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An algorithmic view of pseudochaos

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Abstract

The controversial concept of *quantum chaos* – the dynamical chaos in bounded mesoscopic quantum systems – is presented as the most important and universal instance of a new generic dynamical phenomenon: *pseudochaos*. The latter characterizes the irregular behaviour of dynamical systems with *discrete* energy and/or frequency spectrum, which include classical systems with *discrete phase space*. The question of randomness is addressed in terms of the algorithmic theory of dynamical systems, using the Arnold cat map for illustration. The relationship between the algorithmic theory and ergodic theory is discussed.

Keywords: Quantum chaos; Pseudochaos; Algorithmic complexity theory; Ergodic theory

1. Introduction: the quantum–classical correspondence

A recent publication [1] has provided the opportunity to reconsider the very controversial problem of the so-called *quantum chaos*, i.e., the statistical properties of bounded mesoscopic quantum systems. In spite of intensive studies into this problem (see, e.g., the conference proceedings [2–10] and also [67–71]), the physical meaning and interpretation of quantum chaos remain vague, to the extent that there is as yet no agreement even on the most basic question: does any chaos at all exist in quantum mechanics? We should clarify from the outset that the trivial affirmative answer to this question (quantum mechanics is fundamentally random) is irrelevant to our discussion. For, even though the random element in quantum mechanics ('quantum jumps') is unavoidable, it can be singled out and separated from the 'proper' quantum processes. The intrinsic randomness in quantum mechanics is related only to a very specific event – the *quantum measurement* – which is, in a sense, foreign to the quantum system itself (see [11–14] for a discussion). This allows us to divide the problem of quantum dynamics into two qualitatively different parts

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- The proper quantum dynamics as described by the time-evolution of the wave function ψ, which obeys some purely dynamical (deterministic) equation, for example, the Schrödinger equation. The discussion below will be restricted to this aspect only.
- The quantum measurement, which includes the registration of the result, whence the collapse of the ψ-function. Because no dynamical description of the quantum measurement is as yet available, this issue remains vague. In particular, there is no agreement on the question of whether this is a real physical problem or an ill-posed one.

Recent advances in the studies of quantum chaos have mostly relied on the aforementioned device of separating out the dynamical part of quantum mechanics. Such a philosophy is accepted – at least implicitly – by most researchers in this field.

While the general aspects of quantum dynamics were extensively studied since the early days of quantum mechanics, the problem of quantum chaos arose only much later, after the simpler phenomenon of classical chaos was discovered and assimilated [11,12]. Classical chaos has been central to the development of the theory of dynamical systems, particularly of ergodic theory [15–17].

As a general mathematical theory, ergodic theory need not be restricted to classical mechanics only, and indeed its application to quantum dynamics has led to surprising results. For instance, it was discovered at an early stage [18–20], and subsequently confirmed [11,12,21], that quantum mechanics does not typically allow 'true' (classical-like) chaos. This is because in quantum mechanics the energy and frequency spectra of any bounded system are discrete, so that the motion is almost periodic. Hence, according to ergodic theory, such quantum dynamics belongs to the limiting case of regular motion, quite the opposite of dynamical chaos.

The ultimate origin of quantum almost-periodicity lies in the discreteness of the phase space itself (or, more formally, in the noncommutative geometry of the latter), which is the basis of quantum physics and which is directly related to the uncertainty principle. At the same time, another fundamental principle – the correspondence principle – requires the transition to classical mechanics in all cases, and in particular when the classical system is chaotic. This dichotomy lies at the heart of the present controversy on quantum chaos.

An insight into this problem [11,12,14,22,23] was gained from the simple observation (well-known in principle, though perhaps not in practice), that the sharp border between discrete and continuous spectrum is physically meaningful only in the limit $|t| \to \infty$, a regime routinely considered in ergodic theory. The study of the *finite time* statistical properties of dynamical systems – quantal as well as classical – thus becomes a new strategic aim; accordingly, the existing ergodic theory will require some modifications, so as to incorporate finite time explicitly.

Over a finite time, a discrete spectrum is dynamically equivalent to a continuous one, thereby providing much stronger statistical properties of motion than one would expect from asymptotic considerations. In some cases, motions with discrete spectrum may exhibit all the statistical properties of classical chaos, albeit only on some finite time scales.

The most important time scale in quantum chaos is the *relaxation time* t_{R}, which refers to one of the main properties of chaos, the statistical relaxation to some steady state. The relaxation time is given by the following general estimate [11,12,22,23] (see also [70])

$$t_{\mathrm{R}} \sim \frac{Q}{\omega} \sim \rho_0 \leq \rho_{\mathrm{H}}. \tag{1}$$

The quantity Q stands for some appropriate quantum parameter, which is large in the semiclassical region, e.g., the total number of states for a bounded quantum motion is $Q \sim \Gamma/(2\pi)^F$, where Γ is the volume of the phase space and F is the number of degrees of freedom. The parameter ω is the characteristic frequency of the motion, while the meaning of ρ_0 and ρ_{H} will be explained below. (In what follows, we always assume that Planck's constant is equal to one.)

The physical meaning of the relaxation time t_R originates directly from the uncertainty principle ($\Delta t \cdot \Delta E \sim 1$), as implemented in Eq. (1). One sees that t_R is bounded by the *Heisenberg time* ρ_H, which is the full average density of energy levels. For $t \lesssim t_R$, the discrete spectrum is not resolved, and the statistical relaxation follows the classical (limiting) behaviour. This is precisely the regime (which is unaccounted for in ergodic theory) where quantum pseudochaos manifests itself. A more accurate estimate of t_R can be obtained by considering the average density ρ_0 restricted to the so-called *operative eigenstates*, i.e., those which are actually present in a particular quantum state ψ, and which control its dynamics.

It is a mathematical expedient to consider in place of finite time relations (of true physical significance), the following *conditional limit*

$$t, Q \to \infty, \qquad \tau_R = \frac{t}{t_R(Q)} = \text{const}, \tag{2}$$

where τ_R is a new dimensionless time. The *double limit* Eq. (2) (unlike the single limit $Q \to \infty$) is not the limit of classical mechanics which, in this representation, holds true for $\tau_R \lesssim 1$ and with respect to the statistical relaxation only. For $\tau_R \gtrsim 1$, the behaviour remains essentially quantum even in the limit $Q \to \infty$, and it is called *mesoscopic*. Specifically, the quantum steady state is generally quite different from the classical statistical equilibrium, in that the former may be *localized* (under certain conditions), i.e., *nonergodic* in spite of classical ergodicity [11,12,22,23,67–70].

Another important difference is found in the *fluctuations*, which are also a characteristic property of chaotic behaviour. By comparison with classical mechanics, the quantum $\psi(t)$ plays, in this respect, an intermediate role between the classical trajectory with large relative fluctuations, and the coarse-grained classical phase space density with no fluctuations at all. Unlike either, the fluctuations of $\psi(t)$ (or rather, those of averages in a quantum state $\psi(t)$) are typically of the order $d_H^{-1/2}$, where $d_H \lesssim Q$ – the *Hilbert dimension* of the state ψ – is the number of operative eigenstates associated with the quantum state ψ. In other words, the chaotic wave function represents statistically a *finite ensemble* of approximately d_H statistically independent systems, even though formally $\psi(t)$ describes a *single* system in a pure state.

The relaxation time scale t_R should not be confused with the *Poincaré recurrence time* t_P, which is typically much longer, and which sharply increases when the size of the recurrence domain decreases. The time scale t_P characterizes large fluctuations for both the classical trajectory and the quantum ψ (but not for the phase density), of which recurrence is a particular case. By contrast, t_R characterizes the average relaxation process. Rare recurrences (the larger the quantum parameter Q, the rarer the recurrences), make quantum relaxation similar to the classical non-recurrent one.

Statistical properties which are stronger than relaxation and fluctuations are related to the local exponential instability of motion. The importance of these stronger properties for statistical mechanics is not completely clear. Nevertheless, in accordance with the correspondence principle, those properties are present in quantum chaos as well, but on a much shorter random time scale t_r

$$\Lambda t_r \sim \ln Q \sim \ln(\omega t_R) \ll t_R. \tag{3}$$

Here, Λ is the classical Lyapunov exponent characterizing the instability rate. This random time scale was discovered and partly explained in [18] (see also [11,12,19,22,23,71]). According to the well-known Ehrenfest theorem, the motion of a narrow wave packet follows an ensemble of classical trajectories as long as the packet remains narrow, and hence it is as random as in the classical limit. Since the instability range is limited both from below (the minimal size of the quantum packet – the coherent state – is of order unity, due to the uncertainty principle) as well as from above (for bounded motions), the time interval t_r of quantum instability is logarithmically short, in agreement with the estimate (3). Nonetheless, it grows indefinitely as $Q \to \infty$, and therefore a temporary, finite time quantum pseudochaos converges slowly to the classical dynamical chaos.

Again, we may consider the conditional limit associated to the random time scale t_{r},

$$t,\, Q \to \infty, \qquad \tau_r = \frac{t}{t_{\mathrm{r}}(Q)} = \mathrm{const}, \tag{4}$$

which gives rise to the new scaled time τ_r. Note that the latter differs from the scaled time τ_R of Eq. (2).

Specifically, if we fix the time t, then in the limit $Q \to \infty$, we obtain the transition to the classical instability in accordance with the correspondence principle, while for Q fixed, and $t \to \infty$, we have the proper quantum time-evolution. For example, the quantum Lyapunov exponent Λ_{q} is given by [11,12,24]

$$\Lambda_{\mathrm{q}}(\tau_r) \to \begin{cases} \Lambda & \text{if } \tau_r \ll 1, \\ 0 & \text{if } \tau_r \gg 1. \end{cases}$$

In the semiclassical region ($Q \gg 1$), the time scale t_{r} is much shorter than the relaxation time t_{R}. This leads to the surprising phenomenon that quantum diffusion and relaxation are *dynamically stable*, contrary to the classical behaviour. Therefore, the instability of motion is in general not important during statistical relaxation: what is crucial for the statistical properties of quantum motion, is the correlation decay, that takes place during the short *initial* random time scale t_{r},

The dynamical stability of quantum diffusion has been demonstrated in striking numerical experiments involving time-reversal [25–27]. In a classical chaotic system, the initial state cannot be recovered by reversing the time-evolution of a sufficiently long orbit, because the unavoidable numerical errors (not random!), are amplified by the local instability. By contrast, under time-reversal, a quantum state will return to the initial state to a very high accuracy, after which the wave packet will begin to spread again.

2. Chaos and pseudochaos

A much studied model for strongly chaotic motions is the so-called *Arnold cat map*, a linear dynamical system on the two-dimensional torus [15–17,28–35]

$$\begin{aligned} q_{t+1} &\equiv 2q_t + p_t \quad (\mathrm{mod}\,1), \\ p_{t+1} &\equiv q_t + p_t \quad (\mathrm{mod}\,1). \end{aligned} \tag{5}$$

The system is exponentially unstable, with positive Lyapunov exponent $\Lambda = \ln(\lambda)$, where $\lambda = (3+\sqrt{5})/2$ is the largest eigenvalue of the mapping. The spectrum of the motion is continuous, implying the decay of correlations.

The canonical variables q and p are real numbers. However, the periodic orbits of the cat map correspond to points with rational coordinates $q = x/N$, $p = y/N$, with N, x, y integers, and x and y reduced modulo N. Clearing the denominator N, one arrives at the *discretized* Arnold cat map

$$\begin{aligned} x_{t+1} &\equiv 2x_t + y_t \quad (\mathrm{mod}\,N), \\ y_{t+1} &\equiv x_t + y_t \quad (\mathrm{mod}\,N). \end{aligned} \tag{6}$$

The phase space is now a $N \times N$ lattice on the torus. The dynamical and statistical properties of the motion are completely changed, and new mathematical tools – mainly from algebraic and probabilistic number theory – are required for their study [31,33,36].

The discrete cat map may describe three rather different dynamical systems

- A *quantum* system, where the classical trajectories lose physical meaning, but are nonetheless used for the calculation of the time-evolution of a quantum state. Loosely speaking, the latter corresponds to an ensemble of discrete orbits.
- A restriction of the *classical* system (5) to a discrete subset of the continuous phase space. Its orbits are periodic orbits of the continuous system (5).
- A discrete dynamical system in its own right, important for computational theories, where arithmetical phenomena – both deterministic and probabilistic – gain significance at the expenses of metric and topological considerations.

The motion's spectrum for the mapping (6) is not only discrete, but equidistant, which implies periodic recurrences for all trajectories of the quantum state [28,32]. Systems of this type are regular and can possess at most the weakest statistical property, ergodicity. For this reason, there is growing consensus that the quantum manifestations of chaos must differ from the classical ones. This has led many to believe that 'true' chaos can only exist in classical mechanics (the authors of [1], however, take a different view – see Section 6).

The aforementioned position is unsatisfactory, in that physics is quantal, and hence the only true physical chaos is quantum chaos. Since the latter is certainly rather different from the former, quantum chaos was termed *pseudochaos* (see, e.g., [13]) to emphasize the difference.

Besides quantum mechanics, there are examples of pseudochaos abound in digital computers, which behave like discrete classical dynamical systems (see, e.g., [13,14]). Their dynamical properties are extremely important, in view of the enormous range of applications. In particular, many studies of classical chaos have relied crucially on computer experiments, where in fact pseudochaos is all, that one normally observes (see Section 3). Therefore, it seems to us that the debate over quantum chaos should be broadened to include considerations about pseudochaos in discrete systems, with particular reference to computer representations of chaotic systems with continuous cooordinates.

The computer is, in a sense, an 'overquantized' system in that any quantity is discrete, while in quantum mechanics only the product of two conjugated variables has that property. The large 'quantum' parameter Q is given here, by the largest representable integer N, whereas the short time scale $t_{\mathrm{r}} \sim \ln N$ (Eq. (3)) is the number of computer digits [22,23]. Owing to discreteness, any dynamical trajectory in a computer becomes eventually periodic, a well-known phenomenon in the theory and design of pseudorandom number generators, from which the term pseudochaos was borrowed [13]. (Precautions are necessary to exclude such computer artifacts in numerical experiments, see, e.g., [37–41].)

The question of randomness in discrete systems may be approached from an algorithmic angle, whereby, one attempts to quantify the computational effort necessary to predict the value of dynamical observables, for instance, the points of an orbit. The amount of computations is typically measured in terms of either the size or the running time of an 'efficient' program which is capable of performing the required calculations. One immediately identifies two extreme situations. On one hand, the smallest of programs may still be as large as the data it generates, which would render such computation as ineffective as storing a result known beforehand and printing it. On the other hand, the fastest of programs may still take as long as the system being simulated, in which case the computation would be as ineffective as observing the physical system – itself an analog computer – evolve. In either case, we have a terminal obstruction to predictability.

The word *random* is routinely linked to incompressibility of information: a sequence is termed random when the shortest program that generates it has (essentially) the same size as the sequence itself. Developing this idea is the main concern of algorithmic complexity theory, whose application to dynamical systems has legitimated the use of the locution deterministic randomness in continuous systems [42–45] (for a clear informal presentation, see [32]). Unlike time-asymptotic ergodic theory, the algorithmic theory does include (and indeed originally developed) the notion of finite time randomness, which is crucial for our discussion (see Section 1). Particularly, in both (classical) chaos as well as pseudochaos, an orbit can be algorithmically random.

The incompressibility of information, and the resulting obstruction to predictability, are directly related (indeed equivalent) to the complete independence of different sections of a random symbolic orbit Section 4 . These sections of orbits are independent not only statistically, but also algorithmically (dynamically), i.e., they cannot be calculated from one another by any finite procedure (algorithm).

Whilst the above meaning of randomness is theoretically quite established, in practice, one's inability to compute is often related to excessive computational time, or excessive storage requirements. Besides numerical experiment with chaotic dynamical systems, there are examples abound in cryptography, where finding computationally intractable problems is the main task. There, one typically looks for problems that can only be solved in *non-polynomial time* [46], which make them unassailable (see Section 3). An example of storage problems is the computation of the mysteriously unpredictable digits of a radical (e.g., $\sqrt{2}$). Even though this can be done by small and fast programs (i.e., Newton's method), during the computation such programs become as large as the output, due to the necessity of storing all intermediate data [47]. (Note that in a theoretical setting the latter problem does not occur, since Turing machines feature infinite storage capacity.)

An important observation is that complex phenomena, which are best described in probabilistic terms, routinely emerge alongside any of the computational difficulties mentioned above. In the rest of the paper, we shall articulate some of these issues, in the context of quantum and discrete dynamics, using the discrete cat map (6) for exemplification. Conforming to current usage, we shall reserve the word 'chaos' to denote the kind of asymptotic obstruction to computability in dynamical systems that derives from excessive program size, while the term (algorithmically) 'random' may describe both chaos and pseudochaos.

3. Time complexity and period functions

The importance of time complexity in discrete dynamics originates from the proximity of the latter to number theory and computer science. In this section, we discuss some obstructions to computability that emerge in irregular discrete dynamical systems such as Eqs. (6) and (8). These problems stem from algorithms with excessive (non-polynomial) running time.

We note that the two-dimensional mapping (6) can be rewritten as the single congruence [29]

$$\zeta_{t+1} \equiv \lambda\zeta_t \pmod N, \tag{7}$$

where both $\lambda > 1$ (the largest eigenvalue of Eq. (6)) and $\zeta = (\lambda - 1)x + y$ are *algebraic integers* [1] (transforming Eq. (6) into Eq. (7) is akin to writing a two-dimensional map in complex coordinates). Eq. (7) is structurally identical to

$$x_{t+1} \equiv bx_t \pmod N, \tag{8}$$

where $b > 1$ and x are ordinary integers. This analogy is strong when N is coprime to b, in which case Eq. (8) is *invertible*. The mapping (8) – the discrete version of the so-called Bernoulli shift [48] – forms the basis for an important class of pseudorandom number generators ([49], Vol. 2). In addition, it is a source of the discrete logarithm problem, which is a computationally intractable problem and which finds widespread use in cryptography [50].

The discrete logarithm has a simple dynamical interpretation: it is the time required to reach a given point x in an orbit, starting from the initial condition x_0. By choosing $x_0 = 1$ in Eq. (8), one sees that $x \equiv b^t \pmod N$, which implies that t is the 'logarithm' of $x \pmod N$ to the base b. By the same token, the discrete logarithm is also closely related to the Poincaré recurrence times in these systems (Sections 1 and 5).

[1] An algebraic integer is a root of an irreducible polynomial with integers coefficients and first coefficient equal to 1.

The running time T of the fastest algorithms known to date is non-polynomial in the input size $\log(N)$

$$T = \exp\left(\sqrt[3]{\log(N)} + c\right). \tag{9}$$

Here c is a constant, and the base b and size N are chosen in such a way that the orbit length be of order N (see below). The closer the run time is to a pure exponential (which in Eq. (9) would imply $T = \mathrm{O}(N)$), the closer is the problem to be solvable only with the 'wait-and-see' algorithm, which amounts to null predictive power. Note that in the computation of the discrete logarithm there are no difficulties with respect to program size.

For this reason, the distinction between polynomial and non-polynomial time is the criterion used in computer science to discriminate between tractable and intractable problems. The cryptographical significance of the discrete logarithm lies in the fact that it is a *trapdoor function*: encryption (the exponential, i.e., computing a point in the orbit) can be performed in polynomial-time, while decryption (the logarithm, i.e., computing the recurrence time) cannot be performed.

Among the most complex objects found in discrete dynamics are the *period functions* $P = P(N)$, which characterize the maximal (or average) period of the orbits, as a function of the system size N [31,51–54]. They play an important role in quantum chaos as well (see Section 4). As a rule, period functions feature very large fluctuations, which in modular systems such as Eq. (6) or Eq. (8) have a crisp arithmetical significance. For instance, in the case of Eq. (8), $P(N)$ is the period of the digits of the rational number x_0/N, expressed in base b, a problem first considered by Gauss ([55], Section 6).

One has the bounds

$$\lfloor \log(N+1) \rfloor \leq P(N) \leq N - 1,$$

where $\lfloor x \rfloor$ is the floor function (the largest integer not exceeding x), and the maximum is attainable (not necessarily attained!) only if N is prime and provided that b is not a square. In the latter case, one has a *space-filling orbit*, reaching every point apart from zero. For the cat map, the corresponding bound is [35]

$$\lfloor \log(\lambda) \rfloor + 1 \leq P(N) \leq 3N.$$

The maximal period is now $\mathrm{O}(N)$, rather than $\mathrm{O}(N^2)$, so that space-filling orbits cannot exist: this is an arithmetical consequence of area-preservation [29]. (However, one can still construct linear dynamical systems of the cat map type, which are invertible on a toral lattice even if their determinant is not unity and which have a space-filling orbit [56].)

The fluctuations of $P(N)$ call for averaging. However, rigorous studies of such averages are notoriously difficult, related as they are to a class of number-theoretic problems, centered around the so-called *Artin's conjecture* [57]. Various heuristic estimates are known, but the 'actual' asymptotic formula for the average order of P

$$\langle P \rangle (N) = \frac{1}{N} \sum_{n=1}^{N} P(n) \sim \frac{N}{\log(N)^{(1+\mathrm{O}(1))\log(\log(\log(N)))}} \tag{10}$$

can only be proved assuming the validity of the so-called generalized Riemann hypothesis (see [1], and references therein). The wide gap existing between what is believed to be true and what can actually be proved, underlines the difficulties in developing an ergodic theory of discrete chaotic systems. By contrast, the ergodic theory of the continuous version of Eq. (8) is much simpler.

Unsurprisingly, the computation of the period-function of the maps (6) and (8), turns out to be non-polynomial, for it requires performing the prime decomposition of N. The run time for factorization of algorithms is similar to that of the discrete logarithm. For this reason, factorization – the inverse of multiplication – is another well-known trapdoor function [50].

The link between non-polynomial time problems, and probabilistic phenomena in discrete dynamical systems is largely unexplored. For the cat map, the discrete logarithm is associated with two types of fluctuations, namely those of Poincaré recurrence times mentioned above (see also Section 5), and spectral fluctuations [30]. Within this context, of note is a recent result [58] establishing a central limit theorem for the propagation of *roundoff errors* in uniform spatial discretization of planar rotations (harmonic oscillator). No polynomial-time algorithm has been found for the corresponding period function, which features wild fluctuations ([54], see also [59]).

4. Discrete pseudochaos

In this section, we review the main construction of the algorithmic theory of continuous dynamical systems, and then characterize the extent to which the asymptotic chaos found in continuous dynamics is suppressed in discrete dynamics. The context is that of a discrete dynamical systems with N states, where the period function $P(N)$ introduced in Section. 3 plays a key role. An alternative approach developed in [1,60] will be discussed in Section 6.

The origin of the algorithmic theory of dynamical systems can be traced to the introduction of the notion of *symbolic trajectory*, due to Hadamard [61]. We start from a partition of the phase space into M cells, each labelled by an integer m. We then consider an orbit $x = x_t$ in correspondence to some discrete instants of time which is again labelled by the integers. At the time t, we record the label m_t of the cell to which the orbit belongs, thereby constructing an infinite sequence of symbols

$$\sigma = \sigma(x_0) = (m_0, m_1, \dots, m_t, \dots). \tag{11}$$

(In an invertible system, σ can be made doubly-infinite.) Such symbolic trajectory is a coarse-grained representation of the original trajectory, obtained by projecting the latter onto a finite set.

Representing orbits as strings of symbols, opened the possibility of applying the notion of algorithmic complexity to the study of motions (see [14,62], for an informal introduction). Following Kolmogorov [42], one introduces the complexity $C(t, x_0)$ of the t-string corresponding to x_0, as the length of the shortest algorithm that computes such symbolic string. The quantity C is determined up to a machine-dependent additive constant. One then considers the limit

$$K(x_0) = \lim_{t\to\infty} \frac{C(x_0, t)}{t}, \tag{12}$$

and the sequence $\sigma(x_0)$ is defined to be *asymptotically random* (or chaotic) if such limit exists and is positive [44,62].

Some strongly chaotic dynamical systems, when represented symbolically with respect to a suitable partition, have the remarkable property that the set of symbolic trajectories is complete, i.e., it contains all possible sequences Eq. (11). From this, it is possible to deduce that most symbolic orbits are random [42,44].

Ergodic and algorithmic theories have a limited but significant overlap, based on a prominent result linking exponential instability of motion with randomness: the Alekseev–Brudno theorem [45,62,63]. For almost all initial conditions x_0 (with respect to some invariant measure μ), we have (cf. Eq. (12)).

$$K(x_0) = h_\mu, \tag{13}$$

where

$$h_\mu = \sum_{\Lambda_i > 0} \Lambda_i \tag{14}$$

is called the *metric entropy*, which has the dimension of a frequency and which characterizes the rate of exponential instability of motion. The summands in Eq. (14) are the positive Lyapunov exponents. In a two-dimensional map like Eq. (5), the above sum contains only one term and one finds that $h_\mu = \Lambda$.

The remarkable relation (13) links explicitly an algorithmic concept (left-hand side) to a probabilistic one (right-hand side). The positiveness of the entropy is then taken as a definition of dynamical asymptotic randomness, thereby justifying the informal inference of that from ergodic theory.

The limit of Eq. (13) says that in order to calculate each successive segment of a symbolic trajectory, one needs new information on the initial conditions of the orbit, at a rate determined by the entropy. Consequently, regardless of how much knowledge one has on the coarse-grained past history of the motion, its future evolution ($t > 0$) still remains unpredictable. Of course, this is true for a symbolic trajectory only, as on the 'exact' orbit it is sufficient to fix a single point.

The remnants of determinism still persist in the symbolic trajectories, in the form of some dynamical correlations which take place within a short *dynamical time scale* t_d, which is determined by the *randomness parameter* [64]

$$R = \frac{|t|}{t_\mathrm{d}} \sim \frac{h|t|}{\ln M}. \tag{15}$$

This allows us to follow in time, the building up of asymptotic randomness, and to make some estimates for a finite time (see below). In particular, we have temporary determinism over the time scale $|t| \lesssim t_\mathrm{d}$ where strong dynamical correlations persist in a symbolic code, so that information about the future evolution of an orbit can be inferred from the result of finite-accuracy observations of an orbit segment. On this relatively short time scale, the trajectory can hardly be termed random. On the other hand, for $|t| \gg t_\mathrm{d}$, almost all symbolic orbits become random, and only a statistical description is possible. Even though, in principle, the equations of motion can still be used to derive all statistical properties without any ad hoc hypotheses, the exact trajectory becomes an elusive entity, which can only be observed, yet neither predicted nor reproduced in any way.

The estimate Eq. (15) may be justified as follows. For a given partition M, the complexity of an individual point of a symbolic trajectory is $C_1 \sim \ln M$ (the number of digits needed to specify an element of the partition). Since successive points within the interval $\sim t_\mathrm{d}$ are essentially correlated, the average complexity per iteration of the map is reduced to $\langle C_1 \rangle \sim C_1/t_\mathrm{d} \sim h$. Hence, $t_\mathrm{d} \sim \ln M/h$, in accordance with Eq. (15).

The definition of randomness as given in Eq. (13) is somewhat weaker than the original algorithmic definition [42,44], as the former allows for some dynamical correlations over the time scale t_d. This is inevitable for a continuous time system. For a map, both definitions coincide if $t_\mathrm{d} < 1 (\ln M \lesssim h)$, or if a very special partition is used [15–17].

The situation is altogether different for discrete dynamical systems. If the size N of the systems is finite, all orbits are (eventually) periodic, and the problem is to estimate the complexity of a typical orbit, for times t not exceeding its period. Note that, for fixed N, the question of asymptotic (long time) randomness has a trivially negative answer here, since every orbit will repeat itself, leading to a logarithmic growth of $C(t)$. Thus, in such a system only pseudochaos is possible, at most.

As in the continuum, each orbit is still determined by its initial point x_0, but specifying the latter requires no more than $\log(N)$ bits of information, whence

$$C(x_0, t, N) = \mathrm{O}(\ln N) \quad \text{for} \quad 0 \le t < P(x_0, N), \tag{16}$$

where $P(x_0, N)$ is the period of the orbit through x_0.

With reference to the limiting procedure (12) and the estimate (16), one sees that in the present class of systems, the question of randomness is intimately related to the asymptotic growth rate of the period function $P(N)$, to be averaged over a set of initial conditions whose density approach unity. In particular, if the orbits have a sufficiently long period, i.e., if $P(N)$ admits a lower bound which grows faster than the logarithm of N for 'typical' values of N

(i.e., possibly excluding a set of zero density), then one would conclude that the discrete motions are ‘non-random’, in the sense that the share of random motion is negligible, i.e., logarithmic in the period of the orbit. This is indeed the case for systems such as Eq. (6) or Eq. (8), as long as one accepts the conjectured asymptotic estimate Eq. (10).

However, from a rigorous viewpoint, the question of the lack of randomness of these discrete orbits – as plausible as it is – still remains unsettled (cf. last section in [1]). It seems though that proving lack of randomness (i.e., finding a super-logarithmic lower bound for $P(N)$) should be considerably easier than proving an asymptotic formula for the complexity, which may well require the full force of Riemann hypothesis!

5. Quantum pseudochaos

As to the quantum version of the mapping (6), there is little to add to the analysis of the classical version presented in Section 4. The time-evolution of a quantum state (in the Wigner representation $W(x, p, t)$) is exactly the same as the classical one [32]. This remarkable peculiarity of a linear quantum map greatly simplifies the studies of the quantum dynamics and chaos. The price is the non-generic global periodicity, whence the difficult problem of the associated period function $P(N)$ (see Section. 3).

The main physical distinction of the quantum cat map lies in the strict restrictions on the quantum state W itself, particularly on the initial conditions of quantum motion. Formally, one can interpret the *quasi-probability* W as an ensemble of ‘trajectories’ which is the analog of the classical phase-space density, apart from possible negative values of W. In this picture, each quantum ‘trajectory’ represents a nonseparable part of the quantum state, with a specific W-value, which, nevertheless, follows exactly the classical trajectory! The number N_W of such elements of the ensemble of trajectories must be sufficiently large: $N_W \gtrsim N$, with the lower bound corresponding to the most localized quantum wave packet, a coherent state. Specifically, the total number of quantum states is $Q = N$, to be compared with the N^2 classical states (initial conditions), each of which is physically distinct from all others. The number of different (periodic) trajectories is much less, being $\sim N^2/\langle P\rangle$, where $\langle P\rangle$ is given by Eq. (10). On the other hand, in the quantum case, the initial state is characterized not only by the initial conditions of the corresponding ‘trajectories’, but also by different values of W on those. Again, this is similar to the classical description in terms of the phase-space density. Besides these quantitative restrictions of the quantum state, there is an additional restriction concerning the shape of the function $W(x, p)$. In particular, it is still unknown whether functions W which are non-negative can exist in the discrete cat map model.

With all the above allowances, the quantum dynamics of the cat map is the same as the discrete classical one, and so all the estimates in Section 4 remain unchanged. In regard to the specific problem of the motion period in both cases, what is important from our perspective is that no matter how short is the random time scale Eq. (3), the latter does grow indefinitely with N, thus providing the correspondence principle which is of fundamental importance in the quantum case, and which is crucial for numerical experiments on computer.

We finally stress that the study of globally periodic systems, natural as for discrete representations of classical systems, remains a degenerate case in the quantum problem. In general, the quantum motion is almost periodic, and it is characterized by quasi-periods or Poincaré recurrences. The latter are extremely long, and are related to very large, and rare fluctuations. The regular statistical processes are characterized by the relaxation time scale, whose dependence on N is given by the estimate Eq. (1).

6. Ordering of orbits

This final section is devoted to a discussion of the controversy between [32] and [1], with the former disproving and the latter proving the random character of the quantum cat map.

The authors of [1] observe that the complexity analysis of periodic orbits depends crucially on the ordering with which the orbits are considered. The concern for ordering originates in ergodic theory, where one typically computes averages with respect to some invariant measure. The periodic orbits constitute a natural discrete sample which may converge – in the sense of probability theory – to a measure. Not only different orderings may correspond to different measures, but crucially, orderings corresponding to the same measure may nonetheless give different readings when it comes to complexity.

In [32], the orbits are ordered according to the system size N, which is just the denominator of the rational points supporting the periodic orbits. Instead in [1], the orbits are ordered according to their increasing minimal period P, and then lexicographically within the same period. Both orderings correspond to the Lebesgue measure.

The central question is to establish which portion, if any, of a 'typical' periodic trajectory is algorithmically random. As discussed in Section 4, the ordering by system size yield (essentially) a logarithmic compressibility of information, whence lack of randomness, as long as one accepts the validity of the estimate Eq. (10).

The ordering by period leads to randomness can be seen from the following considerations. To find the periodic points x_0 of period P, one solves the (continuous version of) congruence Eq. (8) for x_0, for fixed P. One finds $x_P \equiv x_0 \equiv b^P x_0 (\mathrm{mod}\, 1)$, whence

$$x_0 = \frac{m}{b^P - 1}, \tag{17}$$

where m is an integer in the range $0 \leq m < b^P - 1$. When m is coprime to $b^P - 1$, the fraction on the right-hand side of Eq. (17) is reduced, whence $N(P) = b^P - 1$ is exponential in P, so that such a point of period P requires $\mathrm{O}(P)$ bits of information to be specified. The set of reduced fractions of the form Eq. (17) has positive density and is uniformly distributed [65], and so its typical member is random. Consequently, the non-repeating part of a typical periodic orbit of period P is also random.

On the other hand, when m and $b^P - 1$ are not coprime, cancellation will occur in the fraction representing x_0. Such values of m correspond to periodic orbits of period P that live on much smaller lattices, which carry most of the weight when enumerated by system size.

The choice between the two orderings depends on the physical system under consideration. In [1], the classical version of the model (5) was considered, whose analysis was based on the properties of individual trajectories. In this case, the ordering by period emerges from ergodic theoretic considerations (even though enumeration by system size seems to be preferable to us from the physical point of view – see Section 3).

Instead in quantum mechanics, the ordering by system size N adopted by [32] is plainly unavoidable, being inextricably linked to the nonseparable quantum 'trajectories' Section 5, and to the semiclassical limit $N \to \infty$. The ordering by period would instead correspond to an artificial collection of distinct systems, each characterized by a specific value of the parameter N, and united solely by the existence of some trajectories with a given period P in each of them. We see no physical meaning in such an arrangement of quantum 'orbits'.

It seems to us that the authors of [1], apparently unaware of the quantum chaos debate, have applied to a quantum problem a machinery which is as impeccable mathematically as it is inappropriate physically, and after deducing with it that the quantum model (5) is chaotic, have accused the authors of [32,66] of 'misinterpretations'.

We instead believe that the misinterpretation lies with [1], and that their criticism of [32,66] is simply irrelevant to the question addressed by J. Ford and coworkers.

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