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Linear and Nonlinear Dynamical Chaos

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Abstract. Interrelations between dynamical and statistical laws in physics on the one hand, and between the classical and quantum mechanics on the other hand, are discussed with the emphasis on the new phenomenon of dynamical chaos.

The principal results of the studies into chaos in classical mechanics are presented in some detail within the general picture of chaos as a specific case of dynamical behavior. These results include the strong local instability and robustness of motion, continuity of both the phase space as well as the motion spectrum, and time reversibility but nonrecurrency of statistical evolution.

The analysis of apparently very deep and challenging contradictions of this picture with the quantum principles is given. The quantum view of dynamical chaos, as an attempt to resolve these contradictions guided by the correspondence principle and based upon the characteristic time scales of quantum evolution, is explained. The picture of the quantum chaos as a new generic dynamical phenomenon is outlined together with a few other examples of such a chaos, including linear (classical) waves and a digital computer.

I conclude with the discussion of two fundamental physical problems: the quantum measurement (ψ–collapse) and the causality principle, which both appear to be related to the phenomenon of dynamical chaos.

1. General Introduction: Statistical Properties of Dynamical Systems

The main purpose of this paper is an overview of recent studies into a new phenomenon (or rather the whole new field of phenomena) known as the *dynamical chaos*, both in classical and especially in quantum mechanics. The concept of dynamical chaos resolves (or, at least, helps to do so) the two fundamental problems in physics and, hence, in all natural sciences:

- Are dynamical laws of a different nature than statistical ones, or do any of them follow from the others?
- Is classical mechanics of a different nature than quantum, or is the latter the most universal and general theory currently available, describing the whole empirical evidence and including the classical mechanics as the limiting case.

The important part of my philosophy in discussing the problem of chaos is the separation of the human from the natural following Einstein's approach to science: *building up a model of the real world*. Clearly, the human is also a part of the world, and moreover the most important one for us as human beings but not as physicists. The whole phenomenon of life is extremely specific, and one should not transfer its peculiarities into other fields of science.

A rather popular human-oriented philosophy in physics nowadays is the information-based representation of natural laws, particularly, substituting the information for entropy (with opposite sign). In the most general way such a philosophy was recently presented by Kadomtsev [1]. Such an approach is possible, and it might be done in a self-consistent way but one should be very careful to avoid many confusions. In my opinion, the information is an adequate conception only for special systems which actually use and process information, like various automata, both natural (living systems) as well as man-made ones. In this case, the information becomes a physical notion rather than a human view of natural phenomena. The same is also true in the theory of measurement, again a very specific physical process, the basic one in our studies of nature but still not a typical one for the nature itself. This is crucially important in quantum mechanics as will be discussed in some detail below (Sections 4 and 5).

One of the major implications from the studies in dynamical chaos is the conception of statistical laws as an intrinsic part of dynamics without any additional statistical hypotheses (for the current state of the theory see, e.g., ref. [2] and a recent collection of papers [3] as well as the introduction to this collection [4]). This basic idea can be traced back to Poincare [5] and Hadamard [6], and even to Maxwell [7], the principal condition for dynamical chaos being strong local instability of motion. In this picture, statistical laws are considered as *secondary* with respect to more fundamental and general *primary* dynamical laws.

Surprisingly, the opposite is also true!

Namely, under certain conditions the dynamical laws were found to be completely contained in the statistical ones. Nowadays this is called 'synergetics' [8] but the principal idea goes back to Jeans [9] who discovered the instability of gravitating gas (a typical example of statistical system) which is the basic mechanism for the formation of galaxies and stars in modern cosmology, and eventually the Solar system, a classical example of a dynamical system. In this case, the resulting dynamical laws proved to be secondary with respect to the primary statistical laws which include the former.

Thus, the whole picture can be represented as a chain of dynamical-statistical inclusions:

$$\ldots ? \ldots \boxed{D \supset S} \supset D \supset S \ldots ? \ldots \tag{1.1}$$

Both ends of this chain, if any, remain unclear. So far the most fundamental (elementary) laws of physics seem to be dynamical (see, however, the discussion of quantum measurement in Sections 4 and 5). This is why I begin the chain (1.1) with some primary dynamical laws.

The strict inclusion in each link of the chain has a very important consequence allowing for the so-called numerical experiments, or computer simulations, of a broad range of natural processes. As a matter of fact, the former (not laboratory experiments) are now the main source of new information in the studies of the secondary laws for both dynamical chaos and synergetics. This might be called *the third way of cognition*, in addition to laboratory experiments and theoretical analysis.

In what follows, I restrict myself to the discussion of just a single fragment of the chain as marked in eq. (1.1). Here I will consider the dynamical chaos separately in classical and quantum mechanics. In the former case, chaos explains the origin and the mechanism of random processes in nature (within the classical approximation). Moreover, this deterministic randomness may occur (and is typical, as a matter of fact) even in a small number of degrees of freedom, $N > 1$ (for Hamiltonian systems), thus enormously expanding the domain of application for the powerful methods of statistical analysis. The latter provides a rather simple (see, however, Section 3) description of essential features for the otherwise highly intricate dynamical motion.

In quantum mechanics, the whole situation is much more tricky and still remains rather controversial. Here we encounter an intricate tangle of various apparent contradictions between the correspondence principle, classical chaotic behavior, and the very foundations of quantum physics. This will be the main topic of my discussion in Section 4.

One way to resolve this tangle is the new general conception — *pseudochaos*, of which the quantum chaos is the most important example. Another interesting example is a digital computer, also very important in view of a broad spectrum of applications in numerical experiments with dynamical systems. On the other hand, the pseudochaos in computers will hopefully help to understand quantum pseudochaos and to accept it as a sort of *chaos* rather than a regular motion as many researchers, even in this field, still tend to believe.

The new and surprising phenomenon of dynamical chaos, especially in quantum mechanics, holds out new hopes for eventually solving some old, longstanding, fundamental problems in physics. In Section 5 I will briefly discuss two of them:

- causality principle (time ordering of cause and effect), and
- ψ-collapse in the quantum measurement.

The conception of dynamical chaos I am going to present here, which is not common as yet, was the result of the long-term Siberian-Italian (SI) collaboration including Giulio Casati and Italo Guarneri (Como), and Felix Izrailev and Dima Shepelyansky (Novosibirsk) with whom I share the responsibility for our joint scientific results and their conceptual interpretation.

2. Chaos in Classical Mechanics: Dynamical Complexity

The classical dynamical chaos, as a part of classical mechanics, was historically the first to have been studied simply because in the time of Boltzmann, Maxwell, Poincare and other founders of statistical mechanics the quantum mechanics did not exist. No doubt, the general mathematical theory of dynamical systems, including the ergodic theory as its modern part describing various statistical properties of motion, has arisen from (and is still conceptually based on) the classical mechanics [10]. Yet, upon construction, it is not necessarily restricted to the latter and can be applied to a much broader class of dynamical phenomena, for example, in quantum mechanics (Section 4).

2.1. Dynamical systems

In classical mechanics, a dynamical system means an object whose motion in some *dynamical space* is *completely* determined by a given interaction and *initial conditions*. Hence a synonym — *deterministic system*. The motion of such a system can be described in two seemingly different ways which, however, prove to be essentially equivalent.

The first one are the *equations of motion*

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t) \tag{2.1}$$

which always have a unique solution

$$\mathbf{x} = \mathbf{x}(t, \mathbf{x_0}). \tag{2.2}$$

Here $\mathbf{x}$ is a finite dimensional vector in the dynamical space and $\mathbf{x_0}$ the initial conditions ($\mathbf{x_0} = \mathbf{x}(0)$). The possible explicit time-dependence in r.h.s. of eq. (2.1) is assumed to be regular, e.g., a periodic one or, at least, that with discrete spectrum.

The most important feature of dynamical systems is the *absence of any random parameters or any noise* in the equations of motion. In particular, for this reason I will consider a special class of dynamical systems, the so-called *Hamiltonian (nondissipative) systems*, which are most fundamental in physics.

Dissipative systems, being very important in many applications, are neither fundamental (because the dissipation is introduced via a crude approximation of the very complicated interaction with some 'heat bath') nor purely dynamical in view of principally inevitable random noise in the heat bath (fluctuation-dissipation theorem). In a more accurate and natural way, the dissipative systems can be described in the frames of the secondary dynamics ($S \supset D$ inclusion in eq. (1.1)), when both dissipation and fluctuations are present from the beginning in the primary statistical laws.

A purely dynamical system is necessarily a *closed* one which is the main object in fundamental physics. Thus, any coupling to the environment is completely neglected. I will come back to this important question below.

In Hamiltonian mechanics the dynamical space, called the *phase space*, is even-dimensional, composed of N pairs of canonically conjugated 'coordinates' and 'momenta', each pair corresponding to one degree of freedom of motion.

In the problem of dynamical chaos the initial conditions play a special role: they completely determine a single trajectory for a given interaction, or a particular realization of dynamical process which may happen to be a very specific, nontypical, one. To get rid of such singularities another description is useful, namely, the Liouville partial differential equation for the *phase space density* or distribution function $f(\mathbf{x}, t)$,

$$\frac{\partial f}{\partial t} = \hat{L} f, \tag{2.3}$$

with the solution

$$f = f(\mathbf{x}, t; f_0(\mathbf{x})). \tag{2.4}$$

Here $\hat{L}$ is a *linear* differential operator, and $f_0(\mathbf{x}) = f(\mathbf{x}, 0)$ the initial density. For any smooth f_0 this description provides the generic behavior of dynamical system via a continuum of trajectories. In the special case $f_0 = \delta(\mathbf{x} - \mathbf{x_0})$ the density describes a single trajectory like the equations of motion (2.1). Notice that even in this limiting case eq. (2.3) is linear with respect to the dynamical variable f.

In any case, the phase space itself is assumed to be *continuous* which is the most important feature of the classical picture of motion and the main obstacle in the understanding of quantum chaos.

2.2. DYNAMICAL CHAOS

Dynamical chaos can be characterized in terms of both individual trajectories and trajectory ensembles, or phase density. Almost all trajectories of a chaotic system are in a sense most complicated (*unpredictable* from the observation of any preceding motion). Exceptional, e.g., periodic trajectories form a set of zero invariant measure, yet it might be everywhere dense.

An appropriate notion in the theory of chaos is *symbolic trajectory*, first introduced by Hadamard [6]. The theory of symbolic dynamics was developed further in refs. [11 – 13]. Symbolic trajectory is a projection of the true (exact) trajectory on a discrete partition of the phase space at discrete instants of time t_n, e.g., such that $t_{n+1} - t_n = T$ fixed. In other words, to obtain a symbolic trajectory we first turn from the differential equations (2.1) to the difference equations over a certain time interval T,

$$\mathbf{x}(t_{n+1}) \equiv \mathbf{x}_{n+1} = M(\mathbf{x}_n, t_n) . \tag{2.5}$$

(2.5) is usually called *mapping* or *map*: $\mathbf{x}_n \to \mathbf{x}_{n+1}$. Then, while running (theoretically) an *exact* trajectory we record each $\mathbf{x}_n$ to a *finite* accuracy: $\mathbf{x}_n \approx m_n$. For a finite partition each m_n can be chosen to be an integer. Hence, the whole infinite symbolic trajectory

$$\sigma = \ldots m_{-n} \ldots m_{-1}\, m_0\, m_1 \ldots m_n \ldots = S(\mathbf{x}_0; T) \tag{2.6}$$

can be represented by a *single* number σ which is in general irrational, and which is some function of the *exact* initial conditions. The symbolic trajectory may be also called a *coarse-grained trajectory*. I remind that the latter is a *projection* of (not the substitution for) the exact trajectory to represent in compact form the global dynamical behavior without unimportant microdetails.

A remarkable property of chaotic dynamics is that the set of its symbolic trajectories is *complete*, that is it actually contains all possible sequences (2.6). Apparently, this is related to the continuity of the function $S(\mathbf{x}_0)$ (2.6). On the contrary, for a regular motion this function is everywhere discontinuous.

In a similar way the *coarse-grained* phase density $\bar{f}(m_n, t)$ is introduced, in addition to the exact or *fine-grained* density, which is also a projection of the latter on some partition of the phase space.

The coarse-grained density represents the global dynamical behavior, in particular, the most important process of *statistical relaxation* of chaotic motion to some *steady state* $f_s(m_n)$ (statistical equilibrium) independent of the initial $f_0(\mathbf{x})$, if the steady state is *stable*. Otherwise, synergetics comes into play giving rise to a secondary dynamics (Section 1). As the relaxation is an aperiodic process the spectrum of chaotic motion is *continuous* which is another obstacle for the theory of quantum chaos.

Relaxation is one of the characteristic properties of statistical behavior. Another one are *fluctuations*. Chaotic motion is the generator of noise, purely *intrinsic* by the definition of dynamical system. Such a noise is a particular manifestation of the complicated dynamics as represented by the symbolic trajectories or by the difference

$$f(\mathbf{x}, t) - \overline{f}(m_n, t) \equiv \tilde{f}(\mathbf{x}, t). \tag{2.7}$$

The relaxation $\overline{f} \to f_s$, apparently asymmetric with respect to time reversal $t \to -t$, gave rise to a longstanding misconception of the notorious *time arrow*. Even now some very complicated mathematical constructions are still being erected (see, e.g., refs. [14]) in attempts to extract somehow statistical irreversibility from reversible mechanics. This is particularly surprising as such 'irreversibility' is based on the separation of the phase density into two parts similar to eq. (2.7). In fact, the time direction is fixed by the additional statistical condition imposed on initial f_0, which is equivalent also to the 'causality condition' (see Section 5).

In the theory of dynamical chaos there is no such problem. The answer turns out to be conceptual rather than physical: one should separate two similar but different notions, *reversibility* and *recurrency*. The exact density $f(\mathbf{x}, t)$ is always *time-reversible* but *nonrecurrent* for chaotic motion, that is, it will never come back to the initial $f_0(\mathbf{x})$ in *both* directions of time $t \to \pm\infty$. In other words, the relaxation, also present in f, is time-symmetric. The projection of f, the coarse-grained $\overline{f}$ which is both nonrecurrent and irreversible, emphasizes nonrecurrency of the exact solution. The apparent violation of the statistical relaxation upon time reversal, as described by the exact $f(\mathbf{x}, t)$, represents in fact the growth of a big fluctuation which eventually will be followed by the same relaxation in the opposite direction of time. This apparently surprising symmetry of the statistical behavior was discovered long ago by Kolmogorov [15]. Another manifestation of that symmetry is the well-known principle of detailed balancing (for discussion see, e.g., ref. [24]).

One can say that instead of the imaginary time arrow there exists the *process arrow* pointing always to the steady state. The following simple example would help, perhaps, overcome this conceptual difficulty. Consider the hyperbolic one-dimensional (1D) motion:

$$x(t) = a \cdot \exp(\Lambda t) + b \cdot \exp(-\Lambda t) \tag{2.8}$$

which is obviously time-reversible, yet remains *unstable* in both directions of time ($t \to \pm\infty$). Apart from its immediate appealing character this example is closely related to the mechanism of chaos which is the instability of motion. Another example of time-reversible chaos will be given in Section 3.

2.3. Instability and chaos: dynamical complexity

Local instability of motion responsible for a very complicated dynamical behavior is described by the *linearized equations*:

$$\frac{d\mathbf{u}}{dt} = \mathbf{u} \cdot \frac{\partial \mathbf{v}(\mathbf{x}^0(t), t)}{\partial \mathbf{x}}. \tag{2.9}$$

Here $\mathbf{x}^0(t)$ is a reference trajectory satisfying eq. (2.1), and $\mathbf{u} = \mathbf{x}(t) - \mathbf{x}^0(t)$ the deviation of a close trajectory $\mathbf{x}(t)$. On average, the solution of eq. (2.9) has a form

$$|\mathbf{u}| \sim \exp(\Lambda t), \tag{2.10}$$

where Λ is called *Lyapunov exponent*. The motion is (exponentially) unstable if $\Lambda > 0$. In a Hamiltonian system of N degrees of freedom there are $2N$ Lyapunov exponents satisfying the condition $\sum \Lambda = 0$. The partial sum of all positive exponents $\Lambda_+ > 0$

$$h = \sum \Lambda_+ \tag{2.11}$$

is called the (dynamical) *metric entropy*. Notice that it has the dimension of frequency and characterizes the instability rate.

The instability of motion is only a necessary but not a sufficient condition for chaos. Another important condition is the *boundedness* of motion, or its oscillatory (in a broad sense) character. Chaos is produced by the combination of these two conditions (also called stretching and folding). Let us again consider an elementary example of 1D map

$$x_{n+1} = 2\,x_n \bmod 1\,, \tag{2.12}$$

where operation mod 1 restricts (folds) x to the interval (0,1). This is not a Hamiltonian system but it can be interpreted as a 'half' of that, namely, as the dynamics of the oscillation phase. This motion is unstable with $\Lambda = \ln 2$ because the linearized equation is the same except taking the fractional part (mod 1). The explicit solution for both reads

$$\begin{aligned} u_n &= 2^n\,u_0 \\ x_n &= 2^n\,x_0 \bmod 1 \end{aligned} \tag{2.13}$$

The first (linearized) motion is unbounded like the hamiltonian hyperbolic motion (2.8) and perfectly regular. The second one is not only unstable but also chaotic just because of the additional operation mod 1 which makes the motion bounded, and which mixes up the points within a finite interval.

The combination of the two above conditions for chaos — exponential instability and boundedness — requires the *equations of motion* to be *nonlinear*. In the latter example (2.12) nonlinearity is provided by the operation mod 1. However, Liouville's eq. (2.3) for the phase density f is always *linear*. Hence, the local stability of f, that is, the variation for a small deviation $\delta f = f - f^0$ is described by the same Liouville's eq. (2.3). The exponential instability of motion ($\Lambda = \pm\Lambda_\pm > 0$) results then in the contraction of the domain occupied by the initial phase density. If the simultaneous stretching in another direction is bounded (owing to nonlinearity of the motion, not Liouville's, equation) the exponentially long domain of conserved volume fills up the whole phase space region allowed by the exact integrals of motion, e.g., the whole energy surface of a conservative system. Eventually, the coarse-grained density $\overline{f}$ attains a homogeneous steady state f_s, while the exact density f keeps fluctuating with a characteristic wave length exponentially decreasing in time. In other words, we may say that in Liouville's description the phase space density evolution is exponentially unstable in the wave number (vector) $\mathbf{k}$ of $f(\mathbf{x})$ rather than in $f(\mathbf{x})$ itself. Notice that for a Hamiltonian system vector $\mathbf{x}$ includes momenta as well.

We may look at the above example (2.12) from a different viewpoint. Let us express the initial x_0 in the binary code as the sequence of two symbols, 0 and 1, and let us make the partition of unit interval also in two equal halves marked by the same symbols. Then the symbolic trajectory will simply repeat x_0, that is, eq. (2.6) takes the form

$$\sigma = x_0 . \qquad (2.14)$$

It implies that, as time goes on, the global motion will eventually depend on ever diminishing details of the initial conditions. In other words, when we formally fix *exact* x_0 we 'supply' the system with infinite complexity which is coming up due to the strong instability of motion. Still another interpretation is that the exact x_0 is the source of *intrinsic noise* amplified by the instability. For this noise to be *stationary* the string of x_0 digits has to be infinite which is only possible in a *continuous* phase space.

A nontrivial part of this picture of chaos is in that the instability must be *exponential* while a power-law instability is insufficient for chaos. For example, linear instability ($|\mathbf{u}| \sim t$) is a generic property of perfectly regular motion of the completely integrable system whose equations of motion are *nonlinear* and, hence, whose oscillation frequencies depend on initial conditions [16, 17]. The character of motion for a faster power–law instability ($|\mathbf{u}| \sim t^{\alpha},\ \alpha > 1$) is unknown.

On the other hand, the exponential instability ($h > 0$) is not invariant with respect to the change of time variable [4] (in this respect the only invariant statistical property is ergodicity [10]). A possible resolution of this difficulty is that the proper characteristics of motion instability, important for dynamical chaos, should be taken with respect to the oscillation phases whose dynamics determines the nature of motion. This implies that the proper time variable must go proportionally to the phases so that the oscillations become stationary [4]. A simple example is the harmonic oscillation with frequency ω recorded at the instances of time $t_n = 2^n t_0$. Then, the oscillation phase $x = \omega t/2\pi$ obeys the map (2.12), which is chaotic. Clearly, the origin of chaos here is not in the dynamical system but in the recording procedure (random t_0). Now, if ω is a parameter (linear oscillator), then the oscillation is exponentially unstable (in new time n) but only with respect to the change of parameter ω, not of the initial x_0 ($x \to x + x_0$). In a slightly 'camouflaged' way, essentially the same effect was considered in ref. [56] with far-reaching conclusions for the quantum chaos (Section 4).

Rigorous results concerning the relation between instability and chaos are concentrated in the Alekseev-Brudno theorem [13] (see also Refs. [4, 18]) which states that the complexity per unit time of almost any symbolic trajectory is asymptotically equal to the metric entropy:

$$\frac{C(t)}{|t|} \to h , \qquad |t| \to \infty \qquad (2.15)$$

Here $C(t)$ is the so-called algorithmic complexity, or in more familiar terms, the information associated with a trajectory segment of length $|t|$.

The transition time from the dynamical to statistical behavior according to eq. (2.15) depends on the partition of the phase space, namely, on the size of a cell μ which is inversely proportional to the biggest integer $M \geq m_n$ in symbolic trajectory (2.6). The transition is controlled by the *randomness parameter* [19]:

$$r = \frac{h\,|t|}{\ln M} \sim \frac{|t|}{t_r} , \qquad (2.16)$$

where t_r is the *dynamical time scale*. As both $|t|$, $M \to \infty$ we have a somewhat confusing situation, typical in the theory of dynamical chaos, when two limits do not commute:

$$M \to \infty\,,\; |t| \to \infty \neq |t| \to \infty\,,\; M \to \infty\,. \tag{2.17}$$

For the left order ($M \to \infty$ first) the parameter $r \to 0$ and we have *temporary determinism* ($|t| \lesssim t_r$), while for the right order $r \to \infty$ and we arrive at the *asymptotic randomness* ($|t| \gtrsim t_r$).

Instead of the above double limit we may consider the *conditional limit*

$$|t|\,,\; M \to \infty\,, \qquad r = \text{const}\,, \tag{2.18}$$

which is also a useful method in the theory of chaotic processes. Particularly, for $r \lesssim 1$ strong dynamical correlations persist in a symbolic trajectory which allows for the prediction of trajectory from a finite-accuracy observation. This is no longer the case for $r \gtrsim 1$ when only statistical description is possible. Nevertheless, the equations of motion can still be used to completely derive all the statistical properties without any *ad hoc* hypotheses. Here the exact trajectory *does exist* as well but becomes the Kantian *thing-in-itself* which can be only observed but neither predicted nor reproduced in any other way.

The mathematical origin of this peculiar property goes back to the famous theorem of Gödel [20] which states (in modern formulation) that *most* theorems in a given mathematical system are unprovable, and which forms the basis of contemporary mathematical logic as well as of the algorithmic theory of dynamical systems (see ref. [21] for a detailed explanation and interesting applications of this relatively less known mathematical achievement). One particular corollary, directly related to symbolic trajectories (2.6), is that almost all real numbers are uncomputable by any finite algorithm. Apart from rational numbers some irrationals like π or e are also known to be computable. Hence, their total complexity, e.g., $C(\pi)$ is finite, and the complexity per digit is zero (cf. eq. (2.15)).

The main object of my discussion here, as well as of the whole physics, is a closed system which requires neglecting the external perturbations. However, in the case of strong instability of motion this is no longer possible, at least, dynamically. What is the impact of a weak perturbation on the statistical properties of a chaotic system? The rigorous answer was given by the robustness theorem due to Anosov [22]: not only statistical properties remain unchanged but, moreover, the trajectories get only slightly deformed providing (and due to) the same strong instability of motion. The explanation of this striking peculiarity is that the trajectories are simply transposed and, moreover, the less the stronger is instability.

In conclusion, let me make a very general remark, far beyond the particular problem of chaotic dynamics (see also Ref.[89]). According to the Alekseev-Brudno theorem (2.15) the source of stationary (new) information is always chaotic. Assuming farther that any creative activity, science included, is supposed to be such a source, we come to an interesting conclusion that any such activity has to be (partly!) chaotic. This is the creative side of the chaos.

3. Chaos in Classical Mechanics: Statistical Complexity

The theory of dynamical chaos does not need any statistical hypotheses, nor does it allow for arbitrary ones. Everything is to be deduced from the dynamical equations. Sometimes the statistical properties turn out to be quite simple and familiar

[2, 23]. This is usually the case if the chaotic motion is also ergodic (on the energy surface). However, quite often, and even typically for a few degrees of freedom chaos, the phase space is divided, and the chaotic component of the motion has a very complicated structure which results in a high complexity not only of individual trajectories (Section 2) but also of the statistical picture of the motion. Before we proceed any further let us consider a few simple examples.

3.1. Simple physical examples of dynamical chaos

In this survey I restrict myself to finite-dimensional systems, where the peculiarities of dynamical chaos are most visible (see Section 4 for some brief remarks on infinite systems). Consider now a few examples of chaos in minimal dimensionality. In a conservative system of one degree of freedom ($N = 1$) chaos is impossible. Such a system is *completely integrable* since there is one integral of motion, the energy, per one degree of freedom. The motion is periodic that is perfectly regular. The solution (2.2) of the equations of motion (2.1) is explicitly expressed in the standard way as the integral of the Hamiltonian. Chaos requires at least two degrees of freedom (for conservative systems). For a regular (quasiperiodic) motion two independent (commuting) and isolating (single-valued) integrals would be necessary which is not always the case. At this point I would like to mention a rather widespread confusion that any equations of motion possess $2N$ integrals, the initial conditions (see eq. (2.2)). This is certainly true but those integrals are nonisolating, in fact they might be infinitely many-valued. In the latter case, the trajectory is not restricted to an *invariant surface* of lower dimension, and may even be *ergodic*, that is occupy the whole energy surface. Let me also mention that there are some minor differences among several possible definitions of the (complete) integrability. One is based on the integrals of motion in some particular dynamical space. Another one (narrower) corresponds to the stronger condition of the existence of the integrals of motion in *action-angle* variables[1] $\mathbf{n}$, ϕ. In this case, the invariant surface of the completely integrable system is an N-dimensional *torus*. Below I will assume the latter definition of integrability.

In the case of time-dependent Hamiltonian $H(n, \phi, t)$, chaos is possible even in one degree of freedom. This is because such a system is equivalent to a conservative one of two degrees of freedom in the *extended phase space* n_1, n_2, ϕ_1, ϕ_2 with the new Hamiltonian [2]

$$\overline{H}(n_1, n_2, \phi_1, \phi_2) = H(n_1, \phi_1, \phi_2/\Omega) + \Omega n_2 = 0, \qquad (3.1)$$

where $n_2 = -H/\Omega$, and $\phi_2 = \Omega t$ (for periodic time-dependence of frequency Ω). In this case, one speaks of one-and-a-half degrees of freedom ($N = 1.5$) as the time dependence is fixed.

3.1.1. *Charged particle confinement in adiabatic magnetic traps*

This is Budker's problem [24, 25], very important in the study of controlled nuclear fusion. A simple model of two degrees of freedom (axisymmetric magnetic field) is

[1] I denote actions by n having in mind the subsequent quantization in Section 4 when they become integers if $\hbar = 1$.

described by the Hamiltonian

$$H = \frac{p^2}{2} + \frac{(1 + x^2)\, y^2}{2}. \tag{3.2}$$

Here the magnetic field is $B = \sqrt{1 + x^2}$, $p^2 = \dot{x}^2 + \dot{y}^2$, x describes the motion along magnetic line, and y — across the line (a projection of Larmor's rotation).

In ref. [24] a slightly different model with 'potential energy' (which is actually the transverse part of particle's kinetic energy) $U = (1 + x^2)^2\, y^2/2$ was considered in detail. I chose model (3.2) here to apply the results below to a completely different physical system.

Assume the adiabaticity parameter

$$\lambda = \frac{1}{v_0} \sim \omega_y(0) \cdot \tau_x \gg 1, \tag{3.3}$$

where v_0 is the full particle velocity $(H = v_0^2/2)$, $\omega_y(x) = \sqrt{1 + x^2}$ stands for the frequency of transverse oscillation, and $\tau_x \sim 1/v_0$ is the characteristic time for crossing the magnetic field minimum at $x = 0$. Under this condition both actions, n_x and n_y, which are also adiabatic invariants, are approximately conserved. This would imply bounded x-oscillations, that is the confinement of a particle in magnetic trap. However, the adiabatic invariant is only an approximate integral of motion, and Budker's problem is the evaluation of a long-term variation of that, if any, which would result in a leakage of particles out of the trap.

The unperturbed (adiabatic) Hamiltonian for model (3.2) is defined by $n_y = \omega_y a_y^2/2 = \text{const}$ with $y = a_y \cos\phi_y$, and reads

$$H_0 = \frac{p_x^2}{2} + n_y\, \omega_y(x) \approx \left(\frac{3\pi}{4\sqrt{2}}\, n_x\, n_y\right)^{2/3} \approx \text{const}. \tag{3.4}$$

Consider the case of large x-oscillation, with amplitude $a_x = H_0/n_y \gg 1$. Then, the frequency

$$\omega_x \approx \frac{\pi}{2}\sqrt{\frac{n_y}{2a_x}} = \frac{\pi}{2\sqrt{2}}\, \frac{n_y}{\sqrt{H_0}} = \frac{\partial H_0}{\partial n_x}. \tag{3.5}$$

Hence, last expression for H_0 in eq. (3.4) and

$$\langle \omega_y \rangle = \frac{\partial H_0}{\partial n_y} = \frac{2}{3}\, \frac{H_0}{n_y}, \tag{3.6}$$

where the brackets denote averaging over x-oscillation.

Now, the central part of the problem — the evaluation of n_y variation from the equation

$$\frac{\dot{n}_y}{n_y} = \frac{d}{dt} \ln n_y = \frac{x\, \dot{x}}{1 + x^2} \cdot \cos 2\phi, \tag{3.7}$$

where we drop sub y. This equation can be derived either via canonical transformation of the original Hamiltonian (3.2) to action-angle variables or directly from the exact equations of motion.

Under adiabatic condition (3.3) n_y variation is exponentially small in parameter λ. So, this part of the problem is essentially *nonperturbative*, that is, it cannot be solved using conventional perturbation techniques by expanding in an asymptotic power series in small parameter $1/\lambda$. Instead, a new perturbation parameter absorbing nonadiabatic exponential should be introduced. To this end we integrate eq. (3.7) over half-period of x-oscillation substituting in r.h.s the unperturbed solution.

The integration is performed in the complex plane of phase ϕ

$$\Delta \ln n_y = \mathrm{Re} \int \frac{x\,\dot{x}}{1+x^2} \cdot \exp(2i\phi)\, d\phi = \epsilon_a \cdot \sin 2\phi_0 \tag{3.8}$$

around the cut from the branch point at $x = x_p = i$ and $\phi = \phi_p$ to infinity. Here

$$\phi_p = \phi_0 + \int_0^{x_p} \frac{\omega_y\, dx}{\dot{x}} \approx \phi_0 + i\,\frac{\pi}{4v_0}, \tag{3.9}$$

$\dot{x} \approx v_0$, and ϕ_0 is the phase value at $x = 0$. For $\lambda \gg 1$ the integral can be reduced to Γ-function, and we obtain for the amplitude in eq. (3.8)

$$\epsilon_a \approx \frac{2\pi}{3} \exp\left(-\frac{\pi}{2}\lambda\right). \tag{3.10}$$

This is the required perturbation parameter.

Now we can derive a map describing the particle motion over many x-oscillations. Apart from eq. (3.8) we need another one for phase $\varphi = 2\phi_0$. From Eqs. (3.5) and (3.6) we have

$$\Delta\varphi = 2\pi\,\frac{\langle\omega_y\rangle}{\omega_x} = \frac{4}{3}\,\frac{v_0^3}{n_y^2} = G(P), \tag{3.11}$$

where a new variable $P = \ln n_y$ is introduced. Now, from Eqs. (3.8) and (3.11) we arrive at the map $(P,\, \varphi) \to (\overline{P},\, \overline{\varphi})$ over half-period of x-oscillation

$$\begin{aligned} \overline{P} &= P + \epsilon_a \cdot \sin\varphi \\ \overline{\varphi} &= \varphi + G(\overline{P}). \end{aligned} \tag{3.12}$$

In the second equation the new value of momentum $(\overline{P})$ is substituted which determines the change in phase φ up to the next crossing of the plane $x = 0$, where the first equation operates. As $v_0 = \mathrm{const}$ is the exact integral of motion, the map (3.12) is canonical, in particular preserving the phase plane area $d\Gamma_2 = dP \cdot d\varphi$.

The map describes the global dynamics of the model and is relatively simple for further analysis both in numerical experiments as well as by means of asymptotic perturbation series in the new small parameter ϵ_a. It can be still simplified by linearizing the second eq. (3.12) around a resonance at $P = P_r$ such that $G(P_r) = 2\pi r$ with any integer r. Upon dropping the latter term the map is reduced to the so-called *standard map* which (in standard notations) reads [23]

$$\begin{aligned} \overline{p} &= p + k \cdot \sin\varphi, \\ \overline{\varphi} &= \varphi + T \cdot \overline{p}, \end{aligned} \tag{3.13}$$

where $p = P - P_r$, $k = \epsilon_a$, and new parameter

$$T = \frac{dG(P)}{dP} = -\frac{8}{3}\, v_0^3 \exp(-2P_r), \tag{3.14}$$

The term 'standard' emphasizes a universal character of the map to which many (but, of course, not all) various physical models can be reduced as we shall see right below. Both maps, (3.12) and (3.13), can be formally considered as describing a system of one degree of freedom driven by the periodic external perturbation in the form of short δ-pulses. Hence the nickname 'kicked rotator' for the model (3.13). Yet, contrary to the common belief, the map can describe also a *conservative* system as it is the case in our example. Then, it is called the *Poincare map* [2].

Unlike the global map (3.12) the standard map describes the dynamics locally in momentum, e.g., P for eq. (3.12). This dynamics is determined by a single parameter

$$K = |kT| = \frac{16\pi}{9\lambda^3}\exp\left(-\frac{\pi\lambda}{2} - 2P\right) > 1 \qquad (3.15)$$

The latter inequality determines the region of chaotic motion in parameter K for the standard map, and that in phase space for the model (3.2) [23]. In the latter case the chaos condition becomes

$$n_y^2 < \frac{16\pi}{9\lambda^3}e^{-\frac{\pi\lambda}{2}} \quad \text{or} \quad \beta_0 < \left(\frac{64\pi}{9}\right)^{1/4}\lambda^{1/4}e^{-\frac{\pi\lambda}{8}} = \beta_b\,, \qquad (3.16)$$

where $\beta_0 \approx v_y/v_0 \ll 1$ is the so-called pitch-angle at $x = 0$. The second inequality (3.16) determines chaotic cone in particle's velocity space. Thus, the motion in *this model has always a chaotic component which, however, is never ergodic on* the energy surface H = const. The chaotic component is bounded by the *chaos border* at $\beta_0 = \beta_b$.

All particles within the chaos cone will be eventually lost diffusing to smaller β_0 which correspond to large amplitudes $a_x \approx \beta_0^{-2}$. The diffusion rate in p per one iteration of the map is obtained from eq. (3.13),

$$D_p = \langle(\Delta p)^2\rangle \approx \frac{k^2}{2} \approx \frac{2\pi^2}{9}e^{-\pi\lambda}\,. \qquad (3.17)$$

Particle life time within the cone can be roughly estimated in the number of x-oscillations as

$$N_x \sim \frac{P_b^2}{D_p} \sim \lambda^2 e^{\lambda}\,. \qquad (3.18)$$

It is fairly long for big $\lambda \gg 1$. Besides, most particles are in the stable region $\beta_0 > \beta_b \ll 1$ *(3.16)* and are confined there forever. So, Budker's adiabatic magnetic trap turns out to be a very good confinement device indeed (at least for a single particle!).

A peculiar feature of the model (3.2) are 'open' (infinite) energy surfaces ($x^2 \to \infty$ if $y^2 \to 0$). Moreover, ergodic (microcanonical) measure Γ_E of the energy surface is also infinite. It is defined by the integral

$$\Gamma_E = \int \delta(H - E)\, d\Gamma\,, \qquad (3.19)$$

where E is a particular value of energy, and $d\Gamma = dn_x\, dn_y\, d\phi_x\, d\phi_y$ stands for the element of the full phase space. Using $dE/dn_x = \omega_x \sim E/n_x \sim n_y E^{-1/2}$ and integrating over phases and energy we obtain

$$\Gamma_E \sim \sqrt{E} \int \frac{dn_y}{n_y} = \sqrt{E}|P| \to \infty \tag{3.20}$$

which diverges as $n_y \to 0$.

Notice that ergodic measure Γ_E is proportional to the measure Γ_2 for both maps, global (3.12) and local (3.13). This ensures a correct description of the global dynamics by the map. If we changed dynamical variables, e.g. $P \to n_y$, it would no longer be the case, and only the local map could be used. For this reason the special ('preferable') variables (P, φ in our example) are called *ergodic variables* [24].

Generally, the description in discrete time (a map, or difference equations) and in continuous time (differential equations of motion) is not completely identical just because of a different time variable. An interesting and instructive example is Lyapunov exponent Λ (Section 2) in the model under consideration. For the standard map it depends (as anything else) on the single parameter K [23]:

$$\Lambda_l \approx \ln\left(\frac{K}{2}\right), \qquad K > 4. \tag{3.21}$$

However, for the global map (3.12) local Λ_l depends on momentum (3.15) and must be averaged over the whole chaotic component

$$\Lambda = \frac{1}{P}\int_{P_b}^{P} \Lambda_l(P')\,dP' \to |P| \to \infty. \tag{3.22}$$

Thus, Lyapunov exponent for the map (per iteration) diverges as does the measure of the chaotic component (3.20).

The result changes drastically in continuous time. Now, we must divide local Λ_l (3.21) by the half-period of x-oscillation $\pi/\omega_x \sim \sqrt{E}\exp(-P)$. We have

$$\Lambda_t = \frac{1}{P}\int_{P_b}^{P} dP' \cdot \Lambda_l(P') \cdot \omega_x(P')/\pi \to \frac{C}{|P|} \to 0, \tag{3.23}$$

where C is some finite constant. Thus, Lyapunov exponent per unit continuous time is zero! This qualitatively different result seems to imply the violation of the main condition for chaos (Section 2). The resolution of apparent contradiction is that for any *finite time* $t \sim \exp(|P|)$ Lyapunov exponent $\Lambda_t \sim 1/\ln t$ remains finite, and the motion is still chaotic but, apparently, with some unusual statistical properties.

3.1.2. *Internal dynamics of Yang-Mills (gauge) fields in classical approximation*
Surprisingly, this Matinyan's problem [26] for a completely different physical system can be also represented by the Hamiltonian (3.2) with symmetrized 'potential energy'

$$U = \frac{(1+x^2)\,y^2 + (1+y^2)\,x^2}{2}. \tag{3.24}$$

The dynamics is always chaotic with divided phase space, similar to the model (3.2) [27]. Model (3.24) describes the so-called massive gauge field, that is, one with the quanta of nonzero mass (in the classical limit!).

The massless field corresponds to the 'potential energy'

$$U = \frac{x^2 y^2}{2} \tag{3.25}$$

and looks ergodic in numerical experiments. This model can be analysed as a limiting case $\mu \to 0$ of Budker's model (3.2) with an additional parameter μ in the potential energy

$$U_\mu = \frac{(\mu^2 + x_\mu^2)\, y_\mu^2}{2}. \tag{3.26}$$

To this end we change the variables: $x_\mu = \mu x$, $y_\mu = \mu y$, $t_\mu = t/\mu$, which brings the new Hamiltonian $H_\mu = \mu^4 H$ into the form of the old one (3.2), and we can use the results above. In particular, the adiabaticity parameter $\lambda = \mu^2 \lambda_\mu$ (3.3) decreases with μ for a given energy H_μ. Hence, the chaos cone β_b (3.16) rapidly expands covering eventually the whole velocity space in agreement with numerical experiments [26, 27].

For any finite μ the energy surfaces are also open and infinite in measure, and Lyapunov exponent takes the opposite limits in discrete and in continuous time. This is not the case for massive field (3.24). Here, the energy surfaces are closed and finite while Lyapunov's exponent does not qualitatively depend on the time variable albeit it may have different values in both cases, which is not important for the nature of the motion.

3.1.3. *Perturbed Kepler motion*

This is a particular instance of the famous 3-body problem. Now we understand why it has not been solved since Newton: chaos is generally present in such a system. One particular example is the motion of comet Halley perturbed by Jupiter, which was found to be chaotic with estimated life time in the Solar system of the order of 10 Myrs [28], and with very complicated divided phase space.

The simplest model is described by a global map similar to (3.12):

$$\begin{aligned} \overline{E} &= E + \epsilon \cdot F(\varphi) \\ \overline{\varphi} &= \varphi + G(\overline{E}), \end{aligned} \tag{3.27}$$

where $E < 0$ is the comet total energy, and φ stands for Jupiter's phase (angle) on its round orbit of unit radius (and unit velocity) at the moment when the comet is in perihelion, where the perturbation effect is the strongest. The function $G(E) = 2\pi\Omega(-2E)^{-3/2}$, where Ω is Jupiter's orbital frequency, is the Kepler law. In our units $\Omega = 1$ but we will keep it in our expressions for the next example. The perturbation parameter $\epsilon \approx 2 \times 10^{-3}$ is essentially determined by the ratio of Jupiter and the Sun masses ($\approx 10^{-3}$). Actually it is somewhat larger because of close encounters between Jupiter and the comet depending on the relative position of their orbits. For simplicity we assume in eq. (3.27) $F(\varphi) \approx \sin\varphi$ like in eq. (3.12) albeit the actual dependence is somewhat different owing to the same close encounters. A relatively weak perturbation by Saturn was also found to be important for the global dynamics of comet Halley.

Because of negligible comet mass the perturbation from Jupiter is fixed, which corresponds to a time-dependent Hamiltonian, and the phase φ is simply proportional to time. In such case, the ergodic variable was shown to be the energy

(E, canonically conjugated to the map phase φ) rather than the comet action in continuous time [29].

The stability parameter K of the local (standard) map for the model (3.27) reads

$$K = \frac{3\pi\,\epsilon\Omega}{2\sqrt{2}\,|E|^{5/2}} > 1. \tag{3.28}$$

The chaotic component corresponds to higher energies $E > -E_b$ ($|E| < E_b$) and goes up to $E = 0$ when the comet leaves out (or is captured into) the Solar system. In the latter case the whole motion (capture – diffusion – ejection) is a sort of delayed (on the diffusion stage) scattering of the comet by the Solar system.

From eq. (3.28) the chaos border for comet Halley is roughly at $E_b \approx 0.13$ or, in frequency, $\omega_b = (2E_b)^{3/2} \approx 0.13$. The actual comet frequency is now $\omega_H \approx 0.16$ which is close to chaos border where the structure of phase space is very complicated, with many stable domains of various size.

Detailed studies [28] have shown that current ω_H is only 5% apart from the border of a big stable region. Additional perturbations, including ones of unknown nature, both in the future as well as in the past could change the character of the comet motion from chaotic to regular and vice versa. Neglecting this possibility, the comet life time t_H in the Solar system can be roughly estimated from the inhomogeneous diffusion equation (see eq. (3.27))

$$\frac{d\langle(\Delta E)^2\rangle}{dt} \approx \frac{\epsilon^2}{2}\cdot\frac{\omega}{2\pi} \sim 2E\dot{E} = \frac{\omega^{1/3}}{3}\dot{\omega} \tag{3.29}$$

which gives

$$t_H \sim 4\pi\frac{\omega_H^{1/3}}{\epsilon^2} = 1.7\times 10^6 = 3.2\,\mathrm{Myrs}. \tag{3.30}$$

This is essentially less than the result from computer simulation of the map (3.27): $t_H \sim 10^7$ yrs. The difference is explained by an anomalously slow diffusion near the chaos border.

Another example of the perturbed Kepler dynamics is a new, diffusive mechanism for the ionization of Rydberg (highly excited) Hydrogen atom in an external monochromatic electric field. It was discovered in laboratory experiments [30], and has been explained by the dynamical chaos in classical approximation [31]. In this system, a given field plays the role of the third body. The simplest model of the diffusive photoelectric effect has 1.5 degrees of freedom, and is described by exactly the same global map (3.27) [32], now with $F(\varphi) = \sin\varphi$ but, of course, with different perturbation parameter

$$\epsilon \approx \frac{2.6\,f}{\Omega^{2/3}}, \tag{3.31}$$

where f is the field strength and we use now atomic units: $|e| = m = \hbar = 1$. Of course, this is essentially a quantum problem but for a large quantum number $n \gg 1$ (electron action variable) the classical approximation proved to be fairly good [31]. We will come back to quantum effects in this system in Section 4. I remind that in n variable the energy is $E = 1/2n^2$ and Kepler frequency $\omega = 1/n^3$.

The stability parameter

$$K = \frac{8.7\, f\, \Omega^{1/3}}{|E|^{5/2}} = 50 \cdot f_n \cdot \Omega_n^{1/3} > 1 \tag{3.32}$$

is expressed here in dimensionless variables $\Omega_n = \Omega\, n^3$ and $f_n = f n^4$, which are reduced to the values of the corresponding atomic quantities at energy level (action) n. For a given field strength f and frequency Ω parameter K increases with n. Hence, in the chaotic component, the electron is diffusing up to eventual ionization. If $\Omega_n \gtrsim 1$ the critical field $f_n \ll 1$, that is, much less than the atomic field. In the interval $1 \lesssim \Omega_n < n/2 \gg 1$ the field frequency may be considerably lower than that required for the conventional (one-photon) ionization, while the ionization rate is much higher provided that chaos condition (3.32) holds.

3.1.4. *Billiards and cavities*

In a (non-dissipative) billiard of at least two dimensions the ball motion is chaotic for almost any shape of the boundary except for special cases like the circle, the ellipse, the rectangle and some others (see, e.g., refs.[2, 10]). However, the ergodicity (on the energy surface) is only known for singular boundaries (of a singly-connected region). If the boundary is smooth enough the structure of motion becomes a very complicated admixture of chaotic and regular domains of various sizes. In the latter case the description via global and local maps of the kind considered above is very useful (see, e.g., refs. [2, 33] and below).

Another view of a billiard model is the wave cavity in the limit of geometric optics. This provides a helpful bridge between classical and quantum chaos.

In general, the mechanism of exponential instability in billiards is related to the particle scattering from a *convex* (towards the particle) boundary [34]. A simple example is a doubly-connected region with a convex internal boundary. A more important example is the collision of several convex balls within any boundary, which is a classical model for the gas of molecules. The first simple estimate for the Lyapunov exponent in such a model was made already by Poincare [5]:

$$\Lambda \sim \frac{v}{L} \ln\left(\frac{L}{R}\right) \leq \frac{v}{Re}, \tag{3.33}$$

where L is the mean distance between the balls, and v, R are the ball velocity and radius, respectively. The maximal instability rate is reached at $L = Re$.

Surprisingly, a *concave* boundary may also cause the instability if its curvature is large enough [35]. This is explained by the so called 'overfocusing': first, close trajectories converge upon reflection from the boundary but later, after passing the focus, they eventually diverge. A well studied example is the 'stadium', the planar billiard with the boundary composed of two semicircles connected by two straight lines. For any nonzero length of the latter the ball motion is not only chaotic but also ergodic.

Here, we consider two examples of chaotic billiards with moving boundary. In this case, chaos is possible already in one degree of freedom, that is, for the ball motion along a straight line. One example is Ulam's model (see ref. [2]) for the mechanism of cosmic rays acceleration proposed by Fermi [36]. The Fermi model was a 'gas' of huge magnetic clouds and protons in cosmic space. In the steady state the mean energy of both must be equal, which would imply an enormous acceleration of protons. Ulam checked this idea in numerical experiments with a very simple one degree of freedom model: a particle between two parallel walls, L apart, one of which is oscillating with a given velocity $V = V_0 \cdot \sin(\Omega t)$. Surprisingly, the computation showed no significant acceleration beyond the wall velocity V_0. This was explained in ref. [37] using chaos theory just developed at that time.

Under the condition $L \gg l = V_0/\Omega$, particle motion is described again by the global map (3.12) in variables v (the particle velocity), $\varphi = \Omega t$ (at collision time), and with $\epsilon = 2V_0$, $G(v) \approx 2L\Omega/v$. Hence,

$$K \approx \frac{4\,L\Omega\,V_0}{v^2} > 1 \tag{3.34}$$

and the chaotic component is bounded from above. Indeed

$$\frac{v}{V_0} \lesssim 2\sqrt{\frac{L}{l}}\,. \tag{3.35}$$

Acceleration v/V_0 turns out to be the bigger the smaller the amplitude of the wall oscillation! It was a surprising result which would be difficult to imagine without theory. Of course, in the original Fermi model there was no such restriction since the cloud motion was assumed to be random, which was later confirmed by the chaos theory for the gas model mentioned above.

Dynamical variables v, φ are not ergodic. Still, the local map can be used to evaluate the conditions for chaos. If we changed velocity to energy the map would no longer be canonical, and a more complicated map would have to be constructed. It is also assumed that the wall has infinite mass, so that its motion is fixed. We may lift this condition to study the ergodicity of the whole system [38]. Assume that the wall with a finite mass $M \gg m$ (the particle mass) is a linear oscillator of frequency Ω. From the energy conservation $mv^2 + MV_0^2 = 2E = MV_m^2$ and condition (3.34) we can derive the chaos border $V_0 = V_b$ on energy surface in the form

$$\frac{V_b}{V_m} = \sqrt{1+\lambda^2} - \lambda\,, \qquad \lambda = 2\,\frac{m}{M}\cdot\frac{L\Omega}{V_m}\,, \tag{3.36}$$

where λ may be called the ergodicity parameter. The chaotic component corresponds to $V_0 > V_b$ and increases with λ. Yet, the motion is never completely ergodic. The measure of chaotic component can be evaluated using the arguments applied above to eq. (3.19). Since now the wall frequency $\Omega = \text{const}$ is fixed, $\Gamma_E \sim v$ (in continuous time). Chaos is restricted to $v^2 > v_b^2 = (M/m)(V_m^2 - V_b^2)$, where v_b is the border value. Hence, the relative measure of chaotic component is

$$\Gamma_{ch} = \frac{v_b}{v_m} = \sqrt{1 - \left(\frac{V_b}{V_m}\right)^2} \to 1 - \frac{1}{4\,\lambda^2}\,, \tag{3.37}$$

where the latter expression corresponds to a large $\lambda \gg 1$. For given parameters of the model the essential ergodicity is achieved in the low energy limit only.

Another version of Ulam's model was studied in ref. [39] (see also ref. [2]). The new model is an 'open' billiard with only one oscillating wall in a homogeneous field which brings the particle back to the wall. The only difference in the global map is the phase shift between collisions: $G(v) \approx 2v\Omega/g$, where g is the particle acceleration in the field. Then

$$K = \frac{4\Omega V_0}{g} \tag{3.38}$$

is independent of v, and the chaotic acceleration becomes unbounded.

3.1.5. *Reversible chaos in magnetic field*

Magnetic lines can be formally considered as the 'trajectories' of some dynamical system, the distance s along a line playing the role of 'time'. Owing to Maxwell's equation div $\mathbf{B} = 0$ the line dynamics is Hamiltonian. Consider a toroidal magnetic field which is used in magnetic traps for plasma confinement like stellarator or tokamak[40].

Three-dimensional magnetic lines have 1.5 degrees of freedom corresponding, in the latter example, to a one degree of freedom oscillation (line rotation in the plane transverse to the torus closed axis) driven by the external perturbation due to the variation of magnetic field in s (along the axis). The transverse surface plays here the role of the 'phase plane' for the line oscillation which is in general nonlinear, that is, with the frequency depending on initial conditions (the distance r from the axis).

Under certain conditions the lines become chaotic [41], which is called the 'braided' magnetic field. In particular, the lines are 'diffusing',

$$|\Delta r|_l \sim \sqrt{l_r\, s}\,, \tag{3.39}$$

where $l_r \sim \Lambda^{-1}$ is the dynamical scale (2.16), and Λ the Lyapunov exponent for magnetic lines (per unit length). Notice that s here is not restricted by the torus circumference. Instead, $s \to \infty$ and line diffusion is only bounded by chaos border at large r, e.g., near the current wires producing the magnetic field.

In sufficiently strong B the electron Larmor radius ρ is negligibly small, and the electron follows a magnetic line: $s_e \approx v_{||}t$, where $v_{||}$ is the longitudinal velocity. Hence, the electron is also diffusing,

$$|\Delta r|_e \sim \sqrt{l_r\, v_{||} t} \tag{3.40}$$

and eventually it will be lost.

Now, consider the impact of the electron collisions with other particles in plasma [42]. For a small Larmor radius the main collision effect would be the electron *velocity reversal* ($v_{||} \to -v_{||}$), which is equivalent to the *time reversal* for magnetic lines ($s \to -s$). Neglecting again finite Larmor radius, the electron will follow back the same line. If the time reversals were periodic so would be the electron motion, and the diffusion would disappear (in this approximation). However, the collisional time reversal is a random process. Hence, the 'time' spread

$$|\Delta s| \sim \sqrt{l_s\, s}\,, \tag{3.41}$$

where l_s is the mean scattering length, would itself grow only diffusively. This implies an anomalously slow electron diffusion (cf. eq. (3.39)):

$$|\Delta r|_e \sim \sqrt{l_r\, |\Delta s|} \sim \sqrt{l_r \sqrt{l_s\, v_{||} t}}\,. \tag{3.42}$$

Various perturbations destroy the exact reversibility of the *electron* motion. Let us consider the impact of a finite Larmor radius ρ. Then the deviation would grow exponentially up to

$$|\delta r| \sim \rho \cdot \exp(\Lambda\, l_s) \lesssim l_r \tag{3.43}$$

at the next collision. The latter inequality is the condition for exponential, rather than diffusive, divergence of trajectories. This is to be compared with the collision-free diffusion (3.40) for $v_\| t = l_s$:

$$\frac{(\delta r)^2}{(\Delta r)_e^2} \sim (\rho\Lambda)^2 \cdot \frac{\exp(2\Lambda l_s)}{\Lambda l_s} \leq \frac{e}{2}\left(\frac{\rho}{l_s}\right)^2 \ll 1 . \tag{3.44}$$

The minimum is reached at $2\Lambda l_s = 1$ if $l_s \gtrsim \rho$ to satisfy the inequality in eq. (3.43).

Strong diffusion suppression is a striking manifestation of the time reversibility in dynamical chaos. Notice that finite residual *electron* diffusion is the result of the *partial reversal* of its velocity ($v_\| \to -v_\|$ only).

3.2. Critical phenomena in dynamics

The examples considered above suggest that a few degrees of freedom chaotic dynamical system typically has the divided phase space with many chaos borders. Each of those is characterized by the so-called *critical structure* [43] which is a hierarchy of chaotic and regular domains on ever decreasing spatial and frequency scales. This makes statistical description a very difficult problem. In particular, any averaging has to be done over the chaotic component of the motion whose measure is no longer simple Hamiltonian Γ_E (3.19) as for the ergodic motion. Nevertheless, the critical structure can be universally described in terms of *renormalization group* which proved to be so efficient in other branches of theoretical physics. In turn, such a renormgroup may be considered as an abstract dynamical system which describes the variation of the whole motion structure, for the original dynamical system, in dependence on its spatial and temporal scales. The logarithm of the latter plays the role of 'time' (renormtime) in that renormdynamics. At chaos border the latter is determined by the motion frequencies. The simplest renormdynamics is a periodic variation of the structure or, for a renorm-map, the invariance of the structure with respect to the scale [44]. Surprisingly, this scale invariance includes chaotic trajectories as well. The opposite limit — *renormchaos* — is also possible and was found in several models (see ref. [43]). Remarkably, for a two-dimensional map, which may also describe the two degrees of freedom conservative system, an extremely complicated renormdynamics can be reduced to the most simple one-dimensional map

$$\bar{r} = \frac{1}{r} \bmod 1 , \tag{3.45}$$

where r is the so-called rotation number, that is, the ratio of the two motion frequencies [43]. This map was introduced by Gauss in number theory and has been well studied by now [10]. In particular, the Lyapunov exponent (per iteration) is $\Lambda = \pi^2/6\ln 2$, and almost any initial r_0 generates a random trajectory which corresponds to random fluctuations of the motion structure from one scale to the next. Exceptional rationals $r_0 = m/n$ give rise to a periodic oscillation of the structure and, hence, to scale invariance in n steps.

Even though the critical structure occupies a very narrow strip along the chaos border it may qualitatively change the statistical properties of the whole chaotic component. This is because a chaotic trajectory unavoidably enters from time to time the critical region and 'sticks' there the longer the closer it comes to the chaos border. The sticking results in a slow power law, rather than exponential, correlation decay for large time

$$C(\tau) \sim \tau^{-p_C}, \quad \tau \to \infty. \tag{3.46}$$

Moreover, the exponent $p_C < 1$, and for the two-dimensional map it was found numerically to be approximately $p_C \approx 0.5$ in agreement with a simple theoretical analysis [43]. In higher dimensional case the dependence

$$p_C = \frac{1}{2N - 2} \tag{3.47}$$

was conjectured based on the same physical theory. Here, N is the number of linearly independent (incommensurate) frequencies, both internal (unperturbed) and driving.

Slow decaying correlation (3.46) implies a singular power spectrum which is the Fourier transform of $C(\tau)$:

$$S(\omega) \sim \frac{1}{\omega^{p_S}}, \quad \omega \to 0; \qquad p_S = 1 - p_C = \frac{2N-3}{2N-2}. \tag{3.48}$$

As $N \to \infty$ the spectrum approaches that of the 'mysterious' $1/\omega$ noise (see, e.g., ref. [45]). In minimal dimension ($N = 2$) the singular spectrum is $S \sim 1/\sqrt{\omega}$.

The diffusion determined by correlation (3.46) turns out to be anomalously fast [46] as the standard diffusion rate

$$D \sim \int C(\tau)\, d\tau \to \infty \tag{3.49}$$

diverges for $p_C \leq 1$. In such a case the dispersion σ^2 (second moment of the distribution function, e.g., $\sigma^2 = \langle(\Delta P)^2\rangle$ in the example below) is given by the double integral of correlation or by the differential equation

$$\frac{d^2\sigma^2}{d\tau^2} = 2\,C(\tau) \tag{3.50}$$

which can be applied to a more general problem [47]. A particular example is the standard map (3.13) (in variables $P = Tp$, φ) for special values of the parameter $K = K_n \approx 2\pi n$ with any integer $n \geq 1$ [23]. In this case there are two fixed points $\varphi = \varphi_1 =$ const satisfying $K \cdot \sin\varphi_1 = \pm 2\pi n$, and the momentum $|P|$ growing proportionally to (discrete) time. The fixed points are stable but the relative area of both stable domains around them is rather small, $A_n \approx 8/\pi^2 K_n \approx 2/\pi^4 n^2$, and decreases rapidly with n. The biggest one is $A_1 \approx 2\%$ only. Within the stable region the particle is accelerating independent of initial conditions. This is how the so-called microtron works, the first cyclic accelerator for relativistic electrons proposed by Veksler [48].

More interesting is the behavior of a chaotic trajectory. From time to time it approaches the chaos border of a tiny stable domain and sticks there for a while, being accelerated much more rapidly than in the rest (98%!) of the chaotic component. Since there are two stable domains with opposite acceleration the resulting motion would also be diffusive but anomalously fast: for $p_C = 1/2$ the average increase in the momentum becomes $|\Delta P| \sim t^{p_D}$ with $p_D = 2 - p_C = 3/2$ [46]. A more accurate calculation leads to the relation [25]

$$\langle(\Delta P)^2\rangle \approx \frac{\alpha}{2} A_n K_n^2 t^{3/2} \approx \frac{4\alpha}{\pi^2} t^{3/2} . \tag{3.51}$$

Here, t is the discrete time of the map, and $\alpha \approx 0.5$ is taken from numerical experiments [49], where such enhanced diffusion was observed for the first time. Actually, the normal diffusion rate $D = \langle(\Delta P)^2\rangle/t$ was measured and found to be 100 times (!) larger than expected $D = K^2/2$. Remarkably, the rate of anomalous diffusion (3.51) does not depend on the stable area A_n, yet the crossover time $t_a \approx \pi^8 n^4 \sim A_n^{-2}$ from normal to anomalous diffusion does.

In higher dimensional case $p_D(N) = (4N-5)/(2N-2) \to 2$ for $N \gg 1$, and $|\Delta P| \sim t$. This is the fastest homogeneous diffusion possible. The motion would be close to the straight acceleration but in both directions of P variation!

4. Quantum Pseudochaos

The mathematical theory of dynamical chaos — ergodic theory — is selfconsistent. However, this is not the case for the physical theory unless we accept the philosophy of the two separate mechanics, classical and quantum. Even though such a view cannot be excluded at the moment, it has a profound difficulty concerning the border between the two. Nor is it necessary according to recent intensive studies of quantum dynamics. Then we have to understand the mechanics of dynamical chaos from the quantum point of view. Our guiding star will be the *correspondence principle* which requires the complete quantum theory for any classical phenomenon, in the quasiclassical limit, assuming that the whole classical mechanics is but a special part (the limiting case) of currently most general and fundamental physical theory, the quantum mechanics. Now it would be more correct to speak about the quantum field theory but I restrict myself here to finite-dimensional systems only.

4.1. The correspondence principle

In an attempt to build the quantum theory of dynamical chaos we immediately encounter a number of apparently very deep contradictions between the well established properties of classical dynamical chaos and the most fundamental principles of quantum mechanics.

To begin with, quantum mechanics is commonly understood as a *fundamentally statistical theory* which seems to imply *always* some quantum chaos, independent of the behavior in the classical limit. This is certainly true but in some restricted sense only. A novel development here is the *isolation* of this fundamental quantum randomness as solely the characteristic of a very specific quantum process, the measurement, and even as the particular part of that — the so-called *ψ-collapse* which, indeed, has so far no dynamical description.

No doubt, quantum measurement is absolutely necessary for the study of microworld by us — the macroscopic human beings. Yet, the measurement is, in a sense, foreign to the microworld proper which might (and should) be described separately from the former. Explicitly [4], or more often implicitly, such a philosophy has become by now common in the studies of chaos but not yet beyond this field of research (see, e.g., ref. [50]).

This approach allows us to single out the dynamical part of quantum mechanics as represented by a *specific dynamical variable* $\psi(t)$ in the *Hilbert space* satisfying some *deterministic* equation of motion, e.g., the Schrödinger equation. The more difficult and vague statistical part is left behind for a better time. Thus, we temporarily bypass (not resolve!) the first serious difficulty in the theory of quantum chaos. The separation of the first part of quantum dynamics, which is very natural from mathematical viewpoint, was first introduced and emphasized by Schrödinger who, however, certainly underestimated the importance of the second part in physics.

However, another principal difficulty arises. As is well known, the energy (and frequency) spectrum of any quantum motion *bounded in phase space* is always *discrete*. And this is not the property of a particular equation but rather a consequence of the fundamental quantum principle — the *discreteness of phase space* itself, or in a more formal language, the noncommutative geometry of quantum phase space. Indeed, according to another fundamental quantum principle — the uncertainty principle — a single quantum state cannot occupy the phase space volume $V_1 \lesssim \hbar^N \equiv 1$ (in what follows I set $\hbar = 1$). Hence, the motion bounded in a domain of volume V is represented by $V/V_1 \sim V$ eigenstates, the property even stronger than the general discrete spectrum (almost periodic motion).

According to the existing ergodic theory, such a motion is considered to be *regular* which is something opposite to the known chaotic motion with continuous spectrum and exponential instability, again independent of the classical behavior. This seems to *never* imply any chaos or, to be more precise, *any classical-like chaos* as defined in the ergodic theory. Meanwhile, the correspondence principle requires *conditional chaos* related to the nature of motion in the classical limit.

4.2. PSEUDOCHAOS

Now the principal question to be answered reads: where in ergodic theory is the expected quantum chaos? Our answer to this question [51] (not commonly accepted as yet) was concluded from a simple observation (in principle well known but never comprehended enough) that the sharp border between the discrete and continuous spectrum is physically meaningful in the limit $|t| \to \infty$ only, the condition actually assumed in ergodic theory. Hence, to understand the quantum chaos the existing ergodic theory needs some modification by the introduction of a new 'dimension' — the time. In other words, a new central problem in ergodic theory arises: *finite-time statistical properties* of a dynamical system, both quantum as well as classical.

Within a finite time the discrete spectrum is dynamically equivalent to the continuous one, thus providing much stronger statistical properties of motion than it was (and still is) expected in ergodic theory in the case of discrete spectrum. In short, the motion with discrete spectrum may exhibit *all* the statistical properties of the classical chaos but only on some *finite* time scales.

A simple example of (classical) pseudochaos is a symbolic trajectory (Section 2) of some period T_s composed of *random* elements m_i ($i = 1, \ldots, T_s$) whatever the origin of the randomness is. In any event, *most* finite sequences m_i are *random*, indeed, according to the algorithmic theory of dynamical systems [21]. In this example there is a *single* time scale (T_s) for *all* statistical properties while, in general, there are several *different* scales related to a particular property (see below).

The conception of time scale is a fundamental one in our theory of quantum chaos [51]. This is certainly a *new* dynamical phenomenon, related but not identical at all to the classical dynamical chaos.[2] We call it *pseudochaos*, the term *pseudo* intending to emphasize the distinction from the asymptotic (in time) chaos in ergodic theory. Yet, from the physical point of view we accept here, the latter, strictly speaking, does not exist in nature. So, in the common philosophy of universal quantum mechanics the *pseudochaos is the only true dynamical chaos* (cf. the term pseudoeuclidean geometry in special relativity). The asymptotic chaos is but a limiting pattern which is, nevertheless, very important both in the theory to compare with the real chaos and in applications as a very good approximation in macroscopic domain as is the whole classical mechanics. Ford calls it *mathematical chaos* as contrasted to the *real physical chaos* in quantum mechanics [52]. Another curious but impressive term is *artificial reality* [53] which is, of course, a selfcontradictory notion reflecting, in particular, confusion in the interpreting such surprising phenomena as chaos.

Until recently the conception of classical dynamical chaos was completely incomprehensible, especially for physicists. One particular point of confusion was (and still remains to some extent) the Second Law of thermodynamics, the entropy increase in a closed system. Meanwhile, the entropy defined by the *exact* phase density is the integral of motion in any Hamiltonian system. Some physicists are still reluctant to assume thermodynamic entropy determined via the *coarse-grained* density, in which case it may well increase under conditions of dynamical chaos. From many researchers I know that they actually observed dynamical chaos in numerical or laboratory experiments but... did their best to get rid of it as some artifact, noise or other interference! Now the situation in this field is upside down: most researchers (not me!) insist that if an apparent chaos is not like that in the classical mechanics (and in the existing ergodic theory) then it is not a chaos at all. The most controversial conception in today's disputes is just the quantum chaos. The curiosity of the current situation is that in most studies of the 'true' (classical) chaos a digital computer is used where only *pseudochaos* is possible, that is, one like in *quantum* (not classical) mechanics!

The statistical properties of discrete-spectrum motion is not a completely new subject of research, it goes back to the time of intensive studies in the mathematical

[2] There are very special, even exotic I would say, examples of the 'true', classical-like, chaos in quantum systems (see [51, 4] and references therein). In all such cases the quantum motion is not only unbounded in some phase space variables but, moreover, the latter grow exponentially in time.

foundations of statistical mechanics *before* the dynamical chaos was discovered or, better to say, was understood (see, e.g., ref. [54]). We call this early stage of the theory *traditional statistical mechanics* (TSM). It is equally applicable to both classical as well as quantum systems. For the problem under consideration here one of the most important rigorous results with far-reaching implications was the *statistical independence* of oscillations with incommensurate (linearly independent) frequencies ω_n, such that the only solution of the resonance equation

$$\sum_n^N m_n \cdot \omega_n = 0 \tag{4.1}$$

in integers is $m_n \equiv 0$ for all n. This is a generic property of the real numbers, that is, the resonant frequencies (4.1) form a set of zero Lebesgue measure. If we define now $y_n = \cos(\omega_n t)$, the statistical independence of y_n means that trajectory $y_n(t)$ is ergodic in the N-cube $|y_n| \le 1$. This is a consequence of ergodicity of the phase trajectory $\phi_n(t) = \omega_n t \bmod 2\pi$ in the N-cube $|\phi_n| \le \pi$.

Statistical independence is a basic property of a set to which the probability theory is to be applied. In particular, the sum of statistically independent quantities

$$x(t) = \sum_n^N A_n \cdot \cos(\omega_n t + \phi_n), \tag{4.2}$$

which is the motion with discrete spectrum, is a typical object of this theory. However, the familiar statistical properties, like Gaussian fluctuations, postulated (directly or indirectly) in TSM, are reached in the limit $N \to \infty$ only [54] which is called the *thermodynamic limit*. In TSM this limit corresponds to infinite-dimensional models [10] which provide a very good approximation for macroscopic systems, both classical and quantal.

However, what is really necessary for good statistical properties of (4.2) is a large number of frequencies $N_\omega \to \infty$ which makes the discrete spectrum continuous (in the limit). In TSM the latter condition is satisfied by setting $N_\omega = N$. The same holds true for quantum fields which are infinite-dimensional. In quantum mechanics another mechanism, independent of N, works in the quasiclassical region $q \gg 1$ where $q = n/\hbar \equiv n$ is some big quantum parameter, e.g. quantum number, and n stands for a characteristic action of the system. Indeed, if the quantum motion (4.2) (with $\psi(t)$ instead of $x(t)$) is determined by many ($\sim q$) eigenstates we can set $N_\omega = q$ independent of N. The actual number of terms in expansion (4.2) depends, of course, on a particular state $\psi(t)$ under consideration. For example, if it is just an eigenstate the sum reduces to a single term. This corresponds to the special, peculiar trajectories of classical chaotic motion whose total measure is *zero*. Similarly, in quantum mechanics $N_\omega \sim q$ for *most* states if the system is *classically chaotic*. This important condition was found to be certainly *sufficient* for good quantum statistical properties (see ref. [51] and below). Whether it is also a necessary condition remains as yet unclear.

Thus, with respect to the mechanism of the quantum chaos we essentially *come back* to TSM with an exchange of the number of degrees of freedom N for the quantum parameter q. However, in quantum mechanics we are not interested, unlike TSM, in the limit $q \to \infty$ which is simply the classical mechanics. Here, the central problem is the statistical properties for *large but finite* q. This problem does not exist in TSM describing macroscopic systems. Thus, with an old mechanism the new phenomena were understood in quantum mechanics.

The direct relation between these two seemingly different mechanisms of chaos can be traced back in some specific dynamical models [4]. One interesting example is the nonlinear Schrödinger equation [88]. From a physical point of view it describes the motion of a quantum system interacting with many other degrees of freedom, whose state is expressed via the ψ function of the system itself (the so-called mean field approximation). This approximation becomes exact in the limit $N \to \infty$ which is a particular case of the thermodynamic limit. Therefore, the mechanism for chaos in this system is apparently the old one. On the other hand, the nonlinear Schrödinger equation has in general exponentially unstable solutions, hence the mechanism of chaos here seems to be the new one. Thus, for this particular model both mechanisms describe the same physical process. We would like to emphasize that the 'true' chaos present in these apparently few-dimensional models actually refers to infinite-dimensional systems.

4.3. Characteristic time scales in quantum chaos

The existing ergodic theory is asymptotic in time, and hence contains no time scales at all. There are two reasons for this. One is technical: it is much simpler to derive the asymptotic relations than to obtain rigorous finite-time estimates. Another reason is more profound. All statements in the ergodic theory hold true up to measure zero, that is, excluding some peculiar nongeneric sets of zero measure. Even this minimal imperfection of the theory did not seem completely satisfactory but has been 'swallowed' eventually and is now commonly tolerated even among mathematicians, to say nothing about physicists. In a finite-time theory all these exceptions acquire a *small but finite* measure which would be already 'unbearable' for mathematicians. Yet, there is a standard mathematical trick, to be discussed below, for avoiding both these difficulties.

The most important time scale t_R in quantum chaos is given by the general estimate

$$\ln t_R \sim \ln q, \qquad t_R \sim q^\alpha \sim \rho_0 \leq \rho_H, \tag{4.3}$$

where $\alpha \sim 1$ is a system-dependent parameter. This is called the *relaxation time scale* referring to one of the principal properties of the chaos — *statistical relaxation* to some steady state (statistical equilibrium). The physical meaning of this scale is principally simple, and it is directly related to the fundamental uncertainty principle ($\Delta t \cdot \Delta E \sim 1$) as implemented in the second eq. (4.3), where ρ_H is the *full* average energy level density (also called Heisenberg time). For $t \lesssim t_R$ the discrete spectrum is not resolved, and the statistical relaxation follows the classical (limiting) behavior. This is just the 'gap' in the ergodic theory (supplemented with the additional time dimension) where the pseudochaos, particularly quantum chaos, dwells. A more accurate estimate relates t_R to a *part* ρ_0 of the level density. This is the density of the so-called *operative eigenstates* only, that is, those which are actually present in a particular quantum state ψ, and which actually control its dynamics.

The formal trick mentioned above is to consider not the finite-time relations we really need in physics, but rather a special *conditional limit* (cf. eq. (2.18)):

$$t, q \rightarrow \infty, \qquad \tau_R = \frac{t}{t_R(q)} = \text{const}. \tag{4.4}$$

The quantity τ_R is here a new rescaled time which is, of course, nonphysical but very helpful technically. The *double* limit (4.4) (unlike the single one $q \rightarrow \infty$) is *not* the classical mechanics which holds true, in this representation, for $\tau_R \lesssim 1$ and with respect to the statistical relaxation only. For $\tau_R \gtrsim 1$ the behavior becomes essentially quantum (even in the limit $q \rightarrow \infty$!) and is called nowadays *mesoscopic phenomena*. In particular, the quantum steady state is quite different from the classical statistical equilibrium in that the former may be *localized* (under certain conditions), that is, it is *nonergodic* in spite of classical ergodicity.

Another important difference is in *fluctuations* which are also a characteristic property of chaotic behavior. In comparison with classical mechanics the quantum $\psi(t)$ plays, in this respect, an intermediate role between the classical trajectory (exact or symbolic) with big relative fluctuations ~ 1 and the coarse-grained classical phase space density with no fluctuations at all. On the other hand both fluctuations of $\psi(t)$ are $\sim N_\omega^{-1/2}$ which is another manifestation of statistical independence, or *decoherence*, of even pure quantum state (4.2) in case of quantum chaos. In other words, chaotic $\psi(t)$ represents statistically a *finite ensemble* of $\sim N_\omega$ systems even though formally $\psi(t)$ describes a single system. Quantum fluctuations clearly demonstrate also the difference between physical time t and auxiliary variable τ: in the double limit $(t, q \rightarrow \infty)$ the fluctuations vanish, and one needs a new trick to recover them.

The popular term *mesoscopic* means here an *intermediate* behavior between classical $(q \rightarrow \infty)$ and quantum one (e.g., localization). In other words, in a mesoscopic phenomenon both classical and quantum features are combined simultaneously. Again, the correspondence principle requires transition to the completely classical behavior. This is, indeed, the case according to Shnirelman's theorem or, better to say, to a physical generalization of the theorem [55]. Namely, the mesoscopic phenomena occur in the so-called *intermediate quasiclassical asymptotics* where $q \gg 1$ is already very big but still $q \lesssim q_f$ less than a certain critical q_f which determines the border of transition to a fully classical behavior. The latter region, ensured by the above theorems in accordance with the correspondence principle, is called the *far quasiclassical asymptotics*.

The striking well known examples of mesoscopic phenomena are superconductivity and superfluidity. A mesoscopic parameter here is the temperature which determines the behavior of microparticles, electrons and atoms, respectively. The far asymptotics corresponds here to $T > T_f$, where both essentially quantum phenomena disappear.

The relaxation time scale should not be confused with the *Poincare recurrence time* $t_P \gg t_R$ which is typically much longer, and which increases sharply with the decrease of recurrence domain. The time scale t_P characterizes big fluctuations (for both the classical trajectory, but not the phase space density, and the quantum ψ) of which the recurrence is a particular case, while t_R characterizes the average relaxation process.

Stronger statistical properties than relaxation and fluctuations are related in ergodic theory to the exponential instability of motion. Their importance for the statistical mechanics is not completely clear. Nevertheless, in accordance with the correspondence principle, these stronger properties are also present in quantum chaos as well but on a *much shorter* time scale,

$$t_r \sim \frac{\ln q}{h}, \tag{4.5}$$

where h is the classical metric entropy (2.11). This time scale was discovered and partly explained in refs. [57] (see also refs. [51, 4]). We call it *random time scale*. Indeed, according to the Ehrenfest theorem the motion of a narrow wave packet follows the beam of classical trajectories as long as the packet remains narrow, and hence it is as random as in the classical limit. Even though the random time scale is very short, it grows indefinitely as $q \to \infty$. Thus, a temporary finite-time quantum pseudochaos turns into the classical dynamical chaos in accordance with the correspondence principle. Again, we may consider the *conditional limit*

$$t, q \to \infty, \qquad \tau_r = \frac{t}{t_r(q)} = \text{const}. \tag{4.6}$$

Notice that the *scaled* time τ_r is different from τ_R in eq. (4.4).

In particular, if we fix time t, then in the limit $q \to \infty$ we obtain the transition to the classical instability in accordance with the correspondence principle, while for q fixed and $t \to \infty$, we have the proper quantum evolution in time. For example, for the quantum Lyapunov exponent we have

$$\Lambda_q(\tau_r) \to \begin{cases} \Lambda, & \tau_r \ll 1 \\ 0, & \tau_r \gg 1. \end{cases} \tag{4.7}$$

The quantum instability ($\Lambda_q > 0$) was observed in numerical experiments [58, 4]. What does terminate the instability for $t \gtrsim t_r$? A naive explanation that the major size of the originally very narrow quantum packet reaches the full swing of a bounded motion is obviously too simplified. This is immediately clear from the comparison with a classical packet behavior (Section 2). Also, the quantum packet squeezing is not principally restricted since only 2-dimensional area (per degree of freedom) is bounded from below in quantum mechanics. Instead, numerical experiments show that the original wave packet, after a considerable stretching similar to the classical one, is rapidly destroyed. Namely, it gets split into many new small packets. A possible explanation [59] (see also ref. [4]) is related to the discreteness of the action variable in quantum mechanics, which leads to the "rupture" of a very long stretched packet into many pieces. Such a mechanism determines a new *destruction time scale* which, for the quantized standard map (see below), is given by the estimate

$$t_d \sim \frac{|\ln T|}{2\Lambda}. \tag{4.8}$$

This roughly agrees with the results of numerical experiments [58, 4]. As expected $t_d \sim t_r$ (see eq. (4.5)).

There is another mechanism which produces deviation of the quantum packet evolution from the classical motion [59]. We call it *inflation* because of the increase in time of the phase space area occupied by the quantum phase space density (the Wigner function) contrary to the classical density which is conserved (Liouville's theorem). The inflation can be analyzed using the quantum Liouville equation for the Wigner function W [60]. In the case of standard map this equation reduces to

$$\frac{dW(n,\varphi)}{dt} \approx -\frac{1}{24}\frac{\partial^3 H}{\partial\varphi^3}\frac{\partial^3 W}{\partial n^3}, \tag{4.9}$$

and gives the following estimate for the *inflation time scale*:

$$t_{if} \sim \frac{|\ln(TK^2/\Lambda^2)|}{6\Lambda}, \tag{4.10}$$

The inflation time is of the order of destruction time (4.8) and of the random time scale (4.5) as well, which implies in particular, a considerable squeezing of the wave packet.

An important implication of the above picture of the packet time evolution is the rapid and complete destruction of the so-called generalized coherent states [61] in quantum chaos.

In quasiclassical region ($q \gg 1$) scale $t_r \ll t_R$ (4.3). This leads to an interesting conclusion that the quantum diffusion and relaxation are *dynamically stable* contrary to the classical behavior. It suggests, in turn, that the instability of motion is not important *during* statistical relaxation. However, the *foregoing* correlation decay on short random time scale t_r is crucial for the statistical properties of quantum dynamics. Dynamical stability of quantum diffusion has been proved in striking numerical experiments with time reversal [65]. In a classical chaotic system the diffusion is immediately recovered due to numerical "errors" (not random!) amplified by the local instability. On the contrary, the quantum "antidiffusion" proceeds until the system passes, to a very high accuracy, the initial state, and only then the normal diffusion is restored. The stability of quantum chaos on relaxation time scale is comprehensible as the random time scale is much shorter. Yet, the accuracy of the reversal (up to $\sim 10^{-15}$ (!)) is surprising. Apparently, this is explained by a relatively large size of the quantum wave packet as compared to the unavoidable rounding-off errors, unlike in the case of the classical computer trajectory which is just of that size [68]. In the standard map the size of the optimal least-spreading wave packet is $\Delta\varphi \sim \sqrt{T}$ [51]. On the other hand, any quantity in the computer must well exceed the rounding-off error $\delta \ll 1$. In particular, $T \gg \delta$, and $(\Delta\varphi)^2/\delta^2 \gtrsim (T/\delta)\delta^{-1} \gg 1$.

4.4. QUANTUM LOCALIZATION: THE KICKED ROTATOR MODEL

The standard map (3.13) was shown in Section 3 to provide the local description of motion for many more realistic classical models. So, the quantized standard map seems to be a good approach in the studies of quantum chaos as well. This can be done in two ways. The first one is to derive exact unitary operator $\hat{U}_T$ over some time interval T

$$\overline{\psi(t)} \equiv \psi(t+T) = \hat{U}_T\,\psi(t), \qquad \hat{U}_T = \exp\left(-i\int_0^T dt\,\hat{H}\right), \tag{4.11}$$

where $\hat{H}$ is the Hamiltonian operator. Generally, this is a very difficult mathematical problem which will not be discussed (see ref. [62]). Instead, we consider here the second way: the direct quantization of the classical standard map (3.13) which is, of course, only an approximate solution of the whole problem. I am not aware of any thorough analysis of the accuracy and limitations of this simple method. However, the direct comparison of such a quantum map with a numerical solution of the Schrödinger equation for the diffusive photoeffect in Rydberg Hydrogen atom confirms that the former is a reasonable approximation indeed [32].

The quantization of standard map with Hamiltonian

$$H(n, \varphi, t) = \frac{n^2}{2} + k \cdot \cos\varphi \cdot \delta_T(t) \tag{4.12}$$

leads to the unitary operator [63]:

$$\hat{U}_T = \exp\left(-i\,\frac{T\,\hat{n}^2}{2} \right) \cdot \exp\left(-ik \cdot \cos\hat{\varphi} \right), \tag{4.13}$$

where $\delta_T(t)$ is δ-function of period T, and $\hat{n} = -i\frac{\partial}{\partial\varphi}$.

The standard map (4.13) is defined on a cylinder ($-\infty < n < +\infty$), where motion can be unbounded. To describe bounded motion in a conservative system it is more convenient to make use of another version of the standard map, namely, one on a torus with *finite* number of states $L \gg 1$. In momentum representation $\psi(n,\, t)$ it is described by a finite unitary matrix U_{nm}

$$\overline{\psi(n)} = \sum_{m=-L_1}^{L_1} U_{nm}\,\psi(m)\,, \tag{4.14}$$

where $L = 2L_1 + 1 \approx 2L_1$, and

$$U_{nm} = \frac{1}{L}\exp\left(i\frac{T}{4}(n^2+m^2)\right) \cdot \sum_{j=-L_1}^{L_1} \exp\left[-ik \cdot \cos\left(2\pi j/L\right) - 2\pi i(n-m)j/L\right], \tag{4.15}$$

while $T/4\pi = M/2L$ is now rational [64].

There are three quantum parameters in this model: perturbation k, period T and size L in momentum, but only two classical combinations remain: perturbation $K = k{\cdot}T$ and classical size $M = TL/2\pi$ which is the number of resonances over the torus. Notice that quantum dynamics is in general richer than the classical one as the former depends on an extra parameter. It is, of course, another representation of Planck's constant which we have set $\hbar = 1$. This is why in quantized standard map we need both parameters k and T separately and cannot combine them in a single classical parameter K.

The quasiclassical region where we expect quantum chaos corresponds to $T \to 0$, $k \to \infty$, $L \to \infty$ while the classical parameters $K = \text{const}$ and $M = \text{const}$.

A technical difficulty in evaluating t_R for a particular dynamical problem is in that the density ρ_0 depends, in turn, on the dynamics. So, we have to solve a self-consistent problem. For the standard map the answer is known (see ref. [4]):

$$t_R = \rho_0 = 2D_0\,, \tag{4.16}$$

where $D_0 = k^2/2$ is the classical diffusion rate (for $K \gg 1$). The quantum diffusion rate depends on the scaled variable

$$\tau_R = \frac{t}{2\,D_0(k)}\,,$$

and is given by

$$D_q = \frac{D_0}{1+\tau_R} \to \begin{cases} D_0\,, & \tau_R = t/t_R \ll 1 \\ 0\,, & \tau_R \gg 1 \end{cases} \,. \qquad (4.17)$$

This is an example of scaling in discrete spectrum which stops eventually the quantum diffusion.

A simple estimate for t_R in the standard map can be derived as follows [51] (see also ref. [67]). The quantum map as a time-dependent system is characterized by *quasienergies* which are determined modulo $\Omega = 2\pi$, where Ω is the frequency of external perturbation, and where the latter value corresponds to the discrete time with one iteration of the map as the time unit. Then, the mean density of operative eigenstates is $\rho_0 = N_0/\Omega$, where N_0 is the number of the latter. In turn, $N_0 \sim 2\sqrt{D_0 t_R}$ which is also the number of unperturbed states (n) covered by the quantum diffusion until it stops. We assume here that both quantum diffusion as well as the eigenstates are statistically homogeneous, that is, they couple *all* unperturbed states, at least, mesoscopically. This natural assumption is in agreement with all the numerical experiments. Microscopic deviations from homogeneity, the so-called 'scars' and some others (see, e.g., refs. [69]), apparently do not affect mesoscopic quantum properties. Then we arrive at a simple estimate: $t_R \sim D_0$ (cf. eq. (4.16)). Moreover, the same estimate gives also the size, or *localization length*, of the localized steady state (l_s) as well as that of the eigenfunctions (l): $l_s \sim l \sim t_R \sim D_0$. These are remarkable relations since they connect essentially *quantum* characteristics (l_s, l, t_R) with the *classical* diffusion rate D_0. This is just a characteristic feature of the mesoscopic phenomena.

For the standard map on a cylinder quantum diffusion is always localized, the shape of the localized states being approximately exponential (see, e.g., ref. [4]),

$$\psi(n) \approx \frac{\exp\left(-\frac{|n-n_0|}{l}\right)}{\sqrt{l}}\,, \qquad (4.18)$$

and the same for the steady state. Interestingly, the two localization lengths are different [51]:

$$l_s \approx D_0 \quad \text{while} \quad l \approx \frac{D_0}{2} \qquad (4.19)$$

because of big fluctuations.

In general, quantum localization is a nonuniversal but very interesting and important mesoscopic phenomenon because it means the formation of nonergodic or localized states (both a steady state as well as eigenstates) for classically ergodic motion. Moreover, the localized steady state *depends* on the initial state from which the diffusion starts. For the standard map on torus the *ergodicity parameter* controlling localization can be defined as

$$\lambda = \frac{D_0}{L} \sim \left(\frac{t_R}{t_e}\right)^{1/2} \sim \frac{k^2}{L} \sim \frac{K}{M}\cdot k\,, \qquad (4.20)$$

where $t_e \sim L^2/D_0$ is a characteristic time of the classical relaxation to the ergodic steady state $|\psi(n)|^2 \approx$ const.

If $\lambda \gg 1$ the final steady state as well as all the eigenfunctions are ergodic, that is, the corresponding Wigner functions are close to the classical microcanonical distribution in phase space (3.19). This is far quasiclassical asymptotics. It can be reached, in particular, if the classical parameter K/M is kept fixed while the quantum parameter $k \to \infty$.

However, if $\lambda \ll 1$ all the eigenstates and the steady state are nonergodic. It means that their structure remains essentially quantum, no matter how large is the quantum parameter $k \to \infty$. This is intermediate quasiclassical asymptotics or mesoscopic domain. In particular, it corresponds to $K > 1$ fixed, $k \to \infty$ and $M \to \infty$ while $\lambda \ll 1$ remains small.

In terms of localization length the region of mesoscopic phenomena is defined by the double inequality

$$1 \ll l \ll L. \tag{4.21}$$

The left inequality is a macroscopic feature of the state while the right one refers to quantum effects. The combination of both allows, in particular, for a classical description, at least for the standard map, of the statistical relaxation to the quantum steady state by a phenomenological diffusion equation [66, 4] for the Green function

$$\frac{\partial g(\nu, \sigma)}{\partial \sigma} = \frac{1}{4}\frac{\partial^2 g}{\partial \nu^2} + B(\nu)\frac{\partial g}{\partial \nu}. \tag{4.22}$$

Here $g(\nu, 0) = |\psi(\nu, 0)|^2 = \delta(\nu - \nu_0)$ and

$$\nu = \frac{n}{2D_0}, \quad \sigma = \ln(1 + \tau_R), \quad \tau_R = \frac{t}{2D_0}. \tag{4.23}$$

The additional drift term in the diffusion equation with

$$B(\nu) = \mathrm{sign}(\nu - \nu_0) = \pm 1 \tag{4.24}$$

describes the so-called quantum coherent backscattering which is the dynamical mechanism of localization.

The solution of eq. (4.22) reads [4]

$$g(\nu, \sigma) = \frac{1}{\sqrt{\pi\sigma}} \exp\left[-\frac{(\delta + \sigma)^2}{\sigma}\right] + \exp(-4\delta)\cdot \mathrm{erfc}\left(\frac{\delta - \sigma}{\sqrt{\sigma}}\right), \tag{4.25}$$

where $\delta = |\nu - \nu_0|$, and

$$\mathrm{erfc}(u) = \frac{2}{\sqrt{\pi}}\int_u^\infty e^{-v^2}\, dv.$$

Asymptotically, as $\sigma \to \infty$, the Green function $g(\nu, \sigma) \to 2\exp(-4\delta) \equiv g_s$ approaches the localized steady state g_s exponentially in σ but only as a power-law in physical time τ_R or t ($g - g_s \sim 1/\tau_R$). This is the effect of discrete motion spectrum. Numerical experiments confirm the prediction (4.25), at least to the logarithmic accuracy $\sim \sigma \approx \ln \tau_R$ [70, 4].

The quantum diffusion on relaxation time scale depends, in general, on two other conditions. The first one requires a sufficiently strong perturbation. Otherwise the quantum transitions between unperturbed states would be suppressed, which is called *perturbative localization*. This is a well-known quantum effect also related to the discrete quantum spectrum. The opposite case of strong perturbation is called *quasicontinuum* (referring to the same spectrum). For the standard map this condition reads $k \gg 1$ (see eq. (4.13)).

The second condition is especially simple for a bounded map, e.g., $k \lesssim L$ in the case of standard map on a torus. This condition is required in both quantum as well as classical systems. Otherwise the diffusion approximation is no longer valid and a more complicated kinetic equation is necessary for the description of statistical relaxation. In continuous time this condition is formulated in terms of the dynamical time scale of the relaxation process for which the former is just one iteration of the map. The general condition requires the dynamical change of variables to be sufficiently small.

A physical example of localization is the quantum suppression of diffusive photoeffect in Hydrogen atom (Section 3). In quantum analysis it is convenient to change the electron energy E for the number of electric field photons: $E \to n_\phi - (E_0 - |E|)/\Omega$ where $E_0 = 1/2n^2$ is the initial energy. The quantum suppression of diffusive ionization depends on the ratio (cf. eq. (4.20))

$$\lambda_\phi = \frac{l_s}{n_\phi^0} \approx \frac{D_0}{n_\phi^0} \approx \frac{6.6 f_n}{\Omega_n^{7/3}}, \tag{4.26}$$

where $D_0 \approx 3.3 f^2/\Omega^{10/3}$ is the 'classical' diffusion rate, and $n_\phi^0 = E_0/\Omega$ the number of absorbed photons required for ionization. Notice that D_0 does not depend on the quantum number n so that the whole ionization process can be described by the local map, which considerably simplifies the theoretical analysis.

If $\lambda_\phi \gtrsim 1$, localization does not affect the diffusion which eventually leads to the complete ionization of the atom. For $\lambda_\phi \ll 1$ the ionization is strongly (but not completely) suppressed due to quantum effects. Depending on parameters, the suppression may occur no matter how large quantum number n is. Again, this is a typical mesoscopic phenomenon which was predicted by the theory of quantum chaos, and was subsequently observed in laboratory experiments (see ref. [32]).

The mesoscopic domain of localized quantum chaos corresponds to the interval $f_n^{(b)} < f_n < f_n^{(l)}$. Here $f_n^{(l)}$ is the border of localization $\lambda_\phi \approx 1$ (4.26), and $f_n^{(b)}$ the chaos border (3.32). The size of this domain grows rapidly with Ω_n,

$$\frac{f_n^{(l)}}{f_n^{(b)}} \approx 7.6\, \Omega_n^{8/3}. \tag{4.27}$$

The two additional conditions for quantum diffusion mentioned above lead to the restriction $\Omega_n \lesssim n$.

4.5. Examples of pseudochaos in classical mechanics

The pseudochaos is a new generic dynamical phenomenon missed in ergodic theory. No doubt, the most important particular case of pseudochaos is the quantum chaos. Nevertheless, pseudochaos occurs in classical mechanics as well. Here are a few examples of classical pseudochaos which may help to understand the physical nature of quantum chaos, my primary goal in this paper. Besides, it unveils new features of classical dynamics as well.

Linear waves is the closest to quantum mechanics example of pseudochaos (see, e.g., ref. [71]). I remind that only a part of quantum dynamics is discussed here, the one described, e.g., by the Schrödinger equation which is a linear wave equation. For this reason quantum chaos is called sometimes wave chaos [72]. Classical electromagnetic waves are used in laboratory experiments as a physical model for quantum chaos [73]. The 'classical' limit corresponds here to the geometric optics, and the 'quantum' parameter $q = L/\lambda$ is the ratio of a characteristic size L of the system to wave length λ. As is well known in optics, no matter how large the ratio L/λ is, the diffraction pattern prevails at a sufficiently far distance $R \gtrsim L^2/\lambda$. This is a sort of relaxation scale: $R/\lambda \sim q^2$.

Linear oscillator (multidimensional) is also a particular representation of waves (without dispersion). A broad class of quantum systems can be reduced to this model [74]. Statistical properties of linear oscillator, particularly in the thermodynamic limit ($N \to \infty$), were studied in ref. [75] in the frames of TSM. On the other hand, the theory of quantum chaos suggests a richer behavior for a big but finite N, in particular, the characteristic time scales for the harmonic oscillator motion [76], the number of degrees of freedom N playing a role of the 'quantum' parameter.

Completely integrable nonlinear systems also reveal pseudochaotic behavior. An example of statistical relaxation in the Toda lattice was presented in ref. [77] much before the problem of quantum chaos arose. Moreover, the strongest statistical properties in the limit $N \to \infty$, including one equivalent to the exponential instability (the so-called K-property), were rigorously proved just for (infinite) completely integrable systems (see ref. [10]).

Digital computer is a very specific classical dynamical system whose dynamics is extremely important in view of the ever growing interest in numerical experiments covering now all branches of science and beyond. The computer is an 'overquantized' system since *any* quantity here is *discrete* while in quantum mechanics only the product of two conjugated variables is. The 'quantum' parameter is here $q = M$ which is the largest computer integer, and the short time scale (4.5) $t_r \sim \ln M$ which is the number of digits in computer word [51]. Owing to the discreteness, any dynamical trajectory in a computer becomes eventually periodic, the effect well known in the theory and practice of the so-called pseudorandom number generators. The term 'pseudochaos' itself was borrowed from just this particular example [68, 4]. One should take all necessary precautions to exclude this computer artifact in numerical experiments (see, e.g., [78] and references therein). On the mathematical part, periodic approximations in dynamical systems are also studied in ergodic theory, apparently without any relation to pseudochaos in quantum mechanics or computers [10].

The computer pseudochaos is the best answer to those who refuse to accept the quantum chaos as, at least, a kind of chaos, and who still insist that only the classical-like (asymptotic) chaos deserves this name, the same chaos which was (and is) studied to a large extent just on computers, that is, the chaos inferred from a pseudochaos!

5. Conclusion: Old Challenges and New Hopes

The discovery and understanding of the new surprising phenomenon — dynamical chaos — opened up new horizons in solving many other problems including some of the long-standing ones. Here I can give only a preliminary consideration of possible new approaches to such problems together with some plausible conjectures (see also ref. [4]).

Let us begin with the problem directly related to quantum dynamics, namely, the quantum measurement or, to be more correct, the specific stage of the latter, the *ψ-collapse*. It is just the part of quantum dynamics I bypassed above. This part still remains very vague to the extent that there is no common agreement even on the question whether it is a real physical problem or an ill-posed one so that the Copenhagen interpretation of (or convention in) quantum mechanics gives satisfactory answers to all *admissible* questions. In any event, there exists as yet no dynamical description of the quantum measurement including ψ-collapse. The quantum measurement, as far as the result is concerned, is commonly understood as a fundamentally random process. However, there are good reasons to hope that this randomness can be interpreted as a particular manifestation of dynamical chaos [79].

The Copenhagen convention was (and still remains) very important as a phenomenological link between very specific quantum theory and laboratory experiments. Without this link the studies of microworld would be simply impossible. The Copenhagen philosophy perfectly matches the standard experimental setup of two measurements: the first one fixes the initial quantum state, and the second one records the changes in the system. However, it is less clear how to deal with *natural processes* without any man-made measurements, that is, without the notorious *observer*. Since the beginning of quantum mechanics such a question has been considered ill-posed (meaning nasty). However, now there is a revival of interest in a deeper insight into this problem (see, e.g., ref. [79]). In particular, Gell-Mann and Hartle put a similar question, true, in the context of a very specific and global problem — the quantum birth of the Universe [80]. In my understanding, such a question arises as well in much simpler problems concerning any natural quantum processes. What is more important, the answer [80] does not seem to be satisfactory. Essentially, it is the substitution of an automaton (information gathering and utilizing system) for a standard human observer. Neither seems to be a generic construction in the microworld.

The theory of quantum chaos allows us to solve, at least, the (simpler) half of the ψ-collapse problem. Indeed, the measurement device is by purpose a macroscopic system for which the classical description is a very good approximation. In such a system strong chaos with exponential instability is quite possible. Chaos in the classical measurement device is not only possible but unavoidable since the measurement system has to be, by purpose again, a highly unstable system where a microscopic intervention produces a macroscopic effect. The importance of chaos for the quantum measurement is that it destroys the coherence of initial pure quantum state to be measured, converting it into a incoherent mixture. In the present theories of quantum measurement this is described as the effect of external noise (see, e.g., ref. [81]). True, the noise is sufficient to destroy the quantum coherence, yet it is not necessary at all [82]. The chaos theory allows one to get rid of the unsatisfactory effect of external noise and to develop a purely dynamical theory for the loss of quantum coherence. Yet this is not the whole story. If we are satisfied with the *statistical* description of quantum dynamics (measurement included) then the decoherence is all we need. However, *individual* behavior includes the second (main) part of ψ-collapse, namely, the *concentration* of ψ in a single state of the original superposition

$$\psi = \sum_n c_n \psi_n \rightarrow \psi_k , \qquad \sum_n |c_n|^2 = 1 .$$

This is the proper ψ-collapse to be understood.

Also, it is another challenge to the correspondence principle. For the quantum mechanics to be universal it must also explain a very specific classical phenomenon of an *event* which does happen and remains forever in the classical records, and which is completely foreign to the quantum mechanics proper. It is just the effect of ψ-collapse.

All these problems could be resolved by a hypothetical phenomenon of *selfcollapse*, that is, the collapse without any 'observer', human or automatic. Recently some attempts to resolve this problem were undertaken [83] yet they are still to be understood and evaluated. So far I would like simply to mention that these attempts are trying to make use of nonlinear "semiquantum" equations like the well-studied nonlinear Schrödinger equation (for discussion see refs. [4, 84]).

We come now to even more difficult problem of the *causality principle*, that is, the universal time ordering of events. This principle has been well confirmed by numerous experiments in all branches of physics. It is frequently used in the construction of various theories but, to my knowledge, any general relation of causality to the rest of physics was never studied.

This principle looks as a statistical law (another time arrow), hence a new hope to understand the mechanism of causality via dynamical chaos. Yet, it directly enters the dynamics as a additional constraint on the interaction and/or the solutions of dynamical equations. A well known and quite general example is to keep the retarded solutions of a wave equation only and discarding the advanced ones as 'nonphysical'. However, this is in general impossible because of the boundary conditions. Still, the causality holds true as well.

In some simple classical *dissipative*, models like a driven damped oscillator, the dissipation was shown to imply causality [85, 86]. However, such results were formulated as a restriction on a class of systems showing causality rather than the foundations of the causality principle. Nevertheless, it was already some indication on a possible physical connection between dynamical causality and statistical behavior. To my knowledge, this connection was never studied any farther. To the contrary, the development of the theory went in the opposite way: taking for granted the causality to deduce all possible consequences, in particular, various dispersion relations [86]. In some physical [87] (not mathematical [10]!) theories in TSM the causality principle, sometimes termed modestly as 'causality condition', is used to 'derive' statistical irreversibility from time-reversible dynamics. As was discussed in Section 2, physical chaos theory (and, implicitly, mathematical ergodic theory as well) predicts, instead, nonrecurrent relaxation without any additional 'conditions', causality included. Then the above-mentioned arguments (e.g., in ref. [87]) could be reversed in such a way to derive the causality from dynamical chaos, similar to refs. [85, 86] but for a much more general class of dynamical systems.

The causality relates two qualitatively different kinds of events: *causes* and *effects*. The former may be simply the initial conditions of motion, the point missed in the above-mentioned examples of causality–dissipation relation. The initial conditions not only formally fix a particular trajectory but they are also *arbitrary* which is, perhaps, the key point in the causality problem. Also, this may shed some light on another puzzling peculiarity of *all* known dynamical laws: they describe the motion up to arbitrary initial conditions only. It looks like the dynamical laws already include the causality implicitly, even though not explicitly. In any event, something arbitrary suggests a chaos around.

Again, we arrive at a tangle of interrelated problems. A plausible conjecture how to resolve them might be as follows. Arbitrary cause indicates some statistical behavior while the cause-effect relation points out a dynamical law. Then, we may conjecture that when the cause acts the transition from statistical to dynamical behavior occurs, which separates statistically the cause from the 'past' and fixes dynamically the effect in the 'future'. In this imaginary picture, the 'past' and 'future' are related not with time but rather with cause and effect, respectively. Thus, the causality might be not the time ordering (time arrow) but the *cause-effect ordering*, or the *causality arrow*. The latter is very similar to the process arrow discussed in Section 2, both always pointing in the same direction. Now, the central point is in that the cause is arbitrary while the effect is not whatever the time ordering.

This is, of course, but a raw guess to be developed, carefully analysed, and eventually confirmed or disproved experimentally.

Also, this picture seems to be closer to the statistical (secondary) dynamics (synergetics, or $S \supset D$ inclusion in (1.1)) rather than to dynamical chaos. Does it mean that primary physical laws are statistical or, instead, that the chain of inclusions (1.1) is actually a closed ring with a 'feedback' coupling the secondary statistics to the primary dynamics?

We don't know.

In this paper I have never given the definition of dynamical chaos, either classical or quantal, restricting myself to informal explanations (see ref. [4] for some current definitions of chaos). In a mathematical theory the definition of the main object of the theory precedes the results; in physics, especially in new fields, it is quite often the other way around. First, one studies a new phenomenon like dynamical chaos and only at a later stage, after understanding it sufficiently, we try to classify it, to find its proper place in existing theories and eventually to choose the most reasonable definition.

This time has not yet come.

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