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Nonlinear Dynamics, Chaotic and Complex Systems

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The physics and mathematics of nonlinear dynamics and chaotic and complex systems constitute some of the most fascinating developments of late twentieth century science. It turns out that chaotic behaviour can be understood, and even utilized, to a far greater degree than had been suspected. Surprisingly, universal constants have been discovered. The implications have changed our understanding of important phenomena in physics, biology, chemistry, economics, medicine and numerous other fields of human endeavour. In this book, two dozen scientists and mathematicians who were deeply involved in the 'nonlinear revolution' cover most of the basic aspects of the field. The book is divided into five parts: dynamical systems, bifurcation theory and chaos; spatially extended systems; dynamical chaos, quantum physics and the foundations of statistical mechanics; evolutionary and cognitive systems; and complex systems as an interface between the sciences.

• Several chapters written by founders in the field (including a Nobel Prize winner, Ilya Prigogine) • Wide range of topics covered

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Pseudochaos in statistical physics

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A new generic dynamical phenomenon of *pseudochaos* and its relevance to statistical physics, both modern and traditional, are discussed in some detail. Pseudochaos is defined as a statistical behaviour of the dynamical system with *discrete* energy and/or frequency spectrum. The statistical behaviour, in turn, is understood as a time-reversible but nonrecurrent relaxation to (at average) some steady state, superimposed with irregular fluctuations. Our main attention is payed to the most important and universal example of pseudochaos, so-called *quantum chaos*, that is dynamical chaos in bounded mesoscopic quantum systems. Quantum chaos as a mechanism for implementation of the fundamental correspondence principle is also discussed.

The quantum relaxation localization, a peculiar characteristic implication of pseudochaos, is reviewed in both time-dependent and conservative systems, with special emphasis on the *dynamical decoherence* of quantum chaotic states. Recent results on the peculiar global structure of the energy shell, Green function spectra and eigenfunctions, both localized and ergodic, in a generic conservative quantum system are presented.

Examples of pseudochaos in classical systems are given, including linear oscillators and waves, digital computers and completely integrable systems. A far-reaching similarity between the dynamics of a quantum system with few degrees of freedom at high energy levels ($n \to \infty$), and that of many degrees of freedom ($N \to \infty$) is also discussed.

1. Introduction: the rebirth of pseudochaos

The concept of *pseudochaos* was first introduced explicitly by Chirikov (1991b) in an attempt to interpret the very controversial phenomenon of *quantum chaos*, and to understand its mechanism and physical meaning. The term itself was borrowed from the theory of *pseudorandom number generators* in digital computers. Even though such imitations of 'true' random quantities are widely used in many *numerical experiments*, e.g. those employing Monte–Carlo techniques, this pseudorandomness was always considered of no general relevance for physics. However, in recent numerous attempts to understand quantum chaos, which is attracting the ever growing attention of researchers (see e.g. Casati (1985), Giannoni *et al.* (1991), Heiss (1992) and a collection of papers in Casati & Chirikov (1995a)), it is becoming more and more clear that this *specific mechanism* provides, in fact, a *typical* chaotic behaviour in physical systems.

Moreover, from the viewpoint of fundamental physics, pseudochaos is the only kind of chaos principally possible in physical systems of finite dimensions. In infinite macroscopic systems of traditional statistical mechanics (TSM), both *classical and quantum*, particularly in the principal TSM concept of the thermodynamic limit, $N \to \infty$, where N is the number of the degrees of freedom, this situation is not the case. Namely, it has been rigorously proved (see e.g. Kornfeld *et al.* (1982)) that, roughly speaking, 'true' chaos is a generic phenomenon in this limit even if for any finite N the system is completely integrable!

The discovery of dynamical chaos in finite (and even low-dimensional) *classical* systems, a fundamental breakthrough in recent decades, has crucially changed classical statistical mechanics. By now, this new mechanism for the statistical laws is well understood (but still not very well known), and has acquired firm mathematical foundations in modern ergodic theory, see Kornfeld *et al.* (1982).

In spite of the success of this new mechanism a 'minor' problem still remains: such a mechanism does not work in finite *quantum* systems, whose motion is bounded in *phase space* and, hence, whose energy and frequency spectra are discrete.

The simplest solution to this problem, which nowadays seems to be almost commonly accepted, is that dynamical chaos in such systems is simply impossible. However, this seemingly obvious 'solution' is in fact a trap, as it immediately leads to a sharp and very profound contradiction with the fundamental correspondence principle, see Casati & Chirikov (1995b). We need to choose what to sacrifice, this principle or else 'true' (classical) chaos. I prefer to drop the latter. If the phenomenon of quantum chaos really did violate the correspondence principle, as some physicists suspect, it would indeed be a great discovery, since it would mean that classical mechanics is not the limiting case of quantum mechanics, but a completely different theory. 'Unfortunately', there exists a less radical (but also interesting and important) resolution of this difficulty, pseudochaos, which is the main topic of my talk.

Within such a philosophical framework the central physical problem is to understand the nature and mechanism of dynamical chaos in quantum mechanics. In other words, we need the *quantum* theory of dynamical chaos, including the transition to the classical limit. Certainly, quantum chaos is a new dynamical phenomenon, see Casati & Chirikov (1995b), related but not identical to classical chaos. We call it *pseudochaos*, the term *pseudo* intending to emphasize the difference from the 'classical' chaos in the ergodic theory. From the physical point of view, which I accept here, the latter, strictly speaking, does not exist in Nature. So, within the common philosophy of universal quantum mechanics *pseudochaos is the only true dynamical chaos*. Classical chaos is but a limiting pattern which is, nevertheless, very important, both in theory, to compare with real (quantum) chaos and in applications, and as a very good approximation in a macroscopic domain, as is the whole of classical mechanics. Ford (1995) calls it *mathematical chaos*, as contrasted to *real physical chaos* in quantum mechanics.

I emphasize again that classical chaos is impossible in *finite and closed* quantum systems to which my talk is restricted. Particularly, I am not going to discuss here quantum measurement in which macroscopic (infinite-dimensional) processes are involved, see e.g. Casati & Chirikov (1995b).

Thus, the physical meaning of the term *pseudochaos* is principally different from (and even opposite to) that of *pseudorandom numbers* in a computer. The reason for it is that the original term *pseudo* had a double meaning. At the beginning, the first and only meaning was related to the common belief that (by definition) no dynamical, deterministic, system like a computer can produce anything random. This illusion has been overcome in the theory of dynamical chaos in the field of *real* numbers. However, the digital computer works on a finite lattice of *integers*. This is qualitatively similar to quantum behaviour, see Chirikov *et al.* (1981). Computer numbers, like quantum variables, can at most be *pseudorandom* only, in contrast to the 'true' random classical quantities represented by real numbers. But then, a very special notion of *pseudorandom* was scrambling up to the level of a new fundamental concept in physics.

Quantum chaos is a part of quantum dynamics which, in turn, is a particular class of dynamical systems. It became a real physical problem upon discovery and understanding of classical dynamical chaos. To explain the problem I need to briefly remind you of the main peculiarities of classical chaos, especially those that are crucial in quantum theory.

2. Asymptotic chaos in classical mechanics

There are two equivalent descriptions of classical mechanics or, more generally, of any finite-dimensional dynamical system: via individual trajectories, and via a distribution function, or phase-space density, for Hamiltonian systems (most fundamental).

The trajectory obeys the equations of motion, which in general are nonlinear. It desribes a particular realization of a system's dynamics depending on the initial conditions. The phase density satisfies the Liouville equation, which is always *linear*, whatever the equations of motion, and which usually represents the typical (generic) dynamical behaviour of a given system. In particular, all zero-measure sets of special trajectories are automatically excluded.

Notice, however, that in some special cases, the phase density may display properties absent for trajectories. An interesting example, see Courbage & Hamdan (1995), is the correlation decay (and, hence, continuous spectrum) for a special initial phase density in a completely integrable system. The point is that such decay is related to the correlation between *different* trajectories rather than the behaviour on a given trajectory. The trajectory spectrum remains descrete, and the corresponding correlation persists. An interesting open question concerns the exact conditions for a phase density to represent the trajectory properties.

The strongest statistical properties of a dynamical system are related to the *local* exponential instability of trajectories, as described by the *linearized* equations of motion, provided the motion is *bounded* in phase space. These two conditions are *sufficient* for a rapid mixing of trajectories by the mechanism of 'stretching and folding'. For the linear equations of motion the combination of both conditions is impossible unless the whole phase space of the system is finite. A well-known example of the latter situation is the model described by the *linear* 'Arnold cat map', see Arnold & Avez (1968):

$$\begin{aligned} \overline{p} &= p + x \pmod 1 \\ \overline{x} &= x + \overline{p} \pmod 1 \end{aligned} \tag{2.1}$$

on a unit torus. The motion is exponentially unstable with (positive) Lyapunov exponent $\Lambda = \ln[(3+\sqrt{5})/2] > 0$, and is bounded due to the operation (mod 1). Notice that the linearized motion is described by the same map but *without* (mod 1), that is in the *infinite* plane ($-\infty < \mathrm{d}p, \mathrm{d}x < \infty$). It is unbounded and *globally* unstable but perfectly regular. We have so-called hyperbolic motion:

$$\mathrm{d}p = a\exp(\Lambda t) + b\exp(-\Lambda t), \quad \mathrm{d}x = c\exp(\Lambda t) + d\exp(-\Lambda t), \tag{2.2}$$

where the constants a, b, c, d depend on initial conditions and on Λ, and the integer t is the discrete map time. Remarkably, the motion (2.2) is time-reversible but *unstable* in both senses ($t \to \pm\infty$). This implies time reversibility of all statistical properties for the main system (2.1). This is a surprising conclusion which is still confusing some researchers (see e.g. Misra & Prigogine (1983)).

A nontrivial part of the relation between instability and chaos is in that the instability must be *exponential.* A power law instability is insufficient for chaos. For example, if we replace the first of equations (2.1) by $\overline{p} = p$, the model becomes completely integrable with oscillation frequency depending on the integral of the motion p (nonlinear oscillation). This produces *linear* (in time) instability, but the motion remains regular (with discrete spectrum). This is a typical property of completely integrable nonlinear oscillations, see Casati *et al.* (1980), which leads to a confusing difference in dynamical behaviour between the trajectories and phase densities, as mentioned above. Another open question is how to choose the correct time variable for a particular dynamical problem, see Casati & Chirikov (1995b). A change of time may convert the exponential instability into a power one, and vice versa (see e.g. Blümel (1994) for discussion).

The two above conditions for dynamical chaos can be realized in very simple (e.g. low-dimensional) systems like the model of (2.1). Another simple example, to which I will refer below, is the so-called 'kicked rotator' described by the *standard map*, see Chirikov (1991b), Casati & Chirikov (1995b), Chirikov (1979) and Chirikov *et al.* (1981):

$$\overline{p} = p + k \sin x, \quad \overline{x} = x + T\,\overline{p}, \tag{2.3}$$

also on either a torus (x, p (mod 2π) or a cylinder (x (mod 2π), $-\infty < p < \infty$). This model is also well studied, and has many physical applications. The motion on a cylinder is bounded in one variable only. This, however, is sufficient for chaos.

The exponential instability implies a continuous spectrum of the motion which is equivalent, roughly speaking, to the mixing, or temporal *correlation decay*. Apparently, this is the most important characteristic property in statistical mechanics, underlying the principal and universal statistical phenomenon of *relaxation* to some steady state, or statistical equilibrium.

Aperiodic relaxation is especially clear in the Liouville picture for phase density behaviour (see e.g. Arnold & Avez (1968)). Consider a basis for Liouville's equation, for example

$$\varphi_{mn} = \exp[2\pi \mathrm{i}(mx + np)], \tag{2.4}$$

where m, n are any integers, in a simple example of model (2.1). In other words, we represent the phase density as a Fourier series:

$$f(x,p,t) = \sum_{m,n} F_{mn}(t)\varphi_{mn}(x,p) = \sum_{m,n} F_{mn}(0)\exp[2\pi \mathrm{i}(m(t)x + n(t)p)]. \tag{2.5}$$

Each term in this series, except φ_{00}, has zero total probability, and characterizes the spatial correlation in the phase density. The map (2.1) induces a map for the Fourier amplitudes and for harmonic numbers:

$$\overline{F}_{mn} = F_{\overline{mn}}, \quad \overline{n} = n + m, \quad \overline{m} = m + \overline{n}. \tag{2.6}$$

Remarkably, the variables $m(t)$ and $n(t)$ obey the same map as that for the *linearized* equations of motion in variables $\mathrm{d}x, \mathrm{d}p$, and with the same instability rate Λ on the *infinite* lattice (m, n). The dynamics of the phase density in the Fourier representation, described by the same equation (2.2) (upon substitution of m, n for $\mathrm{d}x, \mathrm{d}p$), is also unbounded, globally unstable, and regular. This is not surprising, as both representations describe the *local* structure of the motion. Dynamical chaos is a *global* phenomenon determined, nevertheless, by the microdetails of the initial conditions, due to the exponential instability of the motion, see Casati & Chirikov (1995b) and Chirikov (1994). Accordingly, in the original phase space the temporal density fluctuations are chaotic, as are almost all trajectories of the map (2.1).

The only *stationary* mode $m = n = 0$ represents, in this picture, the statistical steady state, while all the others describe *nonstationary* fluctuations. The latter are another characteristic property of statistical behaviour. These higher modes can be separated from the average statistical relaxation by so-called coarse-graining, or spatial averaging, which is a projection of the phase density on a finite (and arbitrarily fine) partition of the phase space. The kinetic (particularly, diffusive) description of the statistical relaxation is restricted to such a coarse-grained projection only, while the fluctuations work as a dynamical *generator of noise*.

Another elegant method of separating out the average relaxation is a suppression of the fluctuations using Prigogine's Λ operator, see Misra & Prigogine (1983), which provides an invertible smoothing of the exact phase density, see Kumicak (this conference). True, the inverse operator is an improper one, yet this method could be efficiently used in some theoretical constructions. Contrary to common belief, it has nothing to do with time irreversibility, see Misra & Prigogine (1983) and Goodrich *et al.* (1980). Moreover, unlike the coarse-grained projection, the Λ-smoothed phase density is as reversible as the exact one (in principle but not in practice, of course). The origin of the misunderstanding concerning *irreversibility* is apparently related to the necessary restriction on the initial smoothed density, which was missing in the theory of Goodrich *et al.* (1980). Such a density is a technical rather than actual property of the system, and hence it does not need to be arbitrary. A similar operation is often used in quantum mechanics (for different purposes) to convert the Wigner function (the counterpart of exact classical phase density) into the so-called Husimi distribution, which is an expansion in the coherent states (see e.g. Casati & Chirikov (1995b)).

Nonstationary fluctuations/correlations of the phase density form a *stationary* flow into higher modes $|m|, |n| \to \infty$ (see Prigogine (1963)), and keep the memory of the exact initial conditions (see the first equation of (2.6)) providing time reversibility for the exact density. A stationary correlation flow is only possible for the *continuous* phase space, which is a characteristic feature of classical mechanics. This allows for an asymptotic formulation of the ergodic theory ($t \to \pm\infty$). Notice that both the trajectories and the full density are *time-reversible*. However, the latter, unlike the former, is *nonrecurrent*. Reversed relaxation, and in particular 'antidiffusion', describe the growth of a large fluctuation, which is eventually (as $t \to -\infty$) followed by standard relaxation in the opposite direction of time, see Chirikov (1994).

3. Quantum pseudochaos: a new dimension in ergodic theory

Dynamical chaos is one limiting case of modern general theory of dynamical systems which describes statistical properties of the deterministic motion (see e.g. Kornfeld *et al.* (1982)). No doubt, this theory has been developed on the basis of classical mechanics. Yet, as a general mathematical theory, it does not need to be restricted to classical mechanics only. In particular, it can be applied to quantum dynamics, and indeed it was, with a surprising result. Namely, it had been found at the beginning, see Casati *et al.* (1979), and was subsequently well confirmed, see e.g. Chirikov (1991b), Casati & Chirikov (1995b), Chirikov *et al.* (1981), Cohen (1991) and Casati *et al.* (1986), that quantum mechanics does not typically permit 'true' (classical-like) chaos. This is because in quantum mechanics the energy (and frequency) spectrum of any system, whose motion is bounded in phase space, is discrete, and its motion is almost periodic. Hence, according to the existing ergodic theory, such a quantum dynamics belongs to the limiting case of regular motion which is the opposite of dynamical chaos. The ultimate origin of quantum almost-periodicity is the discreteness of the phase space itself (or, in a more formal language, the noncommutative geometry of this space). This property is the basis of quantum physics, directly related to the fundamental uncertainty principle. Nevertheless, another fundamental principle, the correspondence principle, requires the transition to classical mechanics in all cases, including dynamical chaos with all its peculiar properties.

Now, the principal question to be answered is: where is the expected quantum chaos in ergodic theory? The answer, see Casati & Chirikov (1995b), Chirikov *et al.* (1981) and Chirikov (1994), (not commonly accepted as yet) was concluded from a simple observation (in principle well-known but never fully understood) that the sharp border between the discrete and continuous spectrum is physically meaningful in the limit $|t| \to \infty$ only, the condition actually assumed in ergodic theory. Hence, to understand quantum chaos, the existing ergodic theory needs some modification by introducing a new 'dimension': time. The ***finite-time statistical properties*** of a dynamical system, both quantum and classical, become a new and central problem in ergodic theory.

Within a finite time, the discrete spectrum is dynamically equivalent to the continuous one, thus providing much stronger statistical properties of the motion than was expected in the ergodic theory in the case of a discrete spectrum. It turns out that the motion with discrete spectrum may exhibit *all* statistical properties of classical chaos, but only on *finite* time scales.

The absence of classical-like chaos in quantum dynamics apparently contradicts not only the correspondence principle but also the fundamental statistical nature of quantum mechanics. However, even though the random element in quantum mechanics ('quantum jumps') is unavoidable, it can be singled out and separated from the proper quantum processes. Namely, the fundamental randomness in quantum mechanics is related only to a very specific event – the *quantum measurement* – which, in a sense, is foreign to the proper quantum system itself. This allows us to divide the whole problem of quantum dynamics into two qualitatively different parts:

- Proper quantum dynamics, as described by a very specific dynamical variable, the wavefunction $\psi(t)$, obeying some deterministic equation, for example the Schrödinger equation. The discussion in what follows will be limited to this part only.
- Quantum measurement, including the registration of the result, and hence, the collapse of the function ψ, which still remains a very vague issue in view of the fact that there is no common agreement even on whether this is a real physical problem or an ill-posed one, so that the Copenhagen interpretation of quantum mechanics answers all 'admissible' questions. In any case, up to now there is no dynamical description of the quantum measurement, including the collapse of ψ.

Recently a breakthrough in the understanding of quantum chaos has been achieved, particularly due to the above-mentioned philosophy of separating out the dynamical part of quantum mechanics. This philosophy is accepted, explicitly or more often implicitly, by most researchers in this field.

3.1. *Time scales of pseudochaos*

The existing ergodic theory is asymptotic in time, and thus involves no explicit time scales.† There are two reasons for it. One is technical: it is much easier to derive the asymptotic relations than to obtain rigorous finite-time estimates. The second reason is more profound. All statements in the ergodic theory hold up to measure zero, that is excluding some peculiar nongeneric sets of measure zero. Even this minor imperfection of the theory did not seem completely satisfactory but has been 'swallowed' eventually, and is now commonly tolerated by both physicists and mathematicians. In a finite-time theory all these exceptions acquire a *small but finite* measure, which could not be accepted by mathematicians. Yet, there is a standard mathematical 'trick' for avoiding both these difficulties.

The most important time scale t_{R} in quantum chaos is given by the general estimate, see Casati & Chirikov (1995b) and Chirikov *et al.* (1981):

$$\ln(\omega t_{\mathrm{R}}) \sim \ln Q, \quad t_{\mathrm{R}} \sim \frac{Q^{\alpha}}{\omega} \sim \rho_0 \le \rho_{\mathrm{H}}, \tag{3.7}$$

† Asymptotic statements in the ergodic theory should not always be understood literally to avoid physical misconceptions (see e.g. addendum to Prigogine (1963)). Actually, classical chaos has also its time scales, for example a dynamical one ($\sim \Lambda^{-1}$), see Casati & Chirikov (1995b).

where ω and $\alpha \sim 1$ are system-dependent parameters, and $Q \gg 1$ stands for some large quantum parameter (in semiclassical region). It can be, e.g. a quantum number $Q = I/\hbar$, related to a characteristic action variable I, or the total number of states for the bounded quantum motion in the phase space domain of volume Γ: $Q \approx \Gamma/(2\pi)^N$.‡ The time scale t_R is called the ***relaxation time scale***, referring to one of the principal properties of chaos; *statistical relaxation* to some steady state. The physical meaning of this scale is simple; it is directly related to the fundamental uncertainty principle ($\Delta t\, \Delta E \sim 1$) as implemented in the second equation (3.7), where ρ_H is the *full* average energy level density (also called the Heisenberg time). For $t \lesssim t_R$, the discrete spectrum is not resolved, and statistical relaxation follows the classical (limiting) behaviour. This indeed is the 'gap' in the ergodic theory (supplemented with time as the additional dimension) where pseudochaos, and in particular quantum chaos, dwells. A more accurate estimate relates t_R to a *part* ρ_0 of the level density. This is the density of the so-called *operative eigenstates* only (those which are present in a particular quantum state ψ, and which actually control its dynamics).

The formal trick mentioned above is not to consider finite-time relations, which we really need in physics, but rather a special *conditional limit*:

$$t, Q \to \infty, \quad \tau_R = \frac{t}{t_R(Q)} = \text{const}, \tag{3.8}$$

where τ_R is a new dimensionless time. The *double* limit (3.8) (unlike the single one $Q \to \infty$) is *not* the classical mechanics which holds, in this representation, for $\tau_R \lesssim 1$, and with respect to statistical relaxation only. For $\tau_R \gtrsim 1$, the behaviour becomes essentially quantum (even in the limit $Q \to \infty$!) and nowadays is called *mesoscopic*. In particular, the quantum steady state is in general quite different from classical statistical equilibrium, in that the former may be *localized* (under certain conditions), that is *nonergodic*, in spite of classical ergodicity.

Another important difference is in *fluctuations* which are also a characteristic property of chaotic behaviour. In comparison with classical mechanics, the quantum $\psi(t)$ plays, in this respect, an intermediate role between the classical trajectory with large relative fluctuations (~ 1), and the coarse-grained classical phase density with no fluctuations. Unlike both, the fluctuations of $\psi(t)$, or rather those of averages on a quantum state $\psi(t)$, are typically $\sim d_H^{-1/2}$, where d_H is the number of operative eigenstates associated with the quantum state ψ (d_H can be called the *Hilbert dimension* of the state ψ). In other words, the chaotic $\psi(t)$ represents statistically a *finite ensemble* of some number ($\sim d_H$) of independent systems, even though formally $\psi(t)$ describes a *single* system. The fluctuations clearly demonstrate the difference between physical time t and the auxiliary variable τ: in the double limit $(t, Q \to \infty)$ the fluctuations vanish, and one needs a new 'trick' to recover them for a finite Q.

The relaxation time scale should not be confused with the *Poincaré recurrence time* t_P ($\gg t_R$) which is typically much longer, and which sharply increases with decreasing recurrence domain. The time scale t_P characterizes large fluctuations (for both the classical trajectory and the quantum ψ, but not the phase density). On the contrary, t_R characterizes the average relaxation process. Rare recurrences make quantum relaxation similar to classical (nonrecurrent) relaxation.

Stronger statistical properties (than relaxation and fluctuations) are related in the ergodic theory to the exponential instability of motion. The importance of those stronger properties for statistical mechanics is not completely clear, see Farquhar (1964). Nevertheless, in accordance with the correspondence principle, these stronger properties are

‡ Here and in what follows I put $\hbar = 1$.

also present in quantum chaos, but on a *much shorter* time scale t_{r}:

$$\Lambda t_{\mathrm{r}} \sim \ln Q. \tag{3.9}$$

Here Λ is the classical Lyapunov exponent. This time scale was discovered and partly explained by Berman & Zaslavsky (1978) (see also Casati & Chirikov (1995b) and Chirikov *et al.* (1981)). We call it a *random time scale*. Indeed, according to Ehrenfest's theorem, the motion of a narrow wave packet follows the beam of classical trajectories as long as the packet remains narrow, and hence it is as random as in the classical limit. Even though the random time scale is very short, it grows indefinitely as $Q \to \infty$. Thus, temporary, finite-time quantum pseudochaos turns into classical dynamical chaos in accordance with the correspondence principle.

Again, we may consider the *conditional limit*:

$$t, Q \to \infty, \quad \tau_{\mathrm{r}} = \frac{t}{t_{\mathrm{r}}(Q)} = \mathrm{const.} \tag{3.10}$$

Notice that the new scaled time τ_{r} is different from that entering equation (3.8) (τ_{R}).

If we fix the time t, then in the limit $Q \to \infty$ we obtain the transition to the classical instability in accordance with the correspondence principle, while for Q fixed, and $t \to \infty$, we get the proper quantum evolution in time. For example, the quantum Lyapunov exponent satisfies

$$\Lambda_{\mathrm{q}}(\tau_{\mathrm{r}}) \to \begin{cases} \Lambda, & \tau_{\mathrm{r}} \ll 1 \\ 0, & \tau_{\mathrm{r}} \gg 1. \end{cases} \tag{3.11}$$

Quantum instability ($\Lambda_{\mathrm{q}} > 0$) was observed in numerical experiments, see Casati & Chirikov (1995b) and Toda & Ikeda (1987). What terminates the instability for $t \gtrsim t_{\mathrm{r}}$? A simple explanation is suggested by the classical picture of the phase density evolution on the integer Fourier lattice m, n discussed above for model (2.1). Classical Fourier harmonics m, n are of a kinematical nature, without any *a priori* dynamical restriction. In particular, they can be arbitrarily large (and actually are for a chaotic motion), which corresponds to a continuous classical phase space. On the contrary, the quantum phase space is discrete. At first glance, quantum wave packet stretching/squeezing, as for a classical system, does not seem to be principally restricted, since only a two-dimensional area (per degree of freedom) is bounded in quantum mechanics. However, Fourier harmonics of the quantum phase density (Wigner function) are directly related to quantum dynamical variables, in particular to the action variables, whose values are restricted by the quantum parameter Q, see estimate (3.9). In the simple model (2.1) this is related to a finite size of the whole phase space. In general, in a conservative system, even with an infinite phase space, the restriction is imposed by the energy conservation. Numerical experiments reveal that the original wave packet, after a considerable stretching similar to the classical one, is rapidly destroyed. Namely, it splits into many new small packets, see Casati & Chirikov (1995b) and Toda & Ikeda (1987). The mechanism of this sharp 'disrupture' of the classical-like motion is not quite clear (for a possible explanation see Casati & Chirikov (1995b) and Chirikov (1993)). The resulting picture is qualitatively similar to that for the classical phase density, the main difference being in the spatial fluctuation scale, now bounded from below by $1/Q$. Nevertheless, the quantum phase density can also be decomposed into a coarse-grained average, and fluctuations. An important implication of this picture for the wave packet time evolution is the rapid and complete destruction of the so-called generalized coherent states in quantum chaos, see Perelomov (1986).

In the quasiclassical region ($Q \gg 1$) the scale is $t_{\mathrm{r}} \ll t_{\mathrm{R}}$. This leads to the surprising conclusion that quantum diffusion and relaxation are *dynamically stable*, contrary to the classical behaviour. This, in turn, suggests that, in general, the instability of the motion is not important *during* statistical relaxation. Nevertheless, the *foregoing* correlation decay on the short time scale t_r is crucial for the statistical properties of quantum dynamics.

Dynamical stability of quantum diffusion has been proved in striking numerical experiments with time reversal, see Shepelyansky (1983). In a classical chaotic system the diffusion is immediately recovered due to numerical 'errors' (not random!) amplified by the local instability. On the contrary, the quantum 'antidiffusion' proceeds until the system passes near the initial state to a very high accuracy. Only then is normal diffusion restored. The stability of quantun chaos on a relaxation time scale is comprehensible, as the random time scale is much shorter. Nevertheless, the accuracy of the reversal (up to $\sim 10^{-15}$ (!)) is surprising. Apparently this is explained by the relatively large size of the quantum wave packet as compared to the unavoidable rounding-off errors, unlike the classical computer trajectory whose size is comparable to these errors, see Chirikov (1992b). In the standard map (2.3) (upon quantization) the size of the optimal, least-spreading, wave packet is $\Delta x \sim \sqrt{T}$, see Chirikov *et al.* (1981). On the other hand, any quantity in the computer must well exceed the rounding-off error δ ($\ll 1$), e.g. $T \gg \delta$, and $(\Delta x)^2/\delta^2 \gtrsim (T/\delta)\delta^{-1} \gg 1$.

3.2. *Classical-like relaxation and residual fluctuations*

The relaxation time scale t_R is more important than the time scale t_r for two reasons. First, it is much longer than t_r, and second, it is related to the principal process of statistical relaxation which is the basis of statistical mechanics. The short scale t_r was interpreted by Berman & Zaslavsky (1978) (see also Zaslavsky (1981)) as a limit for the classical-like behaviour of chaotic quantum motion. Subsequently, it was found that the *method* of quasiclassical quantization can be extended to much longer times, see Chirikov *et al.* (1981) and Sokolov (1984). However, the *physics* on both time scales is qualitatively different: dynamical instability on the scale t_r, and statistical relaxation on t_R.

The discrete pseudochaos spectrum is not resolved on the whole scale t_R, and relaxation follows the classical law. Consider, for example, model (2.3), the standard map on a torus with total number of quantum states Q, and p, x being the action–angle variables.

If the perturbation parameter $k \gtrsim Q$, the relaxation to ergodic steady state in this model, as well as in model (2.1), is very quick, with characteristic relaxation time $t_e \sim 1$ (iterations). Such regimes often take place in physical systems. Here I consider another case, more interesting for the problem of pseudochaos, namely *diffusive relaxation*, which occurs for a sufficiently weak perturbation

$$k \ll Q. \tag{3.12}$$

In the classical limit this relaxation is descibed by the standard diffusion equation

$$\frac{\partial f(p,t)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial p}D(p)\frac{\partial f(p,t)}{\partial p}, \tag{3.13}$$

where $f(p,t) = \langle f(p,x,t)\rangle_x$ is a coarse-grained phase density (averaged over x), and

$$D = \frac{\langle(\Delta p)^2\rangle}{t} \approx \frac{1}{2}k^2 \tag{3.14}$$

is the diffusion rate. Equation (3.14) holds for the standard map if $K \equiv kT \gg 1$, which is also the condition for global chaos in this model, see Chirikov (1979). The relaxation to the ergodic steady state $f_s = 1/Q$ is exponential with characteristic time

$$t_e = \frac{Q^2}{2\pi^2 D} \approx \frac{Q^2}{\pi^2 k^2}. \tag{3.15}$$

In the diffusive regime ($k \ll Q$) this time is $t_e \gg 1$. This average relaxation is stable and regular, in spite of the underlying chaotic dynamics.

The quantized standard map $\overline{\psi} = \hat{U}\psi$ is described by the unitary operator

$$\hat{U} = \exp\left(-\mathrm{i}\frac{T\hat{p}^2}{2}\right)\exp(-\mathrm{i}k\cos\hat{x}) \tag{3.16}$$

on a cylinder ($Q \to \infty$), see Casati *et al.* (1979), where $\hat{p} = -\mathrm{i}\partial/\partial x$, and by a similar but somewhat more complicated expression on a torus, see Izrailev (1990).

There are three quantum parameters in this model: the perturbation k, the period T and the size Q, but only two classical combinations remain: the perturbation $K = kT$ and the classical size $M = TQ/(2\pi)$, which is the number of classical resonances over the torus. Notice that the quantum dynamics is in general richer than the classical, as the former depends on an extra parameter. It is, of course, another representation of Planck's constant which I have set equal to one. That is why, in the quantized standard map, we need both parameters k and T, and cannot combine them in a single classical parameter K.

The quasiclassical region, where we expect quantum chaos, corresponds to $T \to 0$, $k \to \infty$, and $Q \to \infty$, while the classical parameters $K = \text{const}$ and $M = \text{const}$ are fixed.

A technical difficulty in evaluating t_R for a particular dynamical problem is that the density ρ_0 depends on the dynamics. So, we have to solve a self-consistent problem. For the standard map the answer is known (see Casati & Chirikov (1995b)):

$$t_\mathrm{R} = \rho_0 = 2D. \tag{3.17}$$

This is a remarkable relation, as it connects essentially *quantum* characteristics (t_R, ρ_0) with the *classical* diffusion rate D, see (3.14).

The quantum diffusion rate depends on the scaled (dimensionless) time τ_R (3.8), and is given by

$$D_\mathrm{q} = \frac{D}{1+\tau_\mathrm{R}} \to \begin{cases} D, & \tau_\mathrm{R} = t/t_\mathrm{R} \ll 1 \\ 0, & \tau_\mathrm{R} \gg 1. \end{cases} \tag{3.18}$$

This is an example of scaling in a discrete spectrum which eventually stops quantum diffusion.

The character of the steady state crucially depends on the ratio $t_\mathrm{R}/t_\mathrm{e}$. Define the *ergodicity parameter* λ (see Casati & Chirikov (1995b)):

$$\lambda = \frac{D}{Q} \sim \left(\frac{t_\mathrm{R}}{t_\mathrm{e}}\right)^{\frac{1}{2}} \sim \frac{k^2}{Q} \sim \frac{K}{M}\,k. \tag{3.19}$$

Consider first the case $\lambda \gg 1$, when the time scale t_R is long enough to allow for the completion of the classical-like relaxation. In that case the final steady state, as well as all the eigenfunctions, are ergodic, i.e. the corresponding Wigner functions are close to the classical microcanonical distribution in the phase space. This region is inevitably reached if the classical parameter K/M is kept fixed, while the quantum parameter $k \to \infty$, in agreement with the Shnirelman theorem, or else with a physical generalization of this theorem, see Shnirelman (1974). This region is called *far quasiclassical asymptotics*.

A principal difference between the quantum ergodic state and the classical state is the existence of *residual fluctuations* in the former. In the quasiclassical region, the chaotic quantum steady state is a superposition of a large number of eigenfunctions. As a result, almost any physical quantity fluctuates in time. Even in the discrete spectrum we are considering here, these fluctuations are very irregular. In the case of a classical-like ergodic steady state all Q eigenfunctions essentially contribute to the fluctuations. Moreover, we would expect their contributions to be statistically almost independent.

Hence, the fluctuations should scale $\sim Q^{-\frac{1}{2}} = d_{\mathrm{H}}^{-\frac{1}{2}}$, where d_{H} is the Hilbert dimension of the ergodic state. That, indeed, is the case, according to numerical experiments, see Casati *et al.* (in preparation). For example, the energy fluctuations were found to follow a simple relation

$$\frac{\Delta E_{\mathrm{s}}}{E_{\mathrm{s}}} \approx \frac{1}{\sqrt{Q}}\,, \tag{3.20}$$

where

$$(\Delta E_{\mathrm{s}})^2 = \overline{E^2(t)} - E_{\mathrm{s}}^2, \quad E_{\mathrm{s}} = \overline{E(t)}, \quad E(t) = \frac{\langle p^2 \rangle}{2}\,. \tag{3.21}$$

Here a bar indicates time averaging over a sufficiently long time interval ($\gg t_{\mathrm{e}}$), and brackets denote the usual average over the quantum state.

Equation (3.20) suggests the complete quantum decoherence in the final steady state for any initial state, even though the steady state is formally a pure quantum state. For $Q \gg 1$ the fluctuations are small, so that *statistically* the quantum relaxation is *nonrecurrent*. The decoherence of a chaotic quantum state is also confirmed by the independence (up to small fluctuations) of the final steady state energy E_{s} of the initial $E(0)$. Since any particular initial quantum state is strongly coherent, the decoherence is a result of the quantum chaos. It is called *dynamical decoherence*. This is one of the most important results in quantum chaos.

3.3. *Mesoscopics: quantum behaviour in the quasiclassical region*

If the ergodicity parameter $\lambda \ll 1$ is small, all the eigenstates and the steady state are non-ergodic, or localized. That is because the scale t_{R} is not long enough to support classical-like diffusion, which stops before classical relaxation is completed. For this reason this situation is also called *quantum diffusion localization*. As a result the structure of eigenfunctions and of the steady state remains essentially quantum, no matter how large the quantum parameter $k \to \infty$. This is called *intermediate quasiclassical asymptotics* or the *mesoscopic* domain. In particular, it corresponds to $K > 1$ fixed, $k \to \infty$ and $M \to \infty$, while $\lambda \ll 1$ remains small.

The popular term 'mesoscopic' means here some intermediate behaviour between classical and quantum. In other words, in mesoscopic phenomenona both classical and quantum features occur. Again, the correspondence principle requires the transition to completely classical behaviour. That, indeed, is the case, as mesoscopic phenomena occur in the region where the quantum parameter $k \gg 1$ is large, but still less than a certain critical value (corresponding to $\lambda \sim 1$) which determines the border of transition to fully classical behaviour (far quasiclassical asymptotics).

If $\lambda \ll 1$ is very small, the shape of localized eigenstates is asymptotically exponential, see Casati & Chirikov (1995b), and can be approximately described by a simple expression, see Casati *et al.* (to be published):

$$f_m(p) = \langle |\psi_m(p)|^2 \rangle \approx \frac{2}{\pi l \cosh[2(p - p_m)/l]}\,. \tag{3.22}$$

The localized steady state has a similar but somewhat more complicated shape, see Chirikov (1981) and Izrailev *et al.* (to be published). This is a simple approximation superimposed with large fluctuations. The parameter l is called the *localization length.* Incidentally, the two localization lengths (l_{s} for steady state and l for eigenfunctions) are different, see Chirikov *et al.* (1981):

$$l_{\mathrm{s}} \approx D \quad \text{while} \quad l \approx \tfrac{1}{2}D, \tag{3.23}$$

because of large fluctuations.

In terms of localization length, the region of mesoscopic phenomena is defined by

$$1 \ll l \ll Q. \tag{3.24}$$

The inequality on the left reflects classical features of the state, while the right hand one refers to quantum effects. The combination allows for a classical description, at least in the standard map, of statistical relaxation to quantum steady state by a phenomenological diffusion equation for the Green function, see Casati & Chirikov (1995b) and Chirikov (1991a):

$$\frac{\partial g(\nu,\sigma)}{\partial \sigma} = \tfrac{1}{4}\frac{\partial^2 g}{\partial \nu^2} + B(\nu)\frac{\partial g}{\partial \nu}\,. \tag{3.25}$$

Here $g(\nu,0) = |\psi(\nu,0)|^2 = \delta(\nu - \nu_0)$ and

$$\nu = \tfrac{p}{2D}, \quad \sigma = \ln(1+\tau_{\mathrm{R}}), \quad \tau_{\mathrm{R}} = \tfrac{t}{2D}\,. \tag{3.26}$$

The additional drift term in the diffusion equation, with

$$B(\nu) \approx \mathrm{sgn}(\nu - \nu_0) = \pm 1, \tag{3.27}$$

describes the so-called quantum coherent back-scattering, which is the dynamical mechanism of localization.

The solution of (3.25) is, see Casati & Chirikov (1995b),

$$g(\nu,\sigma) = \frac{1}{\sqrt{\pi\sigma}}\exp\left[-\frac{(\Delta+\sigma)^2}{\sigma}\right] + \exp(-4\Delta)\,\mathrm{erfc}\left(\frac{\Delta-\sigma}{\sqrt{\sigma}}\right), \tag{3.28}$$

where $\Delta = |\nu - \nu_0|$.

Asymptotically, as $\sigma \to \infty$, the Green function $g(\nu,\sigma) \to 2\exp(-4\Delta) \equiv g_{\mathrm{s}}$ approaches the localized steady state g_{s}, exponentially in σ but only as a power-law in physical time τ_{R} or t ($g - g_{\mathrm{s}} \sim 1/\tau_{\mathrm{R}}$). This is the effect of a discrete motion spectrum. Numerical experiments confirm prediction (3.28), at least up to the logarithmic accuracy $\sim \sigma \approx \ln \tau_{\mathrm{R}}$, see Casati & Chirikov (1995b) and Cohen (1991).

A physical example of localization is the quantum suppression of the diffusive photoeffect in a hydrogen atom, see Casati *et al.* (1987). Depending on parameters, the suppression can occur, no matter how large the atomic quantum numbers are. This is a typical mesoscopic phenomenon which had been predicted by the theory of quantum chaos, and was subsequently observed in laboratory experiments.

One might expect that in the case of localization ($D \ll Q$), the fluctuations scale like $l^{-1/2} \sim k^{-1}$, as the number of coupled eigenfunctions in the localized steady state is $\sim l$. This, however, is *not* the case as was found in the first numerical experiments (Casati & Chirikov (1995b)). According to more accurate data, see Casati *et al.* (in preparation), the fluctuations are described by

$$\frac{\Delta E_{\mathrm{s}}}{E_{\mathrm{s}}} = \frac{A}{k^{\gamma}} = \frac{a}{d_{\mathrm{H}}^{\gamma/2}}\,, \tag{3.29}$$

with fitting parameters $\gamma = 0.55$, $a = 0.65$. For a nonergodic state, the Hilbert dimension can be defined as (Casati & Chirikov (1995b))

$$d_{\mathrm{H}}^{-1} = \tfrac{1}{3}\int |\psi(p)|^4\,\mathrm{d}p = \int f^2(p)\,\mathrm{d}p, \tag{3.30}$$

where $f(p)$ is a smoothed (coarse-grained) density, and the factor $\frac{1}{3}$ accounts for the ψ fluctuations, see Flambaum *et al.* (1994). In the case of exponential localization (3.22), $d_{\mathrm{H}} \approx \pi^2 l/4$. The most important parameter here, γ, is about half the expected value $\gamma = 1$. This result suggests some fractal properties of localized eigenfunctions and/or their spectra. To put it another way, a slow fluctuation decay (3.29) implies *incomplete* quantum decoherence, which can be characterized by the number d_s of statistically independent components in the steady state, see Casati & Chirikov (1995b). Then, from (3.20) and (3.29) we obtain in the two limits:

$$\frac{d_s}{d_H} \approx \begin{cases} 1, & \lambda \gg 1 \\ d_H^{\gamma-1}/a^2 = 2.4\, d_H^{-0.45}, & \lambda \ll 1. \end{cases} \tag{3.31}$$

This result was confirmed by Izrailev *et al.* (to be published) for a band random matrix model.

The phenomenon of quantum diffusion localization also explains the limitation of quantum instability in systems with infinite phase space, like the standard map on a cylinder. Indeed, the maximum number of coupled states here is determined by the localization length, whatever the total number of states in the system. Hence, we should substitute the quantum parameter $Q \sim l \sim k^2$ in estimate (3.9). Even if there is no localization (e.g. for a standard map with the parameter $k(p)$ depending on p, see Casati & Chirikov (1995b) and Chirikov *et al.* (1981)), so that the quantum diffusion goes on and the quantum spectrum becomes continuous, the number of coupled states increases with time as a power law only ($\Delta p \sim \sqrt{t}$), and hence the quantum Lyapunov exponent vanishes on the relaxation time scale, $\Lambda_q \to 0$. Only if the action variables grow exponentially, the instability rate Λ_q remains finite, and quantum chaos becomes asymptotic, as in the classical limit (see Chirikov *et al.* (1981) and Weigert (1990) for such 'exotic' models).

3.4. *Examples of pseudochaos in classical mechanics*

Pseudochaos is a new generic dynamic phenomenon missed in ergodic theory. No doubt, the most important particular case of pseudochaos is quantum chaos. Nevertheless, pseudochaos occurs in classical mechanics as well. A few examples of classical pseudochaos are given below, which may help in understanding the physical nature of quantum chaos. Furthermore, they unveil new features of classical dynamics.

• *Linear waves* are an example of pseudochaos close to quantum mechanics (see e.g. Chirikov (1992a)). Only a part of quantum dynamics is discussed, that described, e.g. by the Schrödinger equation, which is a linear wave equation. For this reason, sometimes quantum chaos is called wave chaos, see Šeba (1990). Classical electromagnetic waves are used in laboratory experiments as a physical model for quantum chaos, see Stöckmann & Stein (1990). The 'classical' limit corresponds here to geometrical optics, and the 'quantum' parameter $Q = L/\lambda$ is the ratio of a characteristic size L of the system to the wave length λ. As is well known from optics, no matter how large the ratio L/λ, the diffraction patterns prevail at sufficiently large distances $R \gtrsim L^2/\lambda$. This gives a sort of relaxation scale: $R/\lambda \sim Q^2$.

• *A linear oscillator* (multi-dimensional) is also a particular representation of waves. A broad class of quantum systems can be reduced to this model, see Eckhardt (1988). Statistical properties of linear oscillators, particularly in the thermodynamic limit ($N \to \infty$), were studied by Bogolyubov (1945) in the framework of TSM. On the other hand, the theory of quantum chaos suggests a richer behaviour for large but finite N, in particular, characteristic time scales for harmonic oscillations, see Chirikov (1986), the number of the degrees of freedoms N playing the role of 'quantum' parameter.

• *Completely integrable nonlinear systems* also reveal pseudochaotic behaviour. An example of statistical relaxation on the Toda lattice was given in Ford *et al.* (1973) much before the problem of quantum chaos arose. Moreover, the strongest statistical properties in the limit $N \to \infty$, including that equivalent to exponential instability (a so-called K-

property), were rigorously proved just for completely integrable systems for any finite N, see Kornfeld *et al.* (1982).

• *A digital computer* is a very specific classical dynamical system, whose properties are extremely important in view of ever growing interest in numerical experiments, covering now all branches of science. The computer is an 'overquantized' system, in which *any* quantity is *discrete*, while in quantum mechanics only the product of two conjugated variables is. The 'quantum' parameter here is $Q = M$, which is the largest computer integer, and the short time scale (3.9), $t_r \sim \ln M$, represents the number of digits in the computer word, see Chirikov *et al.* (1981). Due to discreteness, any dynamical trajectory in a computer eventually becomes periodic, an effect well known in the theory and practice of pseudorandom number generators. One should take all necessary precautions to exclude such computer artifacts in numerical experiments (see e.g. Maddox (1994) and references therein). As for mathematics, the periodic approximations in dynamical systems are also considered in ergodic theory, apparently without any relation to pseudochaos in quantum mechanics or a computer, see Kornfeld *et al.* (1982).

Computer pseudochaos seems to be the most convincing argument for researchers who are still reluctant to accept quantum chaos as at least a kind of chaos. They insist that only classical-like (asymptotic) chaos deserves this name. But this is the same chaos which was (and still is) studied to a large extent on a computer and so is chaos inferred from pseudochaos!

4. Statistical theory of pseudochaos: random matrices

A complete solution of a dynamical quantum problem can be obtained via diagonalization of the Hamiltonian, to find the energy (or quasi-energy) eigenvalues and eigenfunctions. The evolution of any quantity is then expressed as a sum over the eigenfunctions. For example, the energy time dependence is

$$E(t) = \sum_{m,m'} c_m c^*_{m'} E_{mm'} \exp[\mathrm{i}(E_m - E_{m'})t], \tag{4.32}$$

where $E_{mm'}$ are matrix elements, and the initial state in momentum representation is $\psi(n,0) = \sum_m c_m \varphi_m(n)$. For chaotic motion this dependence is in general very complicated, but statistical properties of the motion can be inferred from the statistics of the eigenfunctions $\varphi_m(n)$ (i.e. of the matrix elements $E_{mm'}$) and of the eigenvalues E_m.

Nowadays there exists a well developed random matrix theory (RMT, see e.g. Brody *et al.* (1981)) which describes average properties of a typical quantum system with a given symmetry of the Hamiltonian. At the beginning, the object of this theory was assumed to be a very complicated (in particular multi-dimensional) quantum system as a representative of a certain statistical ensemble. In the course of understanding the phenomenon of dynamical chaos, it became clear that the number of the degrees of freedom of the system is irrelevant. Instead, the number of quantum states (quantum parameter Q) is of importance, providing dynamical chaos in the classical limit.

This approach to the theory of complex quantum systems like atomic nuclei, was adopted by Wigner (1955) 40 years ago, much before the problem of quantum chaos was formulated. He introduced the so-called band random matrices (BRM), which were most suitable to account for the structure of conservative systems. However, due to severe mathematical difficulties, RMT immediately turned to a much simpler case of statistically homogeneous (full) matrices, for which impressive theoretical results have been achieved, see Brody *et al.* (1981). The price was that full matrices describe the local chaotic structure only, a limitation especially inacceptable for atoms, see Flambaum *et* *al.* (1994) and Chirikov (1985). Only recently, interest returned to the original Wigner BRM, see Casati *et al.* (1993) and Feingold *et al.* (1991).

One of the main results in studies of quantum chaos was the discovery of quantum diffusion localization. This mesoscopic, quasiclassical, phenomenon discussed above, has been well studied and confirmed by many researchers for dynamical models described by maps. Contrary to common belief, the maps describe not only time-dependent systems, but also conservative ones in the form of Poincaré maps, see e.g. Bogomolny (1992). Nevertheless, to my knowledge, up to now no direct studies of quantum localization in conservative systems have been undertaken, either in the laboratory or in numerical experiments. Moreover, the very existence of quantum localization in conservative systems is challenged, see Bogomolny (1994). Here, I briefly descibe recent results concerning the structure of localized quantum chaos in the momentum space of a generic conservative system with few degrees of freedom, which is classically strongly chaotic, in particular ergodic on a compact energy surface, see Casati *et al.* (to be published).

In general, RMT is a statistical theory of systems with discrete energy spectra. This is the principal property of quantum dynamical chaos, see Casati & Chirikov (1995b). Thus, RMT turned out to become (accidentally!) a statistical theory for the incoming quantum chaos. Remarkably, this statistical theory does not include any time-dependent noise, that is any coupling to a thermal bath, the standard element of most statistical theories. Moreover, a *single matrix* from a given statistical ensemble represents a typical (generic) *dynamical* system of a given class characterized by the matrix parameters. This makes an important bridge between dynamical and statistical descriptions of quantum chaos. A similarity between the problem of quantum diffusion localization in momentum space, and the well known dual problem of Anderson localization in configurational space of disordered solids, see Anderson (1978) and Fishman *et al.* (1984), is especially clear and instructive in matrix representation.

Consider real Hamiltonian matrices of a general type

$$H_{mn} = H_{nn}\,\delta_{mn} + v_{mn}, \quad m,n = 1,\dots,N, \tag{4.33}$$

where off-diagonal matrix elements $v_{mn} = v_{nm}$ are random and statistically independent, with $\langle v_{mn}\rangle = 0$ and $\langle v_{mn}^2\rangle = v^2$ for $|m-n| \le b$, and are zero otherwise. The most important characteristic of these Wigner band random matrices (WBRM) is the average energy level density ρ defined by

$$\frac{1}{\rho} = \langle H_{mm} - H_{m'm'}\rangle\,, \tag{4.34}$$

where $m' = m-1$. The averaging here and below is understood as performed either on a disorder, i.e. over many random matrices, or within a single sufficiently large matrix. These are equivalent, due to the assumed statistical independence of matrix elements. In other words, many matrices are statistically equivalent to a large one. In general, the quantum numbers m and n are arbitrary, but we will have in mind those related to the action variables, thus considering the quantum structure in momentum space. The basis in which the matrix elements are calculated is usually assumed to correspond to a completely integrable system with N quantum numbers, where N is the number of the degrees of freedom. By ordering the basis states in energy, we can represent N quantum numbers by a single one related to the energy, which is also an action variable.

In the classical limit the definition of WBRM (4.33) corresponds to the standard Hamiltonian $H = H_0 + V$, where the perturbation V is usually assumed to be sufficiently small, while the unperturbed Hamiltonian H_0 is completely integrable.

The quantum model (4.33) is defined by 3 independent physical parameters: ρ, v and

b. The fourth parameter, the matrix size Q, is considered to be technical in this model provided $Q \gg d_e$ (see (4.38) below) is large enough to avoid boundary effects.

In terms of the unperturbed energy E_0, the classical chaotic trajectory of a given total energy E = const fills up the *energy shell* $\Delta E_0 = \Delta V$ with ergodic (microcanonical) measure w_e depending on a particular perturbation function V. In the quantum system this measure characterizes the shape (distribution) of the ergodic eigenfunction (EF) in the unperturbed basis. Conversely, if we keep the unperturbed energy fixed, E_0 = const, the measure w_e describes the band of energy surfaces E = const whose trajectories reach the unperturbed energy E_0. In a quantum system, the measure w_e in the latter case corresponds to the energy spectrum of a Green function (GS) with initial energy E_0. This characteristic was originally introduced also by Wigner (1955) as the 'strength function', the term still in use in nuclear physics. Nowadays, it is also called the 'local density of (eigen)states'.

For a typical perturbation represented by WBRM, w_e depends on the Wigner parameter $q = (\rho v)^2/b$, see Wigner (1955) and also Casati & Chirikov (1995b) and Fyodorov (unpublished),

$$w_e(E) = \begin{cases} \frac{2}{\pi E_{SC}^2}\sqrt{E_{SC}^2 - E^2}, & |E| \le E_{SC}, \quad q \gg 1 \\ \frac{\Gamma/(2\pi)}{E^2+\Gamma^2/4}\,\frac{\pi}{2\arctan[1/(\pi q)]}, & |E| \le E_{BW}, \quad q \ll 1, \end{cases} \tag{4.35}$$

provided $\eta = \rho v \gtrsim 1$, which is the condition for coupling neighboring unperturbed states by the perturbation. In the opposite case, $\eta \ll 1$, the impact of the perturbation is negligible, and is called *perturbative localization.* The latter is a well known *quantum* effect, but not one we are interested in (for chaotic phenomena it was first considered by Shuryak (1976)). What is less known is that for the coupling of *all* unperturbed states within the Hamiltonian band, a stronger condition is required, i.e.

$$\eta \gtrsim \sqrt{b} \quad \text{or} \quad q \gtrsim 1. \tag{4.36}$$

This is a simple estimate in first order perturbation theory. Indeed, the coupling is $\sim V/\delta E$. Within the band in question, energy detuning is $\delta E \sim b/\rho$, while the total random perturbation is $V \sim v\sqrt{b}$, leading to the estimate (4.36). In the opposite case, $q \lesssim 1$, a *partial perturbative localization* takes place. This is also a *quantum* phenomenon, but again not what we have in mind when speaking of quantum localization. The mechanism of perturbative localization is relatively simple and straightforward. This quantum effect is completely absent only in the semicircle (SC) limit of (4.35), where the width of the energy shell $\Delta E = 2E_{SC} = 2\sqrt{8bv^2} = 4\sqrt{2q}E_b \gg E_b$, and $E_b = b/\rho$ is the half-width of the Hamiltonian matrix band in energy. The last inequality allows for *diffusive* quantum motion within the energy shell, as a single random jump is $\sim b \ll \rho\Delta E$. The quantum localization under consideration is just related to the localization (suppression) of quantum diffusion by the interference effects in discrete spectrum, see e.g. Casati & Chirikov (1995b). Notice that the SC width immediately follows from the above estimate for perturbative localization: $\delta E \sim \Delta E \sim v\sqrt{b}$.

In the lower limit of (4.35), that of Breit-Wigner (BW), the full size of the energy shell $\Delta E = 2E_{BW} = 2E_b$ is equal to that of the Hamiltonian band. However, due to the partial perturbative localization explained above, the main peak of the quantum ergodic measure is considerably narrower, with width $\Gamma = 2\pi\rho v^2 = 2\pi q E_b \ll E_b$. This is again in accordance with the simple estimate: $\delta E \sim \Gamma \sim v\sqrt{\rho\Gamma}$.

To my knowledge, the quantum distributions (4.35) were theoretically derived and studied for GSa only. Classically, the measure w_e seems to be the same for both E = const and E_0 = const, as determined by the same perturbation V. One of the main recent results in studies of WBRM, see Casati *et al.* (to be published), is that the classical symmetry between EFs (E = const) and GSa (E_0 = const) is in general lost in quantum mechanics. Namely, in the ergodic case, such a statistical symmetry still persists. However, quantum localization drastically violates the symmetry, producing a very intricate and unusual global structure of quantum chaos.

In a sense, a conservative system is always localized (finite ΔE), even for ergodic motion. This sometimes is the origin of misunderstandings (see e.g. Feingold & Piro (1995)). In fact, such a *classical localization* is a trivial consequence of energy conservation, as was explained above. It persists, of course, in the classical limit as well. Here we are interested in *quantum localization* as explained above. In what follows it will simply be called localization.

As for maps, the localization in conservative systems also depends on the ergodicity parameter λ (see 3.19), where now

$$\lambda = a\frac{b^2}{d_e} = \frac{ab^{3/2}}{4\sqrt{2}c\eta}\,. \qquad (4.37)$$

Here the ergodicity is not related to the total number of states Q, as in maps (3.19), but to that within the energy shell of width ΔE:

$$d_e = c\rho(\Delta E)_{SC} = 4c\eta\sqrt{2b}. \qquad (4.38)$$

The Hilbert dimension d_e is also called the *ergodic localization length*, as it is a measure of the maximum number of basis states (BS) coupled by the perturbation in the case of ergodic motion. The numerical factor $c \approx 0.92$ is directly calculated from the limiting expression (4.35) for a particular definition of d in (3.30). Formally equation (4.38) is only valid in the SC region ($q \gg 1$), but was shown by computations to hold, within the accuracy of a few per cent, down to $q \approx 0.4$.

The parameter λ (4.37) was found in Feingold *et al.* (1991), and implicitely used there (without any reference to ergodicity). Its meaning was explained in detail in Casati *et al.* (1993), where the factor $a \approx 1.2$ was also calculated numerically.

Localization is characterized by the parameter

$$\beta_d = \frac{d}{d_e} \approx 1 - e^{-\lambda} < 1. \qquad (4.39)$$

Here d stands for the actual average localization length of EFs measured according to the same definition (3.30). The empirical relation (4.39) was found in numerical experiments, see Casati *et al.* (1993), to hold in the whole interval $\lambda \leq 2.5$, and even up to $\lambda \approx 7$, see Casati *et al.* (to be published).

In the BW region $d_e = \pi\rho\Gamma = 2\pi^2 bq$ and $\lambda \approx ab/(2\pi^2 q) \gg 1$, as $q \ll 1$ in (4.35) and $b \gg 1$ in a quasiclassical region.

Hence, localization is only possible in the SC domain which was studied in Casati *et al.* (to be published).

The numerical results, see Casati *et al.* (to be published), were obtained from *two* individual matrices: the main one for the localized case, with parameters

$$\lambda = 0.23,\ q = 90,\ Q = 2400,\ v = 0.1,\ b = 10,\ \rho = 300,\ \eta = 30,\ d_e = 500,$$

and an additional one for the ergodic case, with parameters

$$\lambda = 3.6,\ q = 1,\ Q = 2560,\ v = 0.1,\ b = 16,\ \rho = 40,\ \eta = 4,\ d_e = 84.$$

All results are entirely contained in the EF matrix c_{mn}, which relates the eigenfunctions

ψ_m to the unperturbed basis states φ_n:

$$\psi_m = \sum_n c_{mn}\varphi_n, \quad w_m(n) = |\psi(n)|^2 = c_{mn}^2 = w_{mn}, \tag{4.40}$$

in momentum representation, and assuming the eigenvalues $E_m \approx m/\rho$. From the matrix c_{mn} the statistics of EFs and GSa was evaluated. In order to suppress large fluctuations in individual distributions, averaging over 300 in the central part of the matrix was performed in two different ways: with respect to the energy shell center ('global average', localization parameter (4.39) $\beta_d = \overline{\beta}_{\rm g}$), and with respect to the centers of individual distributions ('local average', $\beta_d = \overline{\beta}_{\rm l}$). Furthermore, the average $\langle\beta_d\rangle$ over β_d values from individual distributions was computed.

In the ergodic case, $\lambda = 3.6$, both average distributions for EFs are fairly close to the SC law. This is a remarkable result, as that law was theoretically predicted for *another* distribution, i.e. GSa. More precisely, the bulk ('cap') of the distributions is very close to the limiting SC (4.35), except for the vicinity of the SC singularities. Numerical values of the localization parameter ($\overline{\beta}_{\rm g} = 1.08$, $\overline{\beta}_{\rm l} = 0.94$, $\langle\beta_d\rangle = 0.99$) are in reasonable agreement with the scaled one $\beta_d = 0.97$ for $\lambda = 3.6$, see (4.39).

As expected, GS structure is similar: $\overline{\beta}_{\rm g} = 1.07$, $\overline{\beta}_{\rm l} = 1.06$, $\langle\beta_d\rangle = 0.98$.

For finite q, all distributions are bordered by two symmetric, steep tails which apparently fall off even faster than a simple exponential with characteristic width $\sim b$. A physical mechanism for tail formation is a specific quantum tunneling via intermediate BSs, see Flambaum *et al.* (1994). An asymptotic theory of the tails was developed by Flambaum *et al.* (1994), Wigner (1955) and Silvestrov (1995). Surprisingly, it works reasonably well even near the SC borders.

The structure of the matrix c_{mn} is completely different in the localized case, $\lambda = 0.23$. The EF *local* average is a clear evidence for exponential localization with $\overline{\beta}_{\rm l} = 0.24$, which is again close to the scaled $\beta_d = 0.21$ for $\lambda = 0.23$. However, the *global* average reveals a nice SC (with tails) in spite of localization ($\overline{\beta}_{\rm g} = 0.98$). It shows that, on average, the EFs homogeneously fill up the whole energy shell. In other words, their centers are randomly scattered over the shell.

Unlike the ergodic case, the localized GS structure is quite different from that of EFs. Both averages now well fit the SC distribution ($\overline{\beta}_{\rm g} = 0.98$, $\overline{\beta}_{\rm l} = 0.96$ as compared to $\overline{\beta}_{\rm g} = 0.99$, $\overline{\beta}_{\rm l} = 0.24$ for EFs). So, though GSa look extended, they are localized! This is immediately clear from the third average $\langle\beta_d\rangle = 0.20$. The explanation of this apparent paradox is that even though the GSa are extended over the shell, they are *sparse*, i.e. contain many 'holes'.

In the analysis of the WBRM structure, the theoretical expression (4.38) for the ergodic localization length $d_{\rm e}$ (the energy shell width) was used. In more realistic and complicated physical models this might impede the analysis. In this respect the new method for direct empirical evaluation of $d_{\rm e}$, and hence the important localization parameters β_d and λ in (4.39), from both the average distributions for GSa and the global average for EFs, looks very promising. It was elaborated by Casati *et al.* (to be published).

A physical interpretation of this structure, based upon the underlying chaotic dynamics, is the following. Spectral sparsity decreases the level density of the operative EFs, which is the main condition for quantum localization via decreasing the relaxation time scale, see e.g. Casati & Chirikov (1995b). However, the initial diffusion and relaxation are still classical, similar to the ergodic case, which requires extended GSa. On the other hand, EFs are directly related to the steady state density, see Chirikov *et al.* (1981), both being solid because of a homogeneous diffusion during the statistical relaxation.

This picture allows us to conjecture that for a classically *regular* motion the EFs also become sparse, so that EF/GS symmetry is apparently restored.

5. Conclusion: pseudochaos and traditional statistical mechanics

Quantum chaos is a particular but most important example of a new generic dynamical phenomenon, *pseudochaos*, in almost periodic motion. Statistical properties of motion with discrete spectra is not a completely new subject of research. It goes back to the time of intensive studies on the mathematical foundations of statistical mechanics *before* dynamical chaos was understood (see e.g. Kac (1959)). This early stage of the theory, as well as the whole TSM, was equally applicable to both classical and quantum systems. As for the problem of pseudochaos, one of the most important rigorous results with far-reaching implications was the *statistical independence* of oscillations with incommensurate (linearly independent) frequencies ω_n, such that the only solution of the resonance equation

$$\sum_{n=1}^{N} m_n \omega_n = 0 \tag{5.41}$$

in integers is $m_n \equiv 0$ for all n. This is a generic property of real numbers. In other words, the resonant frequencies (5.41) form a set of zero Lebesgue measure. If we define $y_n = \cos(\omega_n t)$, the statistical independence of y_n means that the trajectory $y_n(t)$ is ergodic in the N-cube $|y_n| \le 1$. This is a consequence of ergodicity of the phase trajectory $\phi_n(t) = \omega_n t$ (mod 2π) on the torus $|\phi_n| \le \pi$.

Statistical independence is the basic property of a set to which the probability theory is to be applied. In particular, the sum of statistically independent quantities,

$$x(t) = \sum_{n=1}^{N} A_n \cos(\omega_n t + \phi_n), \tag{5.42}$$

which is the motion with discrete spectrum, is a typical object of this theory.

However, familiar statistical properties, like Gaussian fluctuations, postulated (directly or indirectly) in TSM, are reached in the thermodynamic limit ($N \to \infty$) only, see Kac (1959). In TSM, this limit corresponds to infinite-dimensional models, see Kornfeld (1982), which provide a very good approximation for macroscopic systems, both classical and quantum.

What is really necessary for good statistical properties of the almost periodic motion (5.42) is a large number of frequencies N_ω, which makes the discrete spectrum continuous (as $N_\omega \to \infty$). In TSM this condition is satisfied by setting $N_\omega = N \to \infty$. The same holds for quantum fields which are infinite-dimensional. In the finite-dimensional quantum mechanics *another* mechanism, independent of N, works in the quasiclassical region $Q \gg 1$. Indeed, if the quantum motion (5.42) (with $x(t)$ replaced by $\psi(t)$) is determined by many ($\sim Q$) eigenstates, we can set $N_\omega = Q$, which is independent of N. The actual number of terms in expansion (5.42) depends on the particular state $\psi(t)$. For example, if it is just an eigenstate, the sum reduces to a single term. This is analogous to some special peculiar trajectories of classical chaotic motion, whose total measure is *zero*. Similarly, in quantum mechanics we have $N_\omega \sim Q$ for *most* states, if the system is *classically chaotic*.

If the motion is regular in the classical limit, the quantity $N_\omega (\ll Q)$ becomes considerably smaller. For example, in the standard map $N_\omega = Q$ in the ergodic case, $N_\omega \sim k^2$ in the case of localization (both cases being classically chaotic, $K > 1$) but only $N_\omega \sim k \ll$

$k^2 \lesssim Q$ for classically regular motion ($K < 1$). The quantum chaos to order transition is not as sharp as the classical one, but the ratio $N_\omega(K > 1)/N_\omega(K < 1) \sim k \to \infty$ increases with the quantum parameter k.

Thus, as far as the mechanism of quantum chaos is concerned, we essentially *come back* from the ergodic theory to an old TSM, with the replacement of the number of the degrees of freedom N by the quantum parameter Q. However, in quantum mechanics, unlike TSM, we are not interested in the limit $Q \to \infty$, which is simply the *classical* mechanics. Here, the central problem is in the statistical properties for *large but finite* Q. This problem does not really exist in TSM, describing macroscopic systems. In *finite-Q* (or finite-N) pseudochaos we have to introduce the basic concept of a *time scale*, see Chirikov *et al.* (1981). This allows for interpretation of quantum chaos as a *new* dynamical phenomenon, related to but not identical with classical dynamical chaos. Hence, the term *pseudochaos*, emphasizing its difference from time asymptotic chaos in the ergodic theory.

In my opinion, the fundamental importance of quantum chaos is precisely in that it reconciles two apparently opposite regimes, regular and chaotic, in the general theory of dynamical systems. Study of quantum chaos helps us to better understand the old mechanism of chaos in multi-dimensional systems. In particular, the existence of characteristic time scales similar to those in quantum systems was conjectured in Casati & Chirikov (1995b).

Is pseudochaos really chaos?

Until recently, even the concept of classical dynamical chaos was rather incomprehensible, especially to physicists. I know that some researchers actually observed dynamical chaos in numerical or laboratory experiments. But did they do their best to get rid of artifacts: noise or other interference! Now the situation in this field is reversed: many researchers insist that if an apparent chaos is not like that in classical mechanics (and in existing ergodic theory), then it is not chaos. This implies sharp disputes over quantum chaos. The peculiarity of the current situation is that in most studies of 'true' (classical) chaos, a digital computer is used. But there only *pseudochaos* is possible, similar to that in *quantum* (not classical) mechanics!

Hopefully, this 'child disease' of quantum chaos will soon be over.

The concept of quantum chaos presented above has been developed in a long-term collaboration with G. Casati, J. Ford, I. Guarneri, F. M. Izrailev and D. L. Shepelyansky.

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