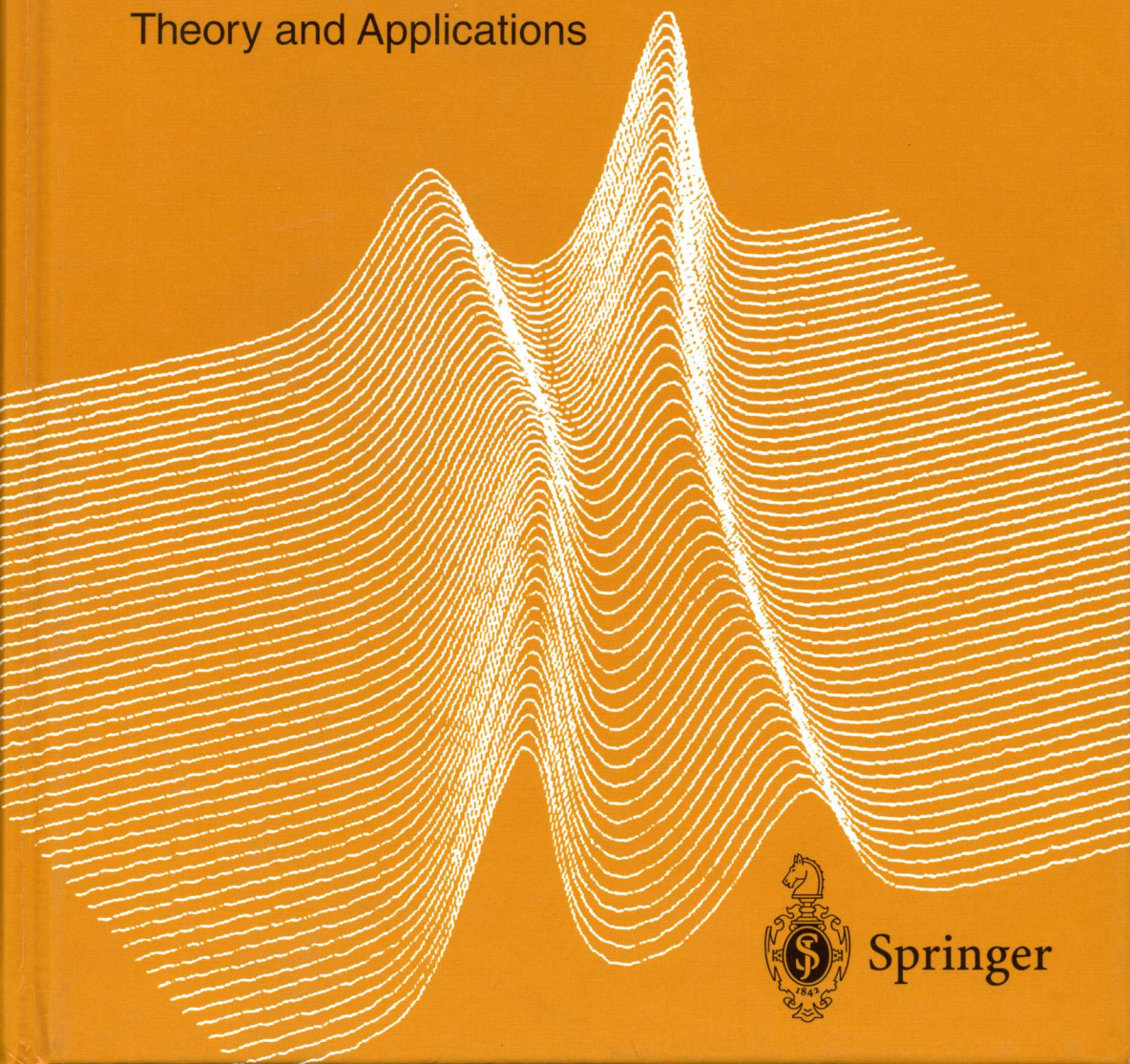


Springer Series in
NONLINEAR DYNAMICS

J. Awrejcewicz (Ed.)

Bifurcation and Chaos

Theory and Applications



Springer

Professor Jan Awrejcewicz

Technical University of Łódź, Division of Dynamics and Control (K-13)
Stefanowskiego 1/15, 90-924 Łódź, Poland

Volume Editor

Professor Miki Wadati

University of Tokyo, Faculty of Science, Department of Physics
7-3-1 Hongo, Bunkyo-ku, Tokyo 113, Japan

ISBN 3-540-58531-1 Springer-Verlag Berlin Heidelberg New York

CIP data applied for

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1995
Printed in Germany

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Typesetting: Camera ready copy from the author/editor using a Springer T_EX macro package
SPIN: 10424400 55/3140 - 5 4 3 2 1 0 - Printed on acid-free paper

Table of Contents

Introduction	1
 Quantum Chaos and Ergodic Theory	
B.Y. Chirikov	9
1. Introduction	9
2. Definition of Quantum Chaos	11
3. The Time Scales of Quantum Dynamics	12
4. The Quantum Steady State	13
5. Concluding Remarks	14
References	15
 On the Complete Characterization of Chaotic Attractors	
R. Stoop	17
1. Introduction	17
2. Scaling Behavior	18
2.1 Scale Invariance	18
2.2 Non-unified Approach	20
3. Unified Approach	22
3.1 The Generalized Entropy Function	22
3.2 Hyperbolic Models with Complete Grammars	25
4. Extensions	31
4.1 The Need for Extensions	31
4.2 Convergence Properties	31
4.3 Nonhyperbolicity and Phase-Transitions	34
5. Conclusions	44
References	44

New Numerical Methods for High Dimensional Hopf Bifurcation Problems

J. Wu and K. Zhou	47
1. Introduction	47
2. Static Bifurcation and Pseudo-Arclength Method	49
3. The Numerical Methods for Hopf Bifurcation	52
4. Examples	65
References	68

Catastrophe Theory and the Vibro-Impact Dynamics of Autonomous Oscillators

G.S. Whiston	71
1. Introduction	71
2. Generalities on Vibro-Impact Dynamics	73
3. The Geometry of Singularity Subspaces	79
4. Continuity of the Poincaré Map of the S/U Oscillator	93
References	95

Codimension Two Bifurcation and Its Computational Algorithm

H. Kawakami and T. Yoshinaga	97
1. Introduction	97
2. Bifurcations of Fixed Point	99
2.1 The Poincaré Map and Property of Fixed Points	99
2.2 Codimension One Bifurcations	100
2.3 Codimension Two Bifurcations	102
3. Computational Algorithms	111
3.1 Derivatives of the Poincaré Map	112
3.2 Numerical Method of Analysis	113
4. Numerical Examples	115
4.1 Circuit Model for Chemical Oscillation at a Water-Oil Interface	116
4.2 Coupled Oscillator with a Sinusoidal Current Source	123
5. Concluding Remarks	129
References	131

Chaos and Its Associated Oscillations in Josephson Circuits

M. Morisue, A. Kanasugi	133
1. Introduction	133
2. Model of Josephson Junction	134
3. Chaos in a Forced Oscillation Circuit	137

4. Autonomous Josephson Circuit	141
4.1 Introduction	141
4.2 Results of Calculation	145
5. Distributed Parameter Circuit	147
6. Conclusion	152
References	152

Chaos in Systems with Magnetic Force

J. Tani	153
1. Introduction	153
2. System of Two Conducting Wires	154
2.1 Formulation of Dynamical Equations	154
2.2 Analytical Procedure	157
2.3 Numerical Simulation of Chaos	158
3. Multi-Equilibrium Magnetoelastic Systems	163
3.1 Theoretical Models	164
3.2 Numerical Simulation	169
3.3 Experiment	173
4. Magnetic Levitation Systems	178
4.1 Formulation of Dynamic Equations	178
4.2 Linearization in Terms of Manifolds	183
4.3 Numerical Simulation	184
4.4 Conclusion	189
References	190

Bifurcation and Chaos in the Helmholtz-Duffing Oscillator

G. Rega	191
1. Mechanical System and Mathematical Model	191
2. Behaviour Chart and Characterization of Chaotic Response	194
3. Prediction of Local Bifurcations of Regular Solutions	198
4. Geometrical Description of System Response Using Attractor-Basin Portraits and Invariant Manifolds	207
5. Conclusions	213
References	214

Bifurcations and Chaotic Motions in Resonantly Excited Structures

S.I. Chang, A.K. Bajaj and P. Davies	217
1. Introduction	217
2. Nonlinear Structural Members	219
2.1 Strings	219

2.2	Beams	220
2.3	Cylindrical Shells and Rings	221
2.4	Plates	222
3.	Resonant Motions of Rectangular Plates with Internal and External Resonances	223
3.1	Equations of Motion	224
3.2	Averaged Equations	226
3.3	Steady-State Constant Solutions	229
3.4	Stability Analysis of Constant Solutions	233
3.5	Periodic and Chaotic Solutions of Averaged Equations	238
4.	Summary and Conclusions	247
	References	247
	Appendix	250

Non-Linear Behavior of a Rectangular Plate Exposed to Airflow

J. Awrejcewicz, J. Mrozowski and M. Potier-Ferry	253
1. Introduction	253
2. Mathematical Model	254
3. Threshold Determination of Periodic Oscillations	262
4. Dynamics Past the Hopf Bifurcation Point	264
5. Summary and Concluding Remarks	270
References	271

Quantum Chaos and Ergodic Theory

B.Y. Chirikov

Budker Institute of Nuclear Physics
630090 Novosibirsk, USSR

Abstract

The conception of quantum chaos is described in some detail. The most striking feature of this novel phenomenon is in that all the properties of classical dynamical chaos are retained but, typically, on finite and different time scales only. The necessary reformulation of the ergodic and algorithmic theories, as parts of the general theory of dynamical systems, is discussed. A number of specific unsolved problems is listed.

1. Introduction

This paper is primarily addressed to mathematicians with the main purpose of explaining new physical ideas in the so-called *quantum chaos* which has recently been attracting ever growing interest of many researchers [1–5, 10].

The breakthrough in understanding of this phenomenon has been achieved, particularly, due to a new philosophy accepted, explicitly or more often implicitly, in most studies of quantum chaos. Namely, the whole physical problem of quantum dynamics was separated into two different parts: (i) the proper quantum motion described by a specific dynamical variable $\phi(t)$ which obeys, e.g., the Schrödinger equation, and (ii) the quantum measurement including ψ collapse which, as yet, has no dynamical description. In this way one can single out the vague problem of the fundamental randomness in quantum mechanics which is related to the second part only, and which in a sense is foreign to the proper quantum system. The remaining first part then fits perfectly the general theory of dynamical systems.

The importance of quantum chaos is not only in that it represents a new unexplored field of nonintegrable quantum dynamics with many applications, but also, and this is most interesting for the fundamental science, in reconciling the two seemingly different dynamical mechanisms for the statistical laws in physics.

Historically, the first mechanism is related to the *thermodynamic limit* $N \rightarrow \infty$ in which the completely integrable system becomes chaotic for typical (random) initial conditions (see, e.g., [6]). A natural question—what happens for large but finite number of freedoms N —has still no rigorous answer but the new phenomenon of quantum chaos, at least, presents an insight into this problem too. We call this mechanism, which is equally applicable in both classical and quantum mechanics, the *traditional statistical mechanics* (TSM).

The second (new) mechanism is based upon the strong (exponential) local instability of motion characterized by positive Lyapunov's exponent $\Lambda > 0$ [6, 7]. It is not at all restricted to large N , and is possible, e.g., for $N > 1$ in a Hamiltonian system. However, this mechanism has been considered, until recently, in the classical mechanics only. We term this the *dynamical chaos* as it does not require any random parameters or any noise in the equations of motion.

The quantum system bounded in phase space has a discrete energy (frequency) spectrum and is similar, in this respect, to the finite- N TSM. Moreover, such quantum systems are even completely integrable in the Hilbert space (see, e.g. [3]). Yet, the fundamental correspondence principle requires the transition to classical mechanics, including dynamical chaos, in the *classical limit* $q \rightarrow \infty$, where q is some quasi-classical parameter, e.g., the quantum number n (the action variable, $\hbar = 1$). Again, a natural physical conjecture is that for finite but large q there must be some chaos similar to finite- N TSM. Yet, in a chaotic quantum system the number of degrees of freedom N does not need to be large similarly to the classical chaos. The quantum counterpart of N is q , both quantities determining the number of frequencies which control the motion. Thus, mathematically, the problem of quantum chaos is the same as that for the finite- N TSM.

The main difficulty here (especially for mathematicians) is that the both problems suggest some chaos in the discrete spectrum which is completely contrary to the existing theory of dynamical systems and to the ergodic theory where such a spectrum corresponds to the opposite limit of regular motion.

The ultimate origin of the quantum integrability is discreteness of the phase space (but not, as yet, of the space-time!) or, in the modern mathematical language, the noncommutative geometry of the former.

As an illustration I will make use of the simple model described classically by the *standard map* (SM) [7, 8]:

$$\bar{n} = n + k \sin \theta ; \quad \bar{\theta} = \theta + T\bar{n} \quad (1)$$

with action-angle variables n, θ , and perturbation parameters k, T . The quantized standard map (QSM) is given by [9, 10]

$$\bar{\psi} = \exp(-ik \cos \bar{\theta}) \exp\left(-i\frac{T}{2}\hat{n}^2\right) \psi, \quad (2)$$

where the momentum operator $\hat{n} = -i\partial/\partial\theta$. To provide the complete boundedness of the motion we consider SM on a torus of circumference (in n)

$$L = \frac{2\pi m}{T} \quad (3)$$

with integer m to avoid discontinuities. The quasi-classical transition corresponds to quantum parameters $k \rightarrow \infty$, $T \rightarrow 0$, $L \rightarrow \infty$ while classical parameters $K = kT = \text{const}$, and $m = LT/2\pi = \text{const}$ remain unchanged.

QSM models the *energy shell* of a conserved system which is the quantum counterpart of the classical energy surface.

In the studies of dynamical systems, both classical and quantal, most problems unreachable for rigorous mathematical analysis are treated “numerically” using the computer as a universal model. With all obvious drawbacks and limitations such “numerical experiments” have very important advantage as compared to the laboratory experiments, namely, they provide the complete information about the system under study. In quantum mechanics this advantage becomes crucial as in the laboratory one cannot observe (measure) the quantum system without a radical change of its dynamics.

2. Definition of Quantum Chaos

The common definition of classical chaos in physical literature is the *strongly unstable motion*, that is one with positive Lyapunov’s exponents $\Lambda > 0$. The Alekseev-Brudno theorem then implies that almost all trajectories of such a motion are unpredictable, or random (see [11]). A similar definition of quantum chaos, which still has adherents among both mathematicians as well as a few physicists, fails because, for the bounded systems, the set of such motions is empty due to the discreteness of the phase space and, hence, of the spectrum.

The common definition of quantum chaos is *quantum dynamics of classically chaotic systems* whatever this might happen to be. Logically, this is a simple and clear definition. Yet, in my opinion, it is completely inadequate from the physical viewpoint just because such a chaos may turn out to be a perfectly regular motion as, for example, in case of the *perturbative localization* [12]. In QSM this corresponds to $k \leq 1$ when all quantum transitions are suppressed independent of classical parameter K which controls the chaos.

I would like to define quantum chaos in such a way as to include some essential part of classical chaos. The best definition I have managed to invent so far reads: the *quantum chaos is statistical relaxation in a discrete spectrum*. This definition is certainly in contradiction to the existing ergodic theory as the relaxation (particularly, correlation decay) requires the mixing, hence, a continuous spectrum. In what follows I will try to explain a new, modified, concept of mixing which is necessary to describe the peculiar phenomena of quantum chaos.

3. The Time Scales of Quantum Dynamics

The first numerical experiments with QSM already revealed the quantum diffusion in n close to the classical one under conditions $K \geq 1$ (classical stability border) and $k \geq 1$ (quantum stability border) [9]. Further studies confirmed this conclusion and showed that the former followed the latter in all details but on a *finite time interval* only [10, 13]. The latter fact was the clue to understanding the dynamical mechanism of the diffusion, which is apparently an aperiodic process, in a discrete spectrum. Indeed, the fundamental uncertainty principle implies that the discreteness of the spectrum is not resolved for sufficiently short time intervals. Whence, the estimate for the *diffusion (relaxation) time scale*:

$$t_R \sim \rho_0 \leq \rho. \quad (4)$$

Here ρ is the density of (quasi)energy levels, and ρ_0 is the same for the *operative eigenstates* which are actually present in the initial quantum state $\psi(0)$. In QSM the quasi-energies are determined mod $2\pi/T$ and, surprisingly, $\rho = LT/2\pi = m$ is a classical parameter (3). As to ρ_0 , it depends on the dynamics and is given by the estimate [10, 13]:

$$\frac{\rho_0}{T} \sim \frac{t_R}{T} \equiv \tau_R \sim D \equiv \frac{\langle (\Delta n)^2 \rangle}{\tau} \leq \frac{m}{T}. \quad (5)$$

Here τ is discrete map's time (the number of iterations), and D is the classical diffusion rate. This remarkable expression relates an essentially quantum characteristic (τ_R) to the classical one (D). The latter inequality in Eq. (5) follows from that in Eq. (4), and is explained by the boundedness of QSM on a torus.

In the quasi-classical region $\tau_R \sim k^2 \rightarrow \infty$ (see Eq.(1)) in accordance with the correspondence principle. Yet, the transition to the classical limit is (conceptually) difficult to understand (and still more to accept) as it involves two limits ($k \rightarrow \infty$ and $t \rightarrow \infty$) which do not commute. The second limit is related to the existing ergodic theory which is asymptotic in t . Meanwhile the new phenomenon of quantum chaos requires the modification of the theory to a finite time which is a difficult mathematical problem still to be solved. The main difficulty is in that even the distinction between the two opposite limits in the ergodic theory—discrete and continuous spectra—is asymptotic only.

In a relatively new *algorithmic theory* of dynamical systems the finite-time trajectories are also considered but, as yet, with the strongest statistical property—the randomness—only, which is generally unnecessary for a meaningful statistical description.

Besides the relatively long time scale (5) there is another one given by the estimate [14, 10]

$$t_r \sim \frac{\ln q}{\Lambda} \rightarrow \frac{T |\ln T|}{\ln(K/2)} \quad (6)$$

where q is some (large) quasi-classical parameter, and where the latter expression holds for QSM. It may be termed the *random time scale* since here the quantum motion of a narrow wave packet is as random as classical trajectories according to the Ehrenfest theorem. This was well confirmed in a number of numerical experiments [15]. The physical meaning of t_r is in the fast spreading of a wave packet due to the strong local instability of classical motion.

Even though the random time scale t_r is very short it grows indefinitely in the quasi-classical region ($q \rightarrow \infty$, $T \rightarrow 0$), again in agreement with the correspondence principle.

The big ratio t_R/t_r implies another peculiarity of quantum diffusion: it is dynamically stable as was demonstrated in striking numerical experiments [16].

4. The Quantum Steady State

As a result of quantum diffusion and relaxation some steady state is formed whose nature depends on the *ergodicity parameter*

$$\lambda = \frac{l_s}{L} \simeq \frac{D}{L}, \quad (7)$$

where l_s is the so-called localization length (see Eq.(10) below). If $\lambda \gg 1$ the quantum steady state is close (on average) to the classical statistical equilibrium which is described by ergodic phase density $g_{cl}(n) = \text{const}$ (for SM on a torus) where n is continuous variable. In quantum mechanics n is integer, and the quantum phase density $g_q(n, \tau)$ in the steady state fluctuates [17, 5], the ergodicity description can be given by relation

$$g_q(n) = \overline{|\psi_s(n, \tau)|^2} = \frac{1}{L}, \quad (8)$$

where the bar denotes time averaging.

According to numerical experiments the ergodicity does not depend on the initial state which implies that all eigenfunctions $\phi_m(n)$ are also ergodic, on average, with Gaussian fluctuations [17, 5]:

$$\langle |\phi_m(n)|^2 \rangle = \frac{1}{L}. \quad (9)$$

This is always the case sufficiently far in the quasi-classical region as $\lambda \sim k^2/L \sim Kk/m \rightarrow \infty$ with $k \rightarrow \infty$ ($K = kT$ and $m = LT/2\pi$ remain constant) in accordance with Shnirelman's theorem [18].

An interesting unsolved problem is the microstructure of ergodic eigenfunctions, particularly, the so-called ‘scars’ [29] which reveal the set of classical periodic trajectories (see [30] for the theory of scars).

Finite fluctuations (9) show that a single chaotic quantum system, described by $\psi_s(n, \tau)$, represents, in a sense, finite statistical ensemble of $M \sim L$ “particles”. The fluctuations can result in partial recurrences toward the initial state but the recurrence time is much longer as compared to the relaxation time scale τ_R and sharply depends on the recurrence domain.

If $\lambda \ll 1$ the quantum steady state is qualitatively different from the classical one. Namely, it is localized in n within the region of size l , around the initial state if the size of the latter $l_0 \ll l_s$. Numerical experiments show that the phase space density, or the *quantum statistical measure*, is approximately exponential [10, 13]

$$g_s(n) \simeq \frac{1}{l_s} \exp\left(-\frac{2|n|}{l_s}\right) ; \quad l_s \simeq D \quad (10)$$

for initial $g(n, 0) = \delta(n)$. The quantum ensemble is now characterized by $M \sim l_s \sim k^2$ “particles”.

The relaxation to this steady state is called *diffusion localization*, and it is described approximately by the diffusion equation [19, 28]

$$\frac{\partial g}{\partial \tau'} = \frac{1}{2} \frac{\partial}{\partial n} D \frac{\partial g}{\partial n} \pm \frac{\partial g}{\partial n} \quad (11)$$

for initial $g(n, 0) = \delta(n)$, where the signs “ \pm ” correspond to $n \neq 1$, and where new time

$$\tau' = \tau_R \ln \left(1 + \frac{\tau}{\tau_R}\right) \quad (12)$$

accounts for the discrete motion spectrum [20]. The last term in Eq. (11) describes “backscattering” of ψ wave propagating in n which eventually results in the diffusion localization. The fitting parameter $\tau_R \sim 2D$ was derived from the best numerical data available (see Ref. [21], where a different theory of diffusion localization was also developed).

5. Concluding Remarks

In conclusion I would like to briefly mention a few important results for unbounded quantum motion. In SM this corresponds to $L \rightarrow \infty$. First, there is an interesting analogy between dynamical localization in momentum space and the celebrated Anderson localization in disordered solids which is a statistical theory. It was discovered in [22] and essentially developed in [23]. The analogy is based upon (and restricted by) the equations for eigenfunctions. The most striking (and less known) difference between the two problems is in

the absence of a diffusion regime in $1D$ solids [24]. This is because the energy level density of the operative eigenfunctions in solids

$$\rho_0 \sim \frac{ldp}{dE} \sim \frac{l}{u} \sim t_R \quad (13)$$

which is the localization (relaxation) time scale, is always of the order of the time interval for a free spreading of the initial wave packet at characteristic velocity u .

Another similarity between the two problems is in that the Bloch extended states in a periodic potential correspond to a peculiar quantum resonance in QSM for rational $T/4\pi$ [9, 10].

An interesting open question is the dynamics for irrational Liouville's (transcendental) $T/4\pi$.

As was proved in [25] the motion can be unbounded in this case unlike a typical irrational value. The latter is the result of numerical experiments, no rigorous proof of localization for $k \gg 1$ has been found as yet.

In [28] the conjecture is put forward, supported by some semiquantitative considerations, that depending on a particular Liouville's number the broad range of motions is possible, from a purely resonant one ($|n| \sim \tau$) down to complete localization ($|n| \leq l$).

If quantum motion is not only unbounded but its rate in unbounded variables is exponential, then "true" chaos (not restricted to a finite time scale) can occur. A few exotic examples together with considerations from different viewpoints can be found in [10, 26]. However, such chaos does not seem to be a typical quantum dynamics.

The final remark is that the quantum chaos, as defined in Sect. 2, comprises not only quantum systems but also any linear, particularly classical, waves [27]. So, it is essentially the *linear wave chaos*. Moreover, a similar mechanism also works in completely integrable nonlinear systems like the Toda lattice, for example [31]. From a mathematical point of view all these new ideas require reconsideration of the existing ergodic theory. Perhaps it is better to say that a new ergodic theory is wanted which, instead of benefiting from the asymptotic approximation ($|t| \rightarrow \infty$ or $N \rightarrow \infty$), could analyze the finite-time statistical properties of dynamical systems. In my opinion, this is the most important conclusion emerging from first attempts to comprehend quantum chaos.

References

1. Proc. Les Houches Summer School on Chaos and Quantum Physics, Elsevier 1991
2. F. Haake: Quantum signatures of chaos. Springer 1991
3. B. Eckhardt: Phys. Reports **163**, 205 (1988)
4. G. Casati and L. Molinari: Suppl. Prog. Theor. Phys. **98**, 287 (1989)

5. F.M. Izrailev: Phys. Reports **196**, 299 (1990)
6. I. Kornfeld, S. Fomin and Ya. Sinai: Ergodic theory. Springer 1982
7. A. Lichtenberg, M. Lieberman: Regular and stochastic motion. Springer 1983;
G.M. Zaslavsky: Chaos in dynamic systems. Harwood 1985
8. B.V. Chirikov: Phys. Reports **52**, 263 (1979)
9. G. Casati et al: Lecture Notes in Physics **93**, 334 (1979)
10. B.Y. Chirikov, F.M. Izrailev and D.L. Shepelyansky: Sov. Sci. Rev. C2 (1981)
209; Physica D **33**, 77 (1988)
11. V.M. Alekseev and M.V. Yacobson: Phys. Reports **75**, 287 (1981)
12. E.V. Shuryak: Zh. Eksp. Teor. Fiz. **71**, 2039 (1976)
13. B.V. Chirikov, D.L. Shepelyansky: Radiofizika **29**, 1041 (1986)
14. G.P. Berman and G.M. Zaslavsky, Physica A **91**, 450 (1978)
15. M. Toda and K. Ikeda: Phys. Lett. A **124**, 165 (1987); A. Bishop et al: Phys.
Rev. B **39**, 12423 (1989)
16. D.L. Shepelyansky, Physica D **8**, 208 (1983); G. Casati et al: Phys. Rev. Lett.
56, 2437 (1986)
17. F.M. Izrailev: Phys. Lett. A **125**, 250 (1987)
18. A.I. Shnirelman: Usp. mat. nauk **29**, No 6, 181 (1974); On the asymptotic
properties of eigenfunctions in the regions of chaotic motion, addendum in: V.F.
Luzutkin: The KAM theory and asymptotics of spectrum of elliptic operators,
Springer 1991
19. B.V. Chirikov, CHAOS **1**, 95 (1991)
20. B.V. Chirikov: Usp. fiz. nauk **139**, 360 (1983); G.P. Berman, F.M. Izrailev:
Operator theory: advances and applications, **46**, 301 (1990)
21. D. Cohen: Quantum chaos, dynamical correlations and the effect of noise on
localization, 1991 (unpublished)
22. S. Fishman et al: Phys. Rev. A **29**, 1639 (1984)
23. D.L. Shepelyansky, Physica D **28**, 103 (1987)
24. E.P. Nakhmedov et al: Zh. Eksp. Teor. Fiz. **92**, 2133 (1987)
25. G. Casati and I. Guarneri: Comm. Math. Phys. **95**, 121 (1984)
26. S. Weigert: Z. Phys. B **80**, 3 (1990); M. Berry: True quantum chaos? An instruc-
tive example. Proc. Yukawa Symposium, 1990; F. Benattz et al: Lett. Math.
Phys. **21**, 157 (1991)
27. B.V Chirikov: Linear chaos. Preprint INP 90-116, Novosibirsk 1990
28. B.V Chirikov: Chaotic quantum systems. Preprint INP 91-83, Novosibirsk 1991
29. E. Heller: Phys. Rev. Lett. **53**, 1515 (1984)
30. E.B. Bogomolny: Physica D **31**, 169 (1988); M. Berry: Proc. Roy. Soc., London,
A **423**, 219 (1989)
31. J. Ford et al: Prog. Theor. Phys. **50**, 1547 (1973)