PSEUDOCHAOS IN STATISTICAL PHYSICS[1]

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Abstract

A new generic dynamical phenomenon of *pseudochaos* and its relevance to the statistical physics both modern as well as traditional one are considered and explained in some detail. The pseudochaos is defined as a statistical behavior of the dynamical system with *discrete* energy and/or frequency spectrum. In turn, the statistical behavior is understood as time–reversible but nonrecurrent relaxation to some steady state, at average, superimposed with irregular fluctuations. The main attention is payed to the most important and universal example of pseudochaos, the so–called *quantum chaos* that is dynamical chaos in bounded mesoscopic quantum systems. The quantum chaos as a mechanism for implementation of the fundamental correspondence principle is also discussed.

The quantum relaxation localization, a peculiar characteristic implication of pseudochaos, is reviewed in both time–dependent and conservative systems with special emphasis on the *dynamical decoherence* of quantum chaotic states. Recent results on the peculiar global structure of the energy shell, the Green function spectra and the eigenfunctions, both localized and ergodic, in a generic conservative quantum system are presented.

Examples of pseudochaos in classical systems are given including linear oscillator and waves, digital computer and completely integrable systems. A far–reaching similarity between the dynamics of a few–freedom quantum system at high energy levels ($n \rightarrow \infty$) and that of many–freedom one ($N \rightarrow \infty$) is also discussed.

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1 Introduction: the second birth of pseudochaos

The conception of *pseudochaos* has been first explicitly introduced [1] in attempts to interpret a very controversial phenomenon of *quantum chaos*, and to understand its mechanism and physical meaning. The term itself has been borrowed from the theory of the well–known 'pseudorandom number generators' in a digital computer. Even though such imitation of the 'true' random quantities is widely used in many 'numerical experiments', e.g., ones employing the Monte–Carlo techniques, this pseudorandomness was always considered as a very specific mathematical model of no general interest for the fundamental physics. However, in recent numerous attempts to understand quantum chaos, which is attracting ever growing attention of many researchers (see, e.g., Proceedings of International Conferences [2–4], and a Collection of papers [5]), it is becoming more and more clear that this 'specific mechanism' provides, in fact, a typical chaotic behavior in physical systems.

Moreover, from the viewpoint of fundamental physics, the pseudochaos is the only kind of chaos principally possible in physical systems of finite dimensions. In infinite macroscopic systems of traditional statistical mechanics (TSM), both *classical and quantal*, particularly, in the principal TSM conception of the thermodinamic limit $N \to \infty$, where N is the number of freedoms, there is no such problem. Namely, it has been rigorously proven (see, e.g., Ref.[6]) that, loosely speaking, the 'true' chaos is a generic phenomenon in this limit even if for any finite N the system is completely integrable!

The discovery of dynamical chaos in finite (and even few–dimensional) *classical* systems – a fundamental breakthrough in recent decades – has crucially changed the classical statistical mechanics. By now, this new mechanism for the statistical laws is well understood (but still not very well known), and received the firm mathematical foundations in the modern ergodic theory [6].

In all success of the latter a 'minor' problem still remains: such a mechanism does not work in finite *quantum* systems that is ones whose motion is bounded in the *phase space* and, hence, whose energy and frequency spectrum is discrete.

The simplest solution of this problem, which seems to be almost commonly accepted currently, is that the dynamical chaos in such systems is simply impossible. However, this seemingly obvious 'solution' is, in fact, a trap as it immediately leads to a sharp and very profound contradiction with the fundamental correspondence principle [7]. We need to choose what to sacrifice, this principle or the 'true' (=classical) chaos. I prefer the latter. If the phenomenon of quantum chaos did really violate the correspondence principle as some physicists suspect it were, indeed, a great discovery since it would mean that the classical mechanics is not the limiting case of quantum mechanics but a different separate theory. 'Unfortunately', there exists a less radical (but also interesting and important) resolution of this difficulty – the pseudochaos – which is the main topic of my talk.

Within such a philosophical framework the central physical problem is to under-

stand the nature and mechanism of dynamical chaos in quantum mechanics. In other words, we need the *quantum* theory of dynamical chaos including the transition to the classical limit. Certainly, the quantum chaos is a new dynamical phenomenon [7], related but not identical to the classical chaos. We call it *pseudochaos*, the term *pseudo* intending to emphasize the difference from the 'classical' chaos in the ergodic theory. From the physical point of view, I accept here, the latter, strictly speaking, does not exist in the Nature. So, in the common philosophy of the universal quantum mechanics the *pseudochaos is the only true dynamical chaos*. The classical chaos is but a limiting pattern which is, nevertheless, very important both in the theory to compare with the real (quantum) chaos and in applications as a very good approximation in macroscopic domain as is the whole classical mechanics. Ford calls it *mathematical chaos* as contrasted to the *real physical chaos* in quantum mechanics [10].

I emphasize again that the classical chaos is impossible in *finite and closed* quantum systems to which my talk is restricted. Particularly, I am not going to discuss here the quantum measurement in which, by purpose, the macroscopic (infinite–dimensional) processes are involved (see, e.g., Ref.[7]).

Thus, the physical meaning of the term 'pseudochaos' is principally different (and even opposite) to that of 'pseudorandom numbers' in computer. The reason for the latter, original, term 'pseudo' was twofold. At the beginning, the first and apparently the only meaning was related to the common belief that no dynamical, deterministic, system like computer can produce anything random, by definition. This delusion has been overcome in the theory of dynamical chaos on the field of *real* numbers. However, the digital computer works on a finite lattice of *integers*. This is qualitatively similar to a quantum behavior [11]. Computer numbers as well as quantum variables can be, at most, 'pseudorandom' only as compared to the 'true' random classical quantities represented by real numbers. But then, a very special notion of 'pseudorandom' is sharply scrambling up to the level of a new fundamental conception in physics.

The quantum chaos is a part of quantum dynamics which, in turn, is a particular class of dynamical systems. It became a hard physical problem upon discovery and understanding of the classical dynamical chaos. To explain the problem I need to briefly remind the main peculiarities of the classical chaos, especially those crucial in the quantum theory.

2   Asymptotic chaos in classical mechanics

There are two equivalent description of classical mechanics or, more generally, of any finite–dimensional dynamical systems: via individual trajectories, and via a distribution function, or phase–space density for Hamiltonian (most fundamental) systems.

The trajectory obeys the motion equations, which are generally nonlinear, and desribes a particular realization of system's dynamics in dependence on the initial conditions. The phase density satisfies the Liouville equation, which is always *linear* whatever the motion equations, and which usually represents the typical (generic) dynamical behavior for a given system. Particularly, all zero–measure sets of special trajectories are automatically excluded.

Notice, however, that in some special cases the phase density may display the properties absent for trajectories. An interesting example [57] is the correlation decay (and, hence, continuous spectrum) for a special initial phase density in a completely integrable system. The point is that such a decay is related to the correlation between *different* trajectories rather than on the same ones. The trajectory spectrum remains descrete, and the corresponding correlation persists. Particularly, this explains a surprising phenomenon known as 'echo' which is the revival of such correlations upon velocity reversal. It was observed in many cases, and has several interesting applications (see, e.g., Ref.[63]). An interesting open question is the exact conditions for a phase density to represent the *trajectory* properties which is the primary problem in dynamics.

The strongest statistical properties of a dynamical system are related to the *local* exponential instability of trajectories, as described by the *linearized* motion equations, provided the motion is *bounded* in phase space. These two conditions are *sufficient* for a rapid mixing of trajectories by the mechanism of 'stretching and folding'. For the linear motion equations the combination of both is impossible unless the whole phase space of the system is finite. A well–known example of the latter is a model described by the *linear* 'Arnold cat map'[8]:

$$\begin{aligned} \overline{p} &= p + x \quad mod \ 1 \\ \overline{x} &= x + \overline{p} \quad mod \ 1 \end{aligned} \tag{1}$$

on a unit torus. The motion is exponentially unstable with (positive) Lyapunov's exponent $\Lambda = \ln\left[(3+\sqrt{5})/2\right] > 0$, and is bounded by operation $mod \ 1$. Notice that the linearized motion is described by the same map but *without mod* 1 that is in *infinite* plane $(-\infty < dp, dx < \infty)$. It is unbounded and *globally* unstable but perfectly regular, the so–called hyperbolic motion:

$$dp = a \cdot \exp\left(\Lambda t\right) + b \cdot \exp\left(-\Lambda t\right); \qquad dx = c \cdot \exp\left(\Lambda t\right) + d \cdot \exp\left(-\Lambda t\right) \tag{2}$$

where constants a, b, c, d depend on the initial conditions and on Λ, and where integer t is discrete map's time. Remarkably, the motion (2) is time–reversible but *unstable* in both senses $(t \to \pm\infty)$. This implies the time reversibility of all the statistical properties for the main system (1), a surprising conclusion which is still confusing some researchers (see, e.g., Ref.[61]).

A nontrivial part of the relation between instability and chaos is in that the instability must be *exponential*. A power–law instability is insufficient for chaos.

For example, if we change first Eq.(1) to $\overline{p} = p$ the model becomes completely integrable with the oscillation frequency depending on the motion integral p (non-linear oscillation). The latter produces *linear* (in time) instability but the motion remains regular (with discrete spectrum). This is a typical property of the completely integrable nonlinear oscillations [58] which leads to a confusing difference in the dynamical behavior of trajectories and of phase densities as mentioned above. Another open question is how to choose the correct time variable for a particular dynamical problem [7]. A change of time may convert the exponential instability into a power–law one, and vice versa (see, e.g., Ref.[59] for discussion).

The two above conditions for dynamical chaos can be realized in very simple (particularly, few–dimensional) systems like model (1). Another simple example, to which I will refer below, is the so–called 'kicked rotator' described by the 'standard map' [1,7,9,11]:

$$\overline{p} = p + k \cdot \sin x; \qquad \overline{x} = x + T \cdot \overline{p} \qquad (3)$$

also on a torus $(x, p \bmod 2\pi)$ or on a cylinder $(x \bmod 2\pi, \ -\infty < p < \infty)$. This model is well studied too, and has many physical applications. The motion on cylinder is bounded in one variable only, yet it is sufficient for chaos.

The exponential instability implies continuous spectrum of the motion which is equivalent, loosely speaking, to the mixing, or temporal *correlation decay*. Apparently, this is the most important characteristic property in the statistical mechanics underlying the principal and universal statistical phenomenon of *relaxation* to some steady state, or statistical equilibrium.

Aperiodic relaxation is especially clear in the Liouville picture for the phase density behavior (see, e.g., Ref.[8]). Consider a certain basis for Liouville's equation, for example

$$\varphi_{mn} = \exp\left[2\pi\, i(mx + np)\right] \qquad (4)$$

where m, n are any integers in a simple example of model (1). In other words, we represent the phase density as a Fourier series:

$$f(x, p, t) = \sum_{m,n} F_{mn}(t)\, \varphi_{mn}(x, p) = \sum_{m,n} F_{mn}(0)\, \exp\left[2\pi\, i(m(t)x + n(t)p)\right] \qquad (5)$$

Except φ_{00} any other term of this series has zero total probability, and characterizes the spatial correlation in the phase density. Map (1) induces a map for the Fourier amplitudes, and for harmonic numbers:

$$\overline{F}_{mn} = F_{\overline{m}\,\overline{n}}; \qquad \begin{array}{l} \overline{n} = n + m \\ \overline{m} = m + \overline{n} \end{array} \qquad (6)$$

Remarkably, variables $m(t)$, $n(t)$ obey the same map as for the *linearized* motion equations in variables dx, dp and with the same instability rate Λ on the *infinite* lattice (m, n). The dynamics of the phase density in the Fourier representation, described by the same Eq.(2) (upon substitution of m, n for dx, dp), is also unbounded,

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globally unstable, and regular. This is of no surprise as both representations describe the *local* structure of motion. Dynamical chaos is a *global* phenomenon determined, nevertheless, by the microdetails of the initial conditions due to the exponential instability of motion [7,13]. Accordingly, in the original phase space the temporal density fluctuations are chaotic as are almost all trajectories of map (1).

The only *stationary* mode $m = n = 0$ with the full probability represents in this picture the statistical steady state while all the others describe *nonstationary* fluctuations. The latter are another characteristic property of statistical behavior. These can be separated from the average statistical relaxation by the so–called coarse–graining, or spatial averaging, which is a projection of the phase density on a finite (and arbitrarily fine) partition of the phase space. The kinetic (particularly, diffusive) description of the statistical relaxation is restricted to such a coarse–grained projection only while the fluctuations work as a dynamical *generator of noise*.

Another elegant method of separating out the average relaxation is the suppression of the fluctuations using Prigogine's Λ operator [61] which provides an invertible smoothing of the exact phase density [60]. True, the inverse operator in a nonproper one, yet this method could be efficiently used in some theoretical constructions. Contrary to a common belief, it has nothing to do with the time irreversibility [61,62]. Moreover, unlike the coarse–grained projection the Λ–smoothed phase density is as reversible as the exact one (principally but not practically, of course). The origin of misunderstanding concerning 'irreversibility' is apparently related to the necessary restriction on the initial smoothed density which was missed in the theory [62]. Such a density is a technical, rather than natural, property of the system, and hence it does not need to be arbitrary. A similar operation is often used in quantum mechanics (for different purposes) to convert the Wigner function (the counterpart of exact classical phase density) into the so–called Husimi distribution which is the expansion in the coherent states (see, e.g., Ref.[7]).

Nonstationary fluctuations/correlations of the phase density form a *stationary* flow into higher modes $|m|, |n| \to \infty$ (cf. Ref.[12]), and keep the memory of the exact initial conditions (see first Eq.(6)) providing time reversibility for the exact density. The stationary correlation flow is only possible for the *continuous* phase space which is a characteristic feature of classical mechanics. This allows for the asymptotic ($t \to \pm\infty$) formulation of the ergodic theory. Notice that both trajectories and the full density are *time–reversible* but the latter, unlike the former, is *nonrecurrent*. Reversed relaxation, particularly 'antidiffusion', describes the growth of a big fluctuation which is eventually (as $t \to -\infty$) followed by the standard relaxation in the opposite direction of time [13].

3 Quantum pseudochaos: a new dimension in the ergodic theory

The dynamical chaos is one limiting case of the modern general theory of dynamical systems which describes the statistical properties of deterministic motion (see, e.g., Ref.[6]). No doubt, this theory has been developed on the basis of classical mechanics. Yet, as a general mathematical theory, it does not need to be restricted to classical mechanics only. Particularly, it can be, and indeed was applied to quantum dynamics with a surprising result. Namely, as had been found from the beginning[14] and was subsequently well confirmed (see, e.g., Ref.[1,7,11,22,28]) the quantum mechanics does not typically permit the 'true' (classical–like) chaos. This is because in quantum mechanics the energy (and frequency) spectrum of any system, whose motion is bounded in phase space, is discrete and its motion is almost periodic. Hence, according to the existing ergodic theory, such a quantum dynamics belongs to the limiting case of regular motion which is opposite to dynamical chaos. The ultimate origin of quantum almost–periodicity is in the discreteness of the phase space itself (or, in a more formal language, in the noncommutative geometry of the latter) which is at the basis of quantum physics and directly related to the fundamental uncertainty principle. Yet, another fundamental principle, the correspondence principle, requires the transition to classical mechanics in all cases including the dynamical chaos with all its peculiar properties.

Now, the principal question to be answered reads: where is the expected quantum chaos in the ergodic theory? The answer to this question [7,11,13] (not commonly accepted as yet) was concluded from a simple observation (principally well–known but never comprehended enough) that the sharp border between the discrete and continuous spectrum is physically meaningful in the limit $|t| \to \infty$ only, the condition actually assumed in the ergodic theory. Hence, to understand the quantum chaos the existing ergodic theory needs some modification by introducing a new 'dimension', the time. In other words, a new and central problem in the ergodic theory becomes the *finite–time statistical properties* of a dynamical system, both quantal as well as classical.

Within a finite time the discrete spectrum is dynamically equivalent to the continuous one, thus providing much stronger statistical properties of the motion than it was (and still is) expected in the ergodic theory in case of discrete spectrum. It turns out that the motion with discrete spectrum may exhibit *all* the statistical properties of the classical chaos but only on some *finite* time scales.

The absence of the classical–like chaos in quantum dynamics apparently contradicts not only the correspondence principle but also the fundamental statistical nature of quantum mechanics. However, even though the random element in quantum mechanics ('quantum jumps') is unavoidable, indeed, it can be singled out and separated from the proper quantum processes. Namely, the fundamental random-

ness in quantum mechanics is related only to a very specific event – the *quantum measurement* – which, in a sense, is foreign to the proper quantum system itself. This allows to divide the whole problem of quantum dynamics in two qualitatively different parts:

- The proper quantum dynamics as described by a very specific dynamical variable, the wavefunction $\psi(t)$ obeying some deterministic equation, for example, the Schrödinger equation. The discussion below will be limited to this part only.

- The quantum measurement including the registration of the result and, hence, the collapse of the ψ function which still remains a very vague issue to the extent that there is no common agreement even on the question whether this is a real physical problem or an ill–posed one, so that the Copenhagen interpretation of quantum mechanics answers all the 'admissible' questions. In any event, there exists as yet no dynamical description of the quantum measurement including the ψ collapse.

Recent breakthrough in the understanding of quantum chaos has been achieved, particularly, due to the above philosophy of separating out the dynamical part of quantum mechanics. Such a philosophy is accepted, explicitly or more often implicitly, by most researchers in this field.

3.1   Time scales of pseudochaos

The existing ergodic theory is asymptotic in time, and thus has no explicit time scales at all [2]. There are two reasons for this. One is technical: it is much simpler to derive the asymptotic relations than to obtain rigorous finite–time estimates. Another reason is more profound. All statements in the ergodic theory hold true up to measure zero that is excluding some peculiar nongeneric sets of zero measure. Even this minimal imperfection of the theory did not seem completely satisfactory but has been 'swallowed' eventually and is now commonly tolerated even among mathematicians, let alone physicists. In a finite–time theory all these exceptions acquire a *small but finite* measure which would be apparently 'unbearable' (for mathematicians). Yet, there is a standard mathematical 'trick' for avoiding both these difficulties.

The most important time scale t_R in quantum chaos is given by the general estimate [7,11]:

$$\ln\left(\omega t_R\right) \sim \ln Q\,, \qquad t_R \sim \frac{Q^{\alpha}}{\omega} \sim \rho_0 \leq \rho_H \tag{7}$$

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[2]Asymptotic statements in the ergodic theory should not be always understood literally to avoid physical misconceptions (see, e.g., Addendum in second Ref.[12]). Actually, the classical chaos has also time scales, for example, a dynamical one ($\sim \Lambda^{-1}$) [7].

where ω and $\alpha \sim 1$ are system–dependent parameters, and $Q \gg 1$ stands for some big (in semiclassical region) quantum parameter. It may be, e.g., a quantum number $Q = I/\hbar$ related to a characteristic action variable I or the total number of states for the bounded quantum motion in a phase space domain of volume Γ: $Q \approx \Gamma/(2\pi)^N$.

Here and below I set $\hbar = 1$.

This scale is called the *relaxation time scale* referring to one of the principal properties of the chaos – *statistical relaxation* to some steady state. The physical meaning of this scale is principally simple, and is directly related to the fundamental uncertainty principle ($\Delta t \cdot \Delta E \sim 1$) as implemented in the second Eq.(7) where ρ_H is the *full* average energy level density (also called Heisenberg time). For $t \lesssim t_R$ the discrete spectrum is not resolved, and the statistical relaxation follows the classical (limiting) behavior. This is just the 'gap' in the ergodic theory (supplemented with the additional dimension, the time) where the pseudochaos, particularly quantum chaos, dwells. A more accurate estimate relates t_R to a *part* ρ_0 of the level density. This is the density of the so–called *operative eigenstates* only, that is those which are actually present in a particular quantum state ψ, and which do actually control its dynamics.

The formal trick mentioned above is to consider not finite–time relations, we really need in physics, but rather the special *conditional limit*:

$$t, Q \rightarrow \infty, \qquad \tau_R = \frac{t}{t_R(Q)} = const \tag{8}$$

where τ_R is a new dimensionless time. The *double* limit (8) (unlike the single one $Q \rightarrow \infty$) is *not* the classical mechanics which holds true, in this representation, for $\tau_R \lesssim 1$ and with respect to the statistical relaxation only. For $\tau_R \gtrsim 1$ the behavior becomes essentially quantum (even in the limit $Q \rightarrow \infty$!) and is called nowadays *mesoscopics*. Particularly, the quantum steady state is generally quite different from the classical statistical equilibrium in that the former may be *localized* (under certain conditions) that is *nonergodic* in spite of classical ergodicity.

Another important difference is in *fluctuations* which are also a characteristic property of chaotic behavior. In comparison with classical mechanics the quantum $\psi(t)$ plays, in this respect, an intermediate role between the classical trajectory with big relative fluctuations ~ 1 and the coarse–grained classical phase density with no fluctuations at all. Unlike both the fluctuations of $\psi(t)$, or rather those of averages in a quantum state $\psi(t)$, are typically $\sim d_H^{-1/2}$ where d_H is the number of operative eigenstates associated with quantum state ψ which the former may be also called the *Hilbert dimension* of state ψ. In other words, chaotic $\psi(t)$ represents statistically a *finite ensemble* of $\sim d_H$ independent systems even though formally $\psi(t)$ describes a *single* system. The fluctuations clearly demonstrate the difference between physical time t and auxiliary variable τ: in the double limit ($t, Q \rightarrow \infty$) the fluctuations vanish, and one needs a new 'trick' to recover them for a finite Q.

The relaxation time scale should not be confused with the *Poincare recurrence*

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time $t_P \gg t_R$ which is typically much longer, and which sharply increases with decreasing the recurrence domain. Time scale t_P characterizes big fluctuations (for both the classical trajectory, but not the phase density, and for the quantum ψ) of which recurrences is a particular case. Unlike this, t_R characterizes the average relaxation process. Rare recurrences, the more rare the larger quantum parameter Q, make quantum relaxation similar to the classical nonrecurrent one.

More strong statistical properties than relaxation and fluctuations are related in the ergodic theory to the exponential instability of motion. The importance of those stronger properties for the statistical mechanics is not completely clear [56]. Nevertheless, in accordance with the correspondence principle, those stronger properties are also present in quantum chaos as well but on a *much shorter* time scale t_r:

$$\Lambda t_r \sim \ln Q \tag{9}$$

where Λ is the classical Lyapunov exponent. This time scale was discovered and partly explained in Ref.[15] (see also Ref.[7,11]). We call it *random time scale*. Indeed, according to the Ehrenfest theorem the motion of a narrow wave packet follows the beam of classical trajectories as long as the packet remains narrow, and hence it is as random as in the classical limit. Even though the random time scale is very short, it grows indefinitely as $Q \to \infty$. Thus, a temporary, finite–time quantum pseudochaos turns into the classical dynamical chaos in accordance with the correspondence principle.

Again, we may consider the *conditional limit*:

$$t, Q \to \infty, \qquad \tau_r = \frac{t}{t_r(Q)} = const \tag{10}$$

Notice that new scaled time τ_r is different from the previous one τ_R in Eq.(8).

Particularly, if we fix time t, then in the limit $Q \to \infty$ we obtain the transition to the classical instability in accordance with the correspondence principle while for Q fixed, and $t \to \infty$ we have the proper quantum evolution in time. For example, the quantum Lyapunov exponent

$$\Lambda_q(\tau_r) \to \begin{cases} \Lambda, & \tau_r \ll 1 \\ 0, & \tau_r \gg 1 \end{cases} \tag{11}$$

The quantum instability ($\Lambda_q > 0$) was observed in numerical experiments [7,16]. What does terminate the instability for $t \gtrsim t_r$? A simple explanation is suggested by the classical picture of the phase density evolution on the integer Fourier lattice m, n discussed above for model (1). Classical Fourier harmonics m, n are of a kinematical nature without any a priori dynamical restriction. Particularly, they can go, and do so for a chaotic motion, arbitrarily large which corresponds to a continuous classical phase space. On the contrary, the quantum phase space is discrete. At first glance, the quantum wave packet stretching/squeezing, similar to the classical one, does

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not seem to be principally restricted since only 2–dimensional area (per freedom) is bounded in quantum mechanics. However, Fourier harmonics of the quantum phase density (Wigner function) are directly related to the quantum dynamical variables, particularly, to the action variables whose values are restricted by the quantum parameter Q, hence estimate (9). In a simple model (1) this is related to a finite size of the whole phase space. Generally, in a conservative system with even infinite phase space the restriction is imposed by the energy conservation. Numerical experiments reveal that the original wave packet, after a considerable stretching similar to the classical one, is rapidly destroyed. Namely, it gets split into many new small packets [7,16]. The mechanism of this sharp 'disrupture' of the classical–like motion is not completely clear (for a possible explanation see Ref.[7,17]). The resulting picture is qualitatively similar to that for the classical phase density, the main difference being in the spatial fluctuation scale bounded now from below by $1/Q$. Nevertheless, the quantum phase density can be also decomposed into a coarse–grained average part, and the fluctuations. An important implication of this picture for the wave packet time evolution is the rapid and complete destruction of the so–called generalized coherent states [18] in quantum chaos.

In quasiclassical region ($Q \gg 1$) the scale $t_r \ll t_R$. This leads to a surprising conclusion that the quantum diffusion and relaxation are *dynamically stable* contrary to the classical behavior. It suggests, in turn, that the motion instability is, generally, not important *during* statistical relaxation. However, the *foregoing* correlation decay on the short time scale t_r is crucial for the statistical properties of quantum dynamics.

Dynamical stability of quantum diffusion has been proved in striking numerical experiments with time reversal [19]. In a classical chaotic system the diffusion is immediately recovered due to numerical 'errors' (not random !) amplified by the local instability. On the contrary, the quantum 'antidiffusion' proceeds until the system passes, to a very high accuracy, the initial state, and only then the normal diffusion is restored. The stability of quantun chaos on relaxation time scale is comprehensible as the random time scale is much shorter. Yet, the accuracy of the reversal (up to $\sim 10^{-15}$ (!)) is surprising. Apparently, this is explained by a relatively large size of the quantum wave packet as compared to the unavoidable rounding-off errors unlike the classical computer trajectory which is just of that size [20]. In the standard map (3) (upon quantization) the size of the optimal, least-spreading, wave packet $\Delta x \sim \sqrt{T}$ [11]. On the other hand, any quantity in the computer must well exceed the rounding–off error $\delta \ll 1$. Particularly, $T \gg \delta$, and $(\Delta x)^2/\delta^2 \gtrsim (T/\delta)\delta^{-1} \gg 1$.

3.2 Classical–like relaxation and residual fluctuations

The relaxation time scale t_R is the most important of the two considered above for two reasons. First, it is much longer than t_r, and second, it is related to the principal

process of statistical relaxation which is the basis of statistical mechanics. The short scale t_r was interpreted in Ref.[15] (see also Ref.[26]) as a limit for the classical–like behavior of chaotic quantum motion. Subsequently, it was found that the *method* of quasiclassical quantization can be extended on a much longer time [11,27]. However, the *physics* on both time scales is qualitatively different: dynamical instability on scale t_r, and statistical relaxation afterwards.

On the whole scale t_R the discrete pseudochaos spectrum is not resolved, and relaxation follows the classical law. Consider, for example, model (3), the standard map on a torus with total number of quantum states Q, and p, x as the action–angle variables.

If perturbation parameter $k \gtrsim Q$ the relaxation to ergodic steady state in this model, as well as in model (1), is very quick, with characteristic relaxation time $t_e \sim 1$ (iterations). Such regime does often take place in many physical systems. Here I consider another, more interesting for the problem of pseudochaos, case, namely the *diffusive relaxation* which occurs for a sufficiently weak perturbation

$$k \ll Q \tag{12}$$

In the classical limit this relaxation is descibed by the standard diffusion equation

$$\frac{\partial f(p, t)}{\partial t} = \frac{1}{2} \frac{\partial}{\partial p} D(p) \frac{\partial f(p, t)}{\partial p} \tag{13}$$

where $f(p, t) = < f(p, x, t) >_x$ is a coarse–grained phase density (averaged over x), and

$$D = \frac{< (\Delta p)^2 >}{t} \approx \frac{k^2}{2} \tag{14}$$

the diffusion rate. The latter expression holds for the standard map if $K = kT \gg 1$ which is also the condition for the global chaos in this model [9]. The relaxation to the ergodic steady state $f_s = 1/Q$ is exponential with characteristic time

$$t_e = \frac{Q^2}{2\pi^2 D} \approx \frac{Q^2}{\pi^2 k^2} \tag{15}$$

In diffusive regime ($k \ll Q$) this time $t_e \gg 1$. That average relaxation is stable and regular in spite of underlying chaotic dynamics.

The quantized standard map $\overline{\psi} = \hat{U}\psi$ is described by a unitary operator

$$\hat{U} = \exp\left(-i\frac{T\hat{p}^2}{2}\right) \cdot \exp\left(-ik \cdot \cos\hat{x}\right) \tag{16}$$

on a cylinder ($Q \to \infty$) [14], where $\hat{p} = -i\,\partial/\partial x$, and by a similar but somewhat more complicated expression on a torus [21].

There are three quantum parameters in this model: perturbation k, period T and size Q but only two classical combinations remain: perturbation $K = k \cdot T$, and

classical size $M = TQ/2\pi$ which is the number of classical resonances over the torus. Notice that the quantum dynamics is generally more rich than the classical one as the former depends on an extra parameter. It is, of course, another representation of Planck's constant which I have set $\hbar = 1$. This is why in the quantized standard map we need both parameters, k and T, separately and cannot combine them in a single classical parameter K.

The quasiclassical region, where we expect quantum chaos, corresponds to $T \rightarrow 0$, $k \rightarrow \infty$, $Q \rightarrow \infty$ while the classical parameters $K = const$ and $M = const$ are fixed.

A technical difficulty in evaluating t_R for a particular dynamical problem is in that the density ρ_0 depends, in turn, on the dynamics. So, we have to solve a self–consistent problem. For the standard map the answer is known (see Ref.[7]):

$$t_R = \rho_0 = 2D \qquad (17)$$

This is a remarkable relation as it connects essentially *quantum* characteristics $(t_R,\ \rho_0)$ with the *classical* diffusion rate D (14).

The quantum diffusion rate depends on the scaled (dimensionless) time τ_R (8), and is given by

$$D_q = \frac{D}{1 + \tau_R} \rightarrow \begin{cases} D, & \tau_R = t/t_R \ll 1 \\ 0, & \tau_R \gg 1 \end{cases} \qquad (18)$$

This is an example of scaling in discrete spectrum which eventually stops the quantum diffusion.

The character of the steady state crucially depends on the ratio t_R/t_e. Define the *ergodicity parameter* λ as [7]

$$\lambda = \frac{D}{Q} \sim \left(\frac{t_R}{t_e}\right)^{1/2} \sim \frac{k^2}{Q} \sim \frac{K}{M} \cdot k \qquad (19)$$

Consider, first, the case $\lambda \gg 1$ when the time scale t_R is long enough to allow for the completion of the classical–like relaxation. In this case the final steady state as well as all the eigenfunctions are ergodic that is the corresponding Wigner functions are close to the classical microcanonical distribution in phase space. This region is inevitably reached if the classical parameter K/M is kept fixed while the quantum parameter $k \rightarrow \infty$ in agreement with the Shnirelman theorem or, better to say, with a physical generalization of that [23]. It is called the *far quasiclassical asymptotics*.

The principal difference of the quantum ergodic state from the classical one is *residual fluctuations* in the former. In quasiclassical region the chaotic quantum steady state is a superposition of very many eigenfunctions. As a result almost any physical quantity fluctuates in time. Even in discrete spectrum we are considering here these fluctuations are very irregular. In case of a classical–like ergodic steady state all Q eigenfunctions essentially contribute to the fluctuations. Moreover, we

would expect their contributions to be statistically almost independent. Hence, the fluctuations should scale $\sim Q^{-1/2} = d_H^{-1/2}$ where d_H is the Hilbert dimension of the ergodic state. This is the case, indeed, according to numerical experiments [29]. For example, the energy fluctuations were found to follow a simple relation

$$\frac{\Delta E_s}{E_s} \approx \frac{1}{\sqrt{Q}} \qquad (20)$$

where

$$(\Delta E_s)^2 = \overline{E^2(t)} - E_s^2; \qquad E_s = \overline{E(t)}; \qquad E(t) = \frac{<p^2>}{2} \qquad (21)$$

Here the bar indicates time averaging over a sufficiently long time interval $(\gg t_e)$, and the brackets denote usual average over the quantum state.

Dependence (20) suggests the complete quantum decoherence in the final steady state for any initial state even though the steady state is formally a pure quantum state. For $Q \gg 1$ the fluctuations are small, so that *statistically* the quantum relaxation is *nonrecurrent*. The decoherence of a chaotic quantum state is also confirmed by the independence (up to small fluctuations) of the final steady–state energy E_s on the initial $E(0)$. Since any particular initial quantum state is strongly coherent the decoherence is a result of the quantum chaos. It is called *dynamical decoherence*. This is one of the most important results of the studies in quantum chaos.

3.3   Mesoscopics: quantum behavior in quasiclassical region

If ergodicity parameter $\lambda \ll 1$ is small all the eigenstates and the steady state are non–ergodic, or localized. This is because the scale t_R is not long enough to support the classical–like diffusion which stops before the classical relaxation is completed. For this reason it is also called the *quantum diffusion localization*. As a result the structure of eigenfunctions and of the steady state remains essentially quantum, no matter how large is the quantum parameter $k \to \infty$. This is called *intermediate quasiclassical asymptotics* or *mesoscopic* domain. Particularly, it corresponds to $K > 1$ fixed, $k \to \infty$ and $M \to \infty$ while $\lambda \ll 1$ remains small.

The popular term 'mesoscopic' means here some intermediate behavior between classical and quantum one. In other words, in mesoscopic phenomenona both classical and quantum features are combined simulteneously. Again, the correspondence principle requires transition to the completely classical behavior. This is, indeed, the case as the mesoscopic phenomena occur in the region where quantum parameter $k \gg 1$ is already very big but still less than a certain critical value (corresponding to $\lambda \sim 1$) which determines the border of transition to the fully classical behavior (far quasiclassical asymptotics).

If $\lambda \ll 1$ is very small the shape of the localized eigenstates is asymptotically

exponential [7], and can be approximately described by a simple expression [41]:

$$f_m(p) \; = \; < |\psi_m(p)|^2 > \approx \; \frac{2/\pi l}{\cosh\left[2\left(p - p_m\right)/l\right]} \tag{22}$$

The localized steady state has a similar but somewhat more complicated shape [11,31]. This is a simple approximation superimposed with big fluctuations. The parameter l is called *localization length*. Interestingly, the two localization lengths (l_s for steady state and l for eigenfunctions) are rather different [11]:

$$l_s \; \approx \; D \quad \text{while} \quad l \approx \; \frac{D}{2} \tag{23}$$

because of big fluctuations.

In terms of localization length the region of mesoscopic phenomena is defined by the double inequality:

$$1 \ll l \ll Q \tag{24}$$

The left inequality is a classical feature of the state while the right one refers to quantum effects. The combination of both allows, particularly, for a classical description, at least in the standard map, of statistical relaxation to the quantum steady state by a phenomenological diffusion equation [7,24] for the Green function:

$$\frac{\partial g(\nu, \sigma)}{\partial \sigma} \; = \; \frac{1}{4} \frac{\partial^2 g}{\partial \nu^2} \; + \; B(\nu) \frac{\partial g}{\partial \nu} \tag{25}$$

Here $g(\nu, 0) = |\psi(\nu,0)|^2 = \delta(\nu - \nu_0)$ and

$$\nu \; = \; \frac{p}{2D}, \quad \sigma \; = \; \ln\left(1 + \tau_R\right), \quad \tau_R \; = \; \frac{t}{2D} \tag{26}$$

The additional drift term in the diffusion equation with

$$B(\nu) \; \approx \; \text{sign}(\nu - \nu_0) \; = \; \pm 1 \tag{27}$$

describes the so–called quantum coherent backscattering which is the dynamical mechanism of localization.

The solution of Eq.(25) reads [7]:

$$g(\nu, \sigma) \; = \; \frac{1}{\sqrt{\pi\sigma}} \exp\left[-\frac{(\Delta + \sigma)^2}{\sigma}\right] \; + \; \exp\left(-4\Delta\right) \cdot \text{erfc}\left(\frac{\Delta - \sigma}{\sqrt{\sigma}}\right) \tag{28}$$

where $\Delta = |\nu - \nu_0|$.

Asymptotically, as $\sigma \to \infty$, the Green function $g(\nu, \sigma) \to 2\exp\left(-4\Delta\right) \equiv g_s$ approaches the localized steady state g_s, exponentially in σ but only as a power–law in physical time τ_R or t ($g - g_s \sim 1/\tau_R$). This is the effect of discrete motion spectrum. Numerical experiments confirm prediction (28), at least, to the logarithmic accuracy $\sim \sigma \approx \ln \tau_R$ [7,22].

A physical example of localization is the quantum suppression of diffusive photo-effect in Hydrogen atom [25]. Depending on parameters the suppression may occur no matter how large the atomic quantum numbers. This is a typical mesoscopic phenomenon which had been predicted by the theory of quantum chaos, and was subsequently observed in laboratory experiments.

One might expect that in case of localization ($D \ll Q$) the fluctuations would scale like $l^{-1/2} \sim k^{-1}$ as the number of eigenfunctions coupled in the localized steady state is $\sim l$. This is, however, *not* the case as was found already in first numerical experiments (see Ref.[7]). According to more accurate data [29] the fluctuations are described by the relation

$$\frac{\Delta E_s}{E_s} = \frac{A}{k^\gamma} = \frac{a}{d_H^{\gamma/2}} \tag{29}$$

with fitting parameters $\gamma = 0.55$, $a = 0.65$. For a nonergodic state the Hilbert dimension can be defined as (see Ref.[7])

$$d_H^{-1} = \frac{1}{3} \cdot \int |\psi(p)|^4 \, dp = \int f^2(p) \, dp \tag{30}$$

where $f(p)$ is smoothed (coarse–grained) density, and the factor $1/3$ accouts for ψ fluctuations [30]. In case of exponential localization (22) $d_H \approx \pi^2 \, l/4$. The most important parameter γ here is about twice as small compared to the expected value $\gamma = 1$. This result suggests some fractal properties of localized eigenfunctions and/or of their spectra. To put it another way, a slow fluctuation decay (29) implies *incomplete* quantum decoherence which can be characterized by the number d_s of statistically independent components in the steady state [7]. Then, from Eqs.(20) and (29) we obtain in the two limits:

$$\frac{d_s}{d_H} \approx \begin{cases} 1, & \lambda \gg 1 \\ \frac{d_H^{\gamma-1}}{a^2} = 2.4 \, d_H^{-0.45}, & \lambda \ll 1 \end{cases} \tag{31}$$

This result was confirmed in Ref.[31] for a band random matrix model.

The phenomenon of quantum diffusion localization explains also the limitation of quantum instability in systems with infinite phase space like the standard map on a cylinder. Indeed, the maximal number of coupled states here is determined by the localization length whatever the total number of states in the system. Hence, we should substitute quantum parameter $Q \sim l \sim k^2$ in estimate (9). Even if localization does not take place (e.g., for standard map with parameter $k(p)$ depending on p, see Ref.[7,11]), so that the quantum diffusion doesn't stop at all and the quantum spectrum becomes continuous, the number of coupled states increases with time as a power law only ($\Delta p \sim \sqrt{t}$), and hence the quantum Lyapunov exponent $\Lambda_q \to 0$ vanishes on the relaxation time scale. Only if the action variables grow exponentially the instability rate Λ_q remains finite, and the quantum chaos becomes asymptotical like in the classical limit (see Ref.[11,32] for such 'exotic' models).

3.4 Examples of pseudochaos in classical mechanics

The pseudochaos is a new generic dynamical phenomenon missed in the ergodic theory. No doubt, the most important particular case of pseudochaos is the quantum chaos. Nevertheless, pseudochaos occurs in classical mechanics as well. Here are a few examples of classical pseudochaos which may help to understand the physical nature of quantum chaos. Besides, it unveils new features of classical dynamics as well.

Linear waves is the most close to quantum mechanics example of pseudochaos (see, e.g., Ref.[33]). I remind that here only a part of quantum dynamics is discussed, one described, e.g., by the Schrödinger equation which is a linear wave equation. For this reason the quantum chaos is called sometime wave chaos [34]. Classical electromagnetic waves are used in laboratory experiments as a physical model for quantum chaos [35]. The 'classical' limit corresponds here to the geometrical optics, and the 'quantum' parameter $Q = L/\lambda$ is the ratio of a characteristic size L of the system to wave length λ. As is well known in optics, no matter how large is the ratio Ł/λ the diffraction patterns prevail at a sufficiently far distance $R \gtrsim L^2/\lambda$. This is a sort of relaxation scale: $R/\lambda \sim Q^2$.

Linear oscillator (many–dimensional) is also a particular representation of waves. A broad class of quantum systems can be reduced to this model [36]. Statistical properties of linear oscillator, particularly in the thermodynamic limit ($N \to \infty$), were studied in Ref.[37] in the framework of TSM. On the other hand, the theory of quantum chaos suggests more rich behavior for a big but finite N, particularly, the characteristic time scales for the harmonic oscillations [38], the number of freedoms N playing a role of the 'quantum' parameter.

Completely integrable nonlinear systems also reveal pseudochaotic behavior. An example of statistical relaxation in the Toda lattice had been presented in Ref.[39] much before the problem of quantum chaos arose. Moreover, the strongest statistical properties in the limit $N \to \infty$, including one equivalent to the exponential instability (the so–called K–property) were rigorously proved just for the systems completely integrable for any finite N (see Ref.[6]).

Digital computer is a very specific classical dynamical system whose properties are extremely important in view of the ever growing interest to numerical experiments covering now all branches of science and beyond. The computer is the 'overquantized' system in that *any* quantity here is *discrete* while in quantum mechanics only the product of two conjugated variables does so. 'Quantum' parameter here $Q = M$ which is the largest computer integer, and the short time scale (9) $t_r \sim \ln M$, the number of digits in the computer word [11]. Owing to the discreteness, any dynamical trajectory in computer becomes eventually periodic, the effect well known in the theory and practice of pseudorandom number generators. One should take all necessary precautions to exclude such computer artifacts in numerical experiments (see, e.g., Ref.[40,64]). On the mathematical part, the periodic

approximations in dynamical systems are also considered in the ergodic theory, apparently without any relation to pseudochaos in quantum mechanics or computer [6].

The computer pseudochaos seems to me most convincing argument for the researchers who are still reluctant to accept the quantum chaos as, at least, a kind of chaos, insisting that only the classical–like (asymptotic) chaos deserves this name, the same chaos which was (and is) studied to a large extent just on computer, that is the chaos inferred from a pseudochaos!

4 Statistical theory of pseudochaos: random matrices

The complete solution of the dynamical quantum problem can be obtained via diagonalization of the Hamiltonian to find the energy (or quasi–energy) eigenvalues and eigenfunctions. The evolution of any quantity is, then, expressed as a sum over the eigenfunctions. For example, the energy time dependence is

$$E(t) = \sum_{mm'} c_m \, c_{m'}^* \, E_{mm'} \, \exp\left[i\left(E_m \, - \, E_{m'}\right)t\right] \tag{32}$$

where $E_{mm'}$ are the matrix elements, and the initial state in momentum representation is $\psi(n, \, 0) = \sum_m c_m \, \varphi_m(n)$. For chaotic motion the dependence is generally very complicated but the statistical properties of the motion can be inferred from the statistics of eigenfunctions $\varphi_m(n)$ (and hence of the matrix elements $E_{mm'}$), and of eigenvalues E_m.

By now, there exists a well–developed random matrix theory (RMT, see, e.g., Ref.[43]) which describes the average properties of a typical quantum system with a given symmetry of the Hamiltonian. At the beginning the object of this theory was assumed to be a very complicated, particularly many–dimensional, quantum system as a representative of a certain statistical ensemble. With understanding the phenomenon of dynamical chaos it became clear that the number of system's freedoms is irrelevant. Instead, the number of quantum states (quantum parameter Q) is of importance provided the dynamical chaos in the classical limit.

This approach to the theory of complex quantum systems like atomic nuclei had been taken by Wigner [42] 40 years ago, much before the problem of quantum chaos was realized. He introduced the so–called band random matrices (BRM) which were most suitable to account for the structure of conservative systems. However, due to severe mathematical difficulties, RMT immediately turned to a much simpler case of statistically homogeneous (full) matrices, for which impressive theoretical results have been achieved [43]. The price was that the full matrices describe the local chaotic structure only, the limitation especially inacceptable in atoms [30,44]. Only

recently the interest of some researchers turned back to the original Wigner BRM [45,48].

One of the main results in the studies of quantum chaos was the discovery of quantum diffusion localization as a mesoscopic quasiclassical phenomenon. This phenomenon, discussed above, has been well studied and confirmed by many researchers for the dynamical models described by maps. Contrary to a common belief the maps describe not only time–dependent systems but also conservative ones in the form of Poincare maps (see, e.g., Ref.[49]). Nevertheless, to my knowledge no direct studies of the quantum localization in conservative systems have been undertaken as yet, either in laboratory or even in numerical experiments. Moreover, the very existence of quantum localization in conservative systems is challenged [50]. Here, I briefly describe recent results concerning the structure of the localized quantum chaos in the momentum space of a generic few–freedom conservative system which is classically strongly chaotic, particularly, ergodic on a compact energy surface [41].

Generally, RMT is a statistical theory of the systems with discrete energy spectrum. This is just the principal property of the quantum dynamical chaos [7]. Thus, RMT turned out to become accidentally (!) a statistical theory for the coming quantum chaos. Remarkably, this statistical theory does not include any time–dependent noise that is any coupling to a thermal bath, the standard element of most statistical theories. Moreover, a *single matrix* from a given statistical ensemble represents the typical (generic) *dynamical* system of a given class characterized by the matrix parameters. This makes an important bridge between dynamical and statistical description of the quantum chaos. In matrix representation the similarity between the problem of quantum diffusion localization in momentum space and the well–known dual problem of Anderson localization in configurational space of disordered solids [53,54] is especially clear and instructive.

Consider real Hamiltonian matrices of a rather general type

$$H_{mn} = H_{nn}\,\delta_{mn} + v_{mn} \qquad m\,, n = 1, ..., N \qquad (33)$$

where off–diagonal matrix elements $v_{mn} = v_{nm}$ are random and statistically independent with $< v_{mn} >= 0$ and $< v_{mn}^2 >= v^2$ for $|m - n| \leq b$, and are zero otherwise. The most important characteristic of these Wigner band random matrices (WBRM) is the average energy level density ρ defined by the relation

$$\frac{1}{\rho} = \langle H_{mm} - H_{m'm'} \rangle \qquad (34)$$

where $m' = m - 1$. The averaging here and below is understood either over disorder that is over many random matrices or within a single sufficiently large matrix. Both are equivalent owing to the assumed statistical independence of matrix elements. In other words, many matrices are statistically equivalent to a big one. Quantum numbers m, n are generally arbitrary but we will have in mind ones related to the action variables, thus considering the quantum structure in the momentum space.

The basis in which the matrix elements are calculated is usually assumed to correspond to a completely integrable system with N quantum numbers where N is the number of freedoms. By ordering the basis states in energy we can represent N quantum numbers by a single one related to the energy which is also an action variable.

In the classical limit the definition of WBRM (33) corresponds to the standard Hamiltonian $H = H_0 + V$ where the perturbation V is usually assumed to be sufficiently small while the unperturbed Hamiltonian H_0 is completely integrable.

Quantum model (33) is defined by 3 independent physical parameters above: ρ, v, and b. The fourth parameter, matrix size Q, is considered to be technical in this model provided $Q \gg d_e$ (see Eq.(38) below) is big enough to avoid the boundary effects.

In terms of unperturbed energy E_0 the classical chaotic trajectory of a given total energy $E = const$ fills up the *energy shell* $\Delta E_0 = \Delta V$ with the ergodic (microcanonical) measure w_e depending on a particular perturbation function V. In the quantum system this measure characterizes the shape (distribution) of the ergodic eigenfunction (EF) in the unperturbed basis. Conversly, if we keep fixed the unperturbed energy $E_0 = const$, the measure w_e describes the band of energy surfaces $E = const$ whose trajectories reach the unperturbed energy E_0. In a quantum system the measure w_e in the latter case corresponds to the energy spectrum of a Green function (GS) with initial energy E_0. This characteristic was originally introduced also by Wigner [42] as the 'strength function', the term still in use in nuclear physics. Now it is called also the 'local density of (eigen)states'.

For a typical perturbation, represented by WBRM, w_e depends on the Wigner parameter [42] $q = (\rho\,v)^2/b$. In the two limits [42] (see also Ref.[7,47])

$$
w_e(E) = \begin{cases} \frac{2}{\pi E_{SC}^2} \sqrt{E_{SC}^2 - E^2}, & |E| \leq E_{SC}, \quad q \gg 1 \\[2ex] \frac{\Gamma/2\pi}{E^2 + \Gamma^2/4} \cdot \frac{\pi}{2 \cdot \arctan(1/\pi q)}, & |E| \leq E_{BW}, \quad q \ll 1 \end{cases} \tag{35}
$$

provided $\eta = \rho v \gtrsim 1$ which is the condition for coupling neighboring unperturbed states by the perturbation. In opposite case $\eta \ll 1$ the impact of perturbation is negligible which is called *perturbative localization*. The latter is a well known *quantum* effect but not one we are interested in (for chaotic phenomena it was first considered in Ref.[51]). What is less known that for the coupling of *all* unperturbed states within the Hamiltonian band a stronger condition is required, namely:

$$
\eta \gtrsim \sqrt{b}, \quad \text{or} \quad q \gtrsim 1 \tag{36}
$$

This is a simple estimate in the first order of perturbation theory. Indeed, the coupling is $\sim V/\delta E$. Within the band the energy detuning $\delta E \sim b/\rho$ while the total random perturbation $V \sim v\sqrt{b}$, hence estimate (36). In opposite case $q \lesssim 1$ a *partial perturbative localization* takes place which is also a *quantum* phenomenon, and again

not one we have in mind speaking about the quantum localization. The mechanism of perturbative localization is relatively simple and straightforward. This quantum effect is completely absent only in the first, semicircle (SC), limit of Eq.(35) where the width of the energy shell $\Delta E = 2E_{SC} = 2\sqrt{8\,b\,v^2} = 4\sqrt{2q}E_b \gg E_b$, and $E_b = b/\rho$ is the half–width of the Hamiltonian matrix band in energy. The latter inequality allows for *diffusive* quantum motion within the energy shell as a single random jump is $\sim b \ll \rho\Delta E$. The quantum localization under consideration here is related just to the localization (suppression) of quantum diffusion by the interference effects in discrete spectrum (see, e.g., Ref.[7]). Notice that the SC width immediately follows from the above estimate for perturbative localization: $\delta E \sim \Delta E \sim v\sqrt{b}$.

In the second, Breit - Wigner (BW), limit of Eq.(35) the full size of the energy shell $\Delta E = 2E_{BW} = 2E_b$ is equal to that of the Hamiltonian band. However, due to the partial perturbative localization explained above the main peak of the quantum ergodic measure is considerably more narrow, with the width $\Gamma = 2\pi\rho v^2 = 2\pi q E_b \ll E_b$. This is again in accordance with the same simple estimate: $\delta E \sim \Gamma \sim v\sqrt{\rho\Gamma}$.

To the best of my knowledge, the quantum distributions (35) were theoretically derived and studied for GSa only. Classically, the measure w_e seems to be the same for both $E = const$ and $E_0 = const$ as determined by the same perturbation V. One of the main recent results [41] in the studies of WBRM is that the classical symmetry between EFs ($E = const$) and GSa ($E_0 = const$) is generally lost in quantum mechanics. Namely, in ergodic case such a statistical symmetry still persists, yet the quantum localization drastically violates the symmetry producing a very intricate and unusual global structure of the quantum chaos.

In a sense, the conservative system is always localized (finite ΔE) even for ergodic motion. This is the origin of misunderstanding sometimes (see, e.g., Ref.[52]). In fact, such a *classical localization* is a trivial consequence of energy conservation as was explained above. It persists, of course, in the classical limit as well. Here we are interested in the *quantum localization* explained above called simply localization below.

Similar to maps the localization in conservative systems also depends on the ergodicity parameter (cf. Eq.(19)):

$$\lambda = a\,\frac{b^2}{d_e} = \frac{ab^{3/2}}{4\sqrt{2}c\eta} \tag{37}$$

Here the ergodicity corresponds not to the total number of states Q as in maps (19) but to that within the energy shell of width ΔE:

$$d_e = c\cdot\rho\,(\Delta E)_{SC} = 4c\eta\sqrt{2b} \tag{38}$$

Hilbert dimension d_e is also called the *ergodic localization length* as a measure of the maximal number of basis states (BS) coupled by the perturbation in case of ergodic motion. Numerical factor $c \approx 0.92$ is directly calculated from the limiting expression

(35) for a particular definition of d in Eq.(30). Relation (38) is valid formally in the SC region ($q \gg 1$) only but, according to computations, has still the accuracy within a few per cent down to $q \approx 0.4$.

Parameter λ (37) had been found in Ref.[48] and implicitely used there (without any relation to ergodicity). It was explained in details in Ref.[45] where factor $a \approx 1.2$ was also calculated numerically.

Localization is characterized by the parameter

$$\beta_d = \frac{d}{d_e} \approx 1 - \mathrm{e}^{-\lambda} < 1 \tag{39}$$

Here d stands for the actual average localization length of EFs measured according to the same definition (30). Empirical relation (39) has been found in numerical experiments [45] to hold in the whole interval $\lambda \leq 2.5$ there, and was confirmed in Ref.[41] up to $\lambda \approx 7$.

In the BW region $d_e = \pi \rho \Gamma = 2\pi^2 b q$, and $\lambda \approx a b/(2\pi^2 q) \gg 1$ as $q \ll 1$ (35) and $b \gg 1$ in quasiclassics. Hence, localization is only possible in the SC domain which was studied in Ref.[41].

The numerical results [41] were obtained from *two* individual matrices: the main one for the localized case with parameters

$$\lambda = 0.23; \ q = 90; \ Q = 2400; \ v = 0.1; \ b = 10; \ \rho = 300; \ \eta = 30; \ d_e = 500$$

and additional one for ergodic case with parameters

$$\lambda = 3.6; \ q = 1; \ Q = 2560; \ v = 0.1; \ b = 16; \ \rho = 40; \ \eta = 4; \ d_e = 84$$

All results are entirely contained in the EF matrix c_{mn} which relates the eigen-functions ψ_m to unperturbed basis states φ_n:

$$\psi_m = \sum_n c_{mn} \cdot \varphi_n, \qquad w_m(n) = |\psi(n)|^2 = c_{mn}^2 = w_{mn} \tag{40}$$

in momentum representation, and in the corresponding string of eigenvalues $E_m \approx m/\rho$. From the matrix c_{mn} the statistics of both EFs as well as GSa was evaluated. In order to suppress big fluctuations in individual distributions the averaging over 300 of them in the central part of the matrix was done in two different ways: with respect to the energy shell center ('global average', localization parameter (39) $\beta_d = \overline{\beta}_g$), and with respect to the centers of individual distributions ('local average', $\beta_d = \overline{\beta}_l$). Besides, the average $< \beta_d >$ over β_d values from the individual distributions was computed.

In the ergodic case $\lambda = 3.6$ both average distributions for EFs are fairly close to the SC law - a remarkable result, because that law was theoretically predicted for the *other* distribution, that of GSa. More precisely, the bulk ('cap') of the distributions are very close to the limiting SC (35), except in the vicinity of the SC singularities.

Numerical values of the localization parameter ($\overline{\beta}_g = 1.08$, $\overline{\beta}_l = 0.94$, $< \beta_d >= 0.99$) are in a reasonable agreement with scaled $\beta_d = 0.97$ for $\lambda = 3.6$, Eq.(39).

As expected, the GS structure is similar: $\overline{\beta}_g = 1.07$, $\overline{\beta}_l = 1.06$, $< \beta_d >= 0.98$.

For finite q all the distributions are bordered by the two symmetric steep tails which apparently fall down even faster than the simple exponential with a characteristic width $\sim b$. The physical mechanism of the tail formation is a specific quantum tunneling via intermediate BSs [30]. The asymptotic theory of the tails was developed in Ref.[30,42,46]. Surprisingly, it works reasonably well even near the SC borders.

The structure of matrix c_{mn} is completely different in the localized case $\lambda = 0.23$. The EF *local* average shows clear evidence for exponential localization with $\overline{\beta}_l = 0.24$ which is again close to scaled $\beta_d = 0.21$ for $\lambda = 0.23$. However, the *global* average reveals a nice SC (with tails) in spite of localization ($\overline{\beta}_g = 0.98$). It shows that, at average, the EFs homogeneously fill up the whole energy shell. In other words, their centers are randomly scattered over the shell.

Unlike ergodic case the localized GS structure is quite different from that of EFs. Both averages now yield similar results which well fit the SC distribution ($\overline{\beta}_g = 0.98$, $\overline{\beta}_l = 0.96$ as compared with $\overline{\beta}_g = 0.99$, $\overline{\beta}_l = 0.24$ for EFs). So, GSa look extended, yet they are localized! This is immediately clear from the third average $< \beta_d >= 0.20$. The explanation of this apparent paradox is that even though the GSa are extended over the shell they are *sparse* that is contain many 'holes'.

In analysis of the WBRM structure theoretical expression (38) for the ergodic localization length d_e (the energy shell width) was used. In more realistic and complicated physical models this might impede the analysis. In this respect the new method for direct empirical evaluation of d_e, and hence the important localization parameters β_d and λ in Eq.(39), from both average distributions for GSa as well as from the global average for EFs [41] looks very promising.

The physical interpretation of this structure based upon the underlying chaotic dynamics is the following. Spectral sparsity decreases the level density of the operative EFs which is the main condition for quantum localization via decreasing the relaxation time scale (see, e.g., [7]). Yet, the initial diffusion and relaxation are still classical, similar to the ergodic case, which requires extended GSa. On the other hand, EFs are directly related to the steady–state density [11], both being solid because of a homogeneous diffusion during the statistical relaxation.

This picture allows to conjecture that for a classically *regular* motion the EFs become also sparse, so that EF/GS symmetry is apparently restored.

5   Conclusion: pseudochaos and traditional statistical mechanics

The quantum chaos is a particular but most important example of the new generic dynamical phenomenon – *pseudochaos* in almost periodic motion. The statistical properties of the discrete–spectrum motion is not a completely new subject of research, it goes back to the time of intensive studies in the mathematical foundations of statistical mechanics *before* the dynamical chaos was discovered or, better to say, was understood (see, e.g., Ref.[55]). This early stage of the theory as well as the whole TSM was equally applicable to both classical and quantum systems. For the problem of pseudochaos one of the most important rigorous results with far–reaching implications was the *statistical independence* of oscillations with incommensurate (linearly independent) frequencies ω_n, such that the only solution of the resonance equation

$$\sum_n^N m_n \cdot \omega_n = 0 \tag{41}$$

in integers is $m_n \equiv 0$ for all n. This is a generic property of the real numbers. In other words, the resonant frequencies (41) form a set of zero Lebesgue measure. If we define now $y_n = \cos(\omega_n t)$ the statistical independence of y_n means that trajectory $y_n(t)$ is ergodic in N–cube $|y_n| \leq 1$. This is a consequence of ergodicity of the phase trajectory $\phi_n(t) = \omega_n t \; mod \; 2\pi$ on a torus $|\phi_n| \leq \pi$.

Statistical independence is the basic property of a set to which the probability theory is to be applied. Particularly, the sum of statistically independent quantities

$$x(t) = \sum_n^N A_n \cdot \cos(\omega_n t + \phi_n) \tag{42}$$

which is the motion with discrete spectrum, is a typical object of this theory. However, the familiar statistical properties like Gaussian fluctuations, postulated (directly or indirectly) in TSM, are reached in the thermodynamic limit $N \to \infty$ only [55]. In TSM this limit corresponds to infinite–dimensional models [6] which provide a very good approximation for macroscopic systems, both classical and quantal.

What is really necessary for good statistical properties of almost periodic motion (42) is a big number of frequencies $N_\omega \to \infty$ which makes the discrete spectrum continuous (in the limit). In TSM the latter condition is satisfied by setting $N_\omega = N \to \infty$. The same holds true for quantum fields which are infinite–dimensional. In the finite–dimensional quantum mechanics *another* mechanism, independent of N, works in the quasiclassical region $Q \gg 1$. Indeed, if the quantum motion (42) (with $\psi(t)$ instead of $x(t)$) is determined by many ($\sim Q$) eigenstates we can set $N_\omega = Q$ independent of N. The actual number of terms in expansion (42) depends, of course, on a particular state $\psi(t)$. For example, if it is just an eigenstate the sum reduces to a single term. This is reminiscent to some special peculiar trajectories of classical

chaotic motion whose total measure is *zero*. Similarly, in quantum mechanics $N_\omega \sim Q$ for *most* states if the system is *classically chaotic.*

For a regular motion in the classical limit the quantity $N_\omega \ll Q$ becomes considerably smaller. For example (see Ref.[14]), in the standard map $N_\omega = Q$ in ergodic case, $N_\omega \sim k^2$ in case of localization (both classically chaotic, $K > 1$) but only $N_\omega \sim k \ll k^2 \lesssim Q$ for classically regular motion ($K < 1$). The quantum chaos–order transition is not as sharp as the classical one but the ratio $N_\omega(K > 1)/N_\omega(K < 1) \sim k \to \infty$ increases with quantum parameter k.

Thus, with respect to the mechanism of the quantum chaos we essentially *come back* from the ergodic theory to old TSM with exchange of the number of freedoms N for quantum parameter Q. However, in quantum mechanics we are not interested, unlike TSM, in the limit $Q \to \infty$ which is simply the *classical* mechanics. Here, the central problem is the statistical properties for *large but finite Q*. This problem does not really exist in TSM describing macroscopic systems. In a *finite–Q* (or finite–N) pseudochaos we have to introduce the basic conception of *time scale* [11]. This allows for interpretation of quantum chaos as a *new* dynamical phenomenon, related but not identical at all to the classical dynamical chaos. Hence, the term *pseudochaos* emphasizing the difference from the asymptotic (in time) chaos in the ergodic theory.

In my opinion, the fundamental importance of quantum chaos is precisely in that it reconciles the two apparently opposite regimes, regular and chaotic, in the general theory of dynamical systems. The studies in quantum chaos help to better understand the old mechanism for chaos in many–dimensional systems. Particularly, the existence of characteristic time scales similar to those in quantum systems was conjectured in Ref.[7].

Is pseudochaos a chaos?

Until recently even the conception of classical dynamical chaos was rather incomprehensible, especially for physicists. I know that some researchers actually observed dynamical chaos in numerical or laboratory experiments but... did their best to get rid of it as some artifact, noise or other interference! Now the situation in this field is upside down: many researchers insist that if an apparent chaos is not like that in the classical mechanics (and in the existing ergodic theory), then it is not a chaos at all. Hence, sharp disputes over the quantum chaos. The peculiarity of the current situation is that in most studies of the 'true' (classical) chaos the digital computer is used where only *pseudochaos* is possible that is one like in *quantum* (not classical) mechanics!

Hopefully, this 'child disease' of quantum chaos will be over before long...

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