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NONLINEAR PHYSICS

Separatrix Conservation Mechanism for Nonlinear Resonance in Strong Chaos

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Abstract—We propose a simple, approximate theory of the fairly general mechanism for the separatrix conservation of nonlinear resonance, which leads to the complete suppression of global diffusion despite the strong local chaos of motion. This theory allows the separatrix splitting angle to be plotted against system parameters and, in particular, yields their values at which the separatrix remains unsplit. We present the results of our numerical experiments confirming theoretical conclusions for a certain class of dynamical Hamiltonian systems. New features of chaos suppression have been found in such systems. In conclusion, we discuss the range of application of the proposed theory. *© 2001 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION: UNEXPECTED STABILITY OF THE SEPARATRIX OF NONLINEAR RESONANCE

The dynamics of nonlinear Hamiltonian systems is governed by the interaction of nonlinear resonances, with each of them, in contrast to a linear resonance, occupying a relatively small region of phase space bounded by the so-called separatrix under a small perturbation (see, e.g., [1-4]). For a single resonance, the separatrix is the trajectory (in general, a surface) that separates phase oscillations (inside the resonance) from phase rotation (outside the resonance).[1] In fact, these are two spatially coincident branches corresponding to going forward and backward in time, respectively. Each branch is a continuous trajectory with an infinite period of motion that goes out of an unstable equilibrium position (saddle) and then asymptotically approaches it. In a typical (i.e., nonintegrable) Hamiltonian system, any arbitrarily small perturbation, for example, from other (at least one) nonlinear resonances, causes the separatrix to split up into two intersecting branches, which go out of the saddle toward each other as before but no longer return to it. The free ends of the branches of the split separatrix form an infinite number of loops with a limitlessly increasing length; these loops fill a narrow region near the unperturbed separatrix to form the so-called chaotic layer. Overlapping of the chaotic layers of all system resonances gives rise to global chaos and, in particular, to diffusion bounded only by the exact integrals of motion, for example, by a surface of constant energy.

The conditions for the formation of global chaos depend on both the magnitude and smoothness of the perturbation (in phase). The latter is characterized by the rate of decrease in its Fourier amplitudes. For an analytic perturbation, the decrease is exponential. In this case, there is always such a critical perturbation magnitude ϵ_{cr} that global diffusion emerges only at $\epsilon \gtrsim \epsilon_{cr}$. If, alternatively, $\epsilon \lesssim \epsilon_{cr}$, chaos is localized in relatively narrow chaotic layers that are formed at any $\epsilon > 0$. For $N > 2$ degrees of freedom, global diffusion is still possible, but only for special initial conditions and with a very low rate (the so-called Arnold diffusion [2]). When $\epsilon \longrightarrow 0$, both the rate of diffusion and the measure of its range decrease exponentially in parameter $1/\epsilon$.

The pattern of motion significantly changes for a smooth Hamiltonian perturbation, which has only a finite number γ of continuous derivatives (see, e.g., [5] and references therein). In this case, there is such a critical smoothness γ_{cr} that global diffusion is suppressed under a sufficiently small perturbation only at $\gamma > \gamma_{cr}$ [6]. Significantly, the converse is generally not true; i.e., at $\gamma < \gamma_{cr}$, global diffusion is commonly observed in numerical experiments, but we know examples when the trajectory remained localized in a part of phase space over the entire long computational time (see, e.g., [7, 8]).

Recently, Ovsyannikov [9] has found a relatively simple, exactly solvable example [see (2.1) below] for which he managed to prove the theorem on the conservation of a single (unsplit) separatrix at special values of the perturbation parameter. This theorem is given in its entirety in [10] (Appendix). Intensive studies of

[1] Below, we use the canonical action-phase variables.

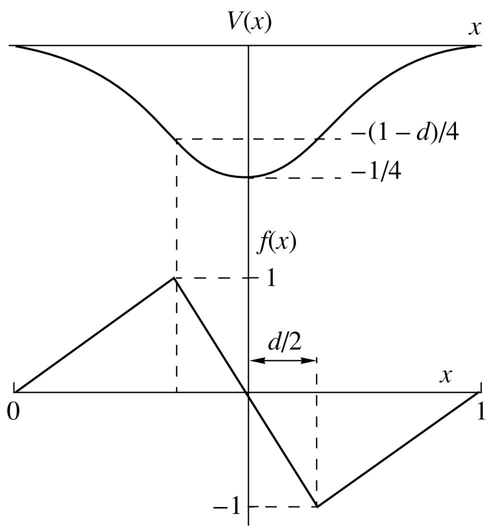

Fig. 1. A scheme of the potential $V(x)$ and force $f(x) = -dV/dx$ with a period of 1 for the family of models (2.2) with parameter d.

model (2.1), which we call below a symmetric piecewise linear mapping, immediately showed that the conservation of the separatrix in strong chaos rather than in the exceptional case of a completely integrable system without any chaos whatsoever turned out to be the most important and unexpected thing in this theorem. Moreover, at the special values of the perturbation parameter found both by Ovsyannikov and by one of us [10–12], the separatrices of nonlinear resonances not only remain unsplit, but form impenetrable barriers for other trajectories; i.e., they completely suppress global diffusion. This takes place despite the fact that the perturbation smoothness in the model of a symmetric piecewise linear mapping is considerably smaller than its critical value, and one might expect global diffusion at any perturbation magnitude.

Meanwhile, an examination of the literature has shown that the same model was mathematically analyzed in detail by Bullett [13] (see also [14]) well before. Although Ovsyannikov proved his theorem independently, this coincidence of the models is not fortuitous, because the solution of a linear (even if piecewise) mapping considerably simplifies the problem. Note that even Ovsyannikov's linear mapping can be completely solved only when the separatrix is conserved, because, otherwise, the two branches of the split separatrix form random trajectories. For the same reason, the model of a symmetric piecewise linear mapping cannot be simplified to a purely linear Arnold-type mapping, in which the separatrices of nonlinear resonances are always split (see Sect. 3). Bullett's and Ovsyannikov's mathematical analyses are therefore restricted to the invariant curves of a new type (with a rational rotation number ν, including the separatrices) themselves, whose first examples were given in [14].

In this paper (as in our previous papers on this subject [10-12]), we rely mainly on numerical experiments, which also allow us to investigate the vicinities of the invariant curves both under various initial conditions of motion and for various model parameters. In this regard, our approach is similar to the study of Hénon and Wisdom [8] for a different model.

2. THE MODEL

Ovsyannikov considered a difference equation that was equivalent to the following two-dimensional mapping in canonical variables "action p–phase x":

$$\bar{p} = p + Kf(x), \quad \bar{x} = x + \bar{p}. \qquad (2.1)$$

Here, $K = \epsilon > 0$ is the perturbation parameter (not necessarily small), and the "force" $f(x)$ has the form of an antisymmetric ($f(-y) = -f(y)$, $y = x - 1/2$) piecewise linear "saw" with a period of 1 [see (2.2) below].

Perhaps, the most unexpected thing in this example is that the smoothness of the Hamiltonian (generating function) for mapping (2.1), $\gamma = 1 < \gamma_{cr} \leq 4$ [6], is considerably smaller than its critical value. In other words, for a certain countable set of special values, $K = K_m$, the unsplit separatrix is "immersed in the sea" of strong chaos; nevertheless, it is conserved and blocks global diffusion [10, 11]!

Since an exact K cannot be specified on a computer, the next crucial step was to analyze the behavior of the (split) separatrix and other trajectories for a small deviation, $|K - K_m| \longrightarrow 0$, which is possible only in numerical experiments. Even the first studies [12] showed that the separatrix splitting angle changed sign with difference $K - K_m$; this angle smoothly passed through zero at odd m and abruptly changed sign at even m (see Fig. 1 from [11] and Fig. 2 below). First, this allowed a set of other special K_m at which the separatrix was conserved to be found immediately and easily. At the same time, such an unusual behavior of the splitting angle also suggested a dynamical mechanism for the conservation of the separatrix, which is the main subject of our discussion here.

It is convenient to simultaneously consider the whole family of sawtooth perturbations specified by the force[2]

$$f(x) = \begin{cases} 2x/(1-d), \text{ if } |x| \leq (1-d)/2 \\ -2y/d, \text{ if } |y| \leq d/2, \end{cases} \qquad (2.2)$$

where $y = x - 1/2$ and $d < 1$ is the distance between the saw teeth $|f(x)| = 1$ at points $y = y_\pm = \pm d/2$. The best studied special case of a symmetric piecewise linear mapping corresponds to $d = 1/2$. At these two points, the

[2] A similar family is briefly mentioned in [13].

force has a singularity, a discontinuity of the first derivative $f' = df/dx$:

$$\Delta f' = \pm\frac{2}{d(1-d)}. \qquad (2.3)$$

The original idea behind the separatrix conservation mechanism is that the perturbation (force) has two singularities, which can interfere between themselves and, in particular, can compensate each other or cancel out, a term introduced in [8] where such a mechanism was apparently first proposed and used. Our approach is peculiar in that we are interested in the action of this mechanism (and give a theory) directly for the separatrix of nonlinear resonance, whereas in [8] such cancellations were determined for arbitrary trajectories and used as a heuristic consideration to search for possible invariant curves (not separatrices) among them by numerical experiments.

To construct a theory, it is convenient to pass from the initial mapping (2.1) to a continuous system with a Hamiltonian that explicitly depends on time (see [2–4, 10]):

$$\begin{aligned} H(x, p, t) &= \frac{p^2}{2} + KV(x)\delta_1(t) \\ &= H_0(x, p) + H_1(x, t), \end{aligned} \qquad (2.4)$$

where $\delta_1(t)$ denotes the δ function of period 1. The unperturbed Hamiltonian

$$H_0 = \frac{p^2}{2} + KV(x) \qquad (2.5)$$

describes the main (integer) resonance in (2.1), and

$$H_1(x, t) = KV(x)(\delta_1(t) - 1) \qquad (2.6)$$

describes its perturbation (with the same period $T_1 = 1$ and frequency $\Omega = 2\pi/T_1 = 2\pi$) from all the remaining integer resonances.

The potential of force (2.2) is

$$\begin{aligned} V(x) &= -\int f(x)dx \\ &= \begin{cases} -x^2/(1-d), & \text{if } |x| \le (1-d)/2 \\ (4y^2-d)/4d, & \text{if } |y| \le d/2. \end{cases} \end{aligned} \qquad (2.7)$$

The maximum potential $V_{max} = 0$ determines the unperturbed separatrix of the main resonance:

$$p_s(x) = \pm\sqrt{-2KV(x)}, \qquad (2.8)$$

while its minimum $V_{min} = -1/4$ gives the total depth U of the unperturbed potential well:

$$U = K(V_{max} - V_{min}) = \frac{K}{4}. \qquad (2.9)$$

Perturbation (2.6) is peculiar in that it is of the order of the unperturbed Hamiltonian, irrespective of the perturbation parameter $K \longrightarrow 0$. Nevertheless, the perturbation theory is generally applicable if the other perturbation parameter is large,

$$\lambda = \frac{\Omega}{\omega_0} \gg 1. \qquad (2.10)$$

Here, $\omega_0 = \sqrt{2K/d}$ is the frequency of small oscillations at the main resonance (2.5) and $\Omega = 2\pi$ is the frequency of the external perturbation. This adiabaticity parameter governs the separatrix splitting. Using the term adiabaticity in this case emphasizes that the effect of a high-frequency perturbation is qualitatively the same as that of a low-frequency perturbation.

To investigate the motion near the separatrix, let us first determine the change in unperturbed Hamiltonian (2.5) for the period of motion in a close vicinity of the unperturbed separatrix. Following [10], we obtain

$$\begin{aligned} \Delta H_0 &= \int_{-\infty}^{\infty} dt\{H_1, H_0\} \\ &\approx K\int_{-\infty}^{\infty} dt\, p_s f(x_s)(\delta_1(t) - 1). \end{aligned} \qquad (2.11)$$

In the latter expression, the motion along a trajectory close to the separatrix was approximated by the motion along the unperturbed separatrix (hence the infinite integration limits).

Since the force $f(x)$ has two singularities (2.3) at points $y_\pm = \pm d/2$, we integrate (2.11) twice by parts, so that

$$\frac{d^2 f(t)}{dt^2} \approx \frac{d^2 f(y)}{dy^2}p^2 = p^2\Delta f'\delta_1(y - y_\pm), \qquad (2.12)$$

where $p = dx/dt$, and only the principal term with the δ function was retained. As a result, we obtain

$$\Delta H_0 \approx Kp_\pm^2\Delta f'[\psi(t_+) - \psi(t_-)] = \frac{K^2}{d}\Delta\psi, \qquad (2.13)$$

where $t_\pm$ are the passage times of the singularities at points $y_\pm$, and the function $\psi(t)$ is given by

$$\ddot{\psi} = \frac{d^2\psi}{dt^2} = \delta_1(t) - 1. \qquad (2.14)$$

To calculate the difference $\Delta\psi = \psi(t_+) - \psi(t_-)$, we change over to new variables Δ and t_0, where

$$\begin{aligned} \Delta &= \frac{t_+ - t_-}{2} = \sqrt{\frac{d}{2K}}\arcsin\sqrt{d} \\ &= \frac{\lambda}{2\pi}\arcsin\sqrt{d} \end{aligned} \qquad (2.15)$$

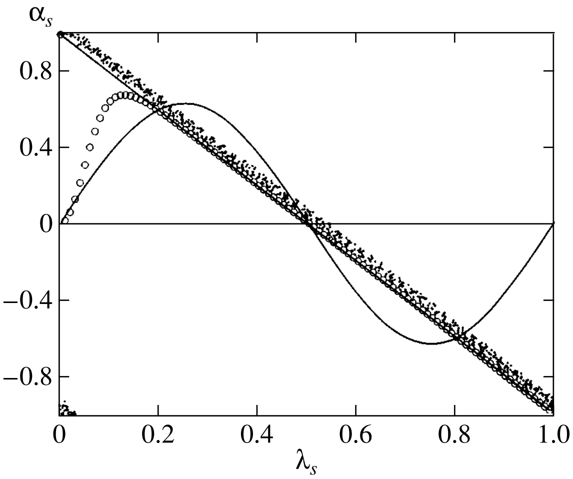

Fig. 2. Periodic dependence of the separatrix splitting angle α on parameters K and d in normalized variables $\alpha_s(\lambda_s)$ (2.20): $d = 0.25, 0.5, 0.75, 0.999$ (dots), $d = 0.01$ (circles) as constructed from our numerical calculations. The solid straight line and the curve represent, respectively, theory (2.19) and the first approximation $\delta_1(t) - 1 \approx 2\cos(2\pi t)$ [see (2.11)]. The argument λ_s is taken modulo 1, so all points (but not circles!) represent many periods of the dependence $\alpha(K, d)$ (see text).

is half the time of motion between the singularities, and

$$t_0 = \frac{t_+ + t_-}{2} \tag{2.16}$$

is the time when the potential minimum is passed ($y = 0$), where the intersection of the branches of the split separatrix is usually analyzed.

In general, $\Delta\psi$ is not factorized in these variables, but this is possible under an additional constraint: $|t_0| \le \Delta$ (> 0). In that case,

$$\Delta\psi = t_0(1 - 2\Delta). \tag{2.17}$$

The angle $\alpha \ll 1$ between the branches of the split separatrix is given by an approximate formula (see [12, 15]),

$$\tan\alpha \approx \alpha \approx \frac{dp}{dy} \approx \frac{dH_0}{p_0^2 dt_0} \approx \frac{2K}{d}(1 - 2\Delta). \tag{2.18}$$

Here, we use the following relations: $dp = dH_0/p$, $dy = pdt$, and

$$\frac{dH_0}{dt} = \Delta\dot{\psi} = 1 - 2\Delta,$$

where all quantities are taken at the point of intersection of the separatrix branches ($y = 0$). The dependence $\alpha(K, d)$ assumes an extremely simple form:

$$\alpha_s \approx (1 - 2\lambda_s) \tag{2.19}$$

in the transformed variables

$$\alpha_s = \alpha\frac{d}{2K} = \alpha\frac{\lambda^2}{4\pi^2},$$
$$\lambda_s = \Delta = \frac{\lambda}{2\pi}\arcsin\sqrt{d} \quad (\text{mod } 1). \tag{2.20}$$

We emphasize that the oscillations of $\alpha(K)$ resulted from the two interfering singularities in the Hamiltonian.

Relation (2.19) is the main result of our analysis. It explains and describes the new phenomenon of the suppression of separatrix chaos and, hence, global diffusion in a certain class of Hamiltonian systems.

A comparison with the results of our numerical experiments is shown in Fig. 2. The very high accuracy of the simple theory (for $K \ll 1$) is limited by a small shift in critical values $K = K_m$. In Ovsyannikov's example (a symmetric piecewise linear mapping, $d = 1/2$), it can be derived even without numerical experiments by using exact expressions for K_m, both predicted in [9] and found later in [12]:

$$K_m = \frac{\pi^2}{16m^2}\left(1 - \frac{\pi^2}{48m^2} + \dots\right) \approx \frac{\pi^2}{16m^2}. \tag{2.21}$$

The last term represents our theory, and a first-order correction is given in parentheses. The theory also explains the unexpected discontinuity of the function $\alpha(\lambda_s)$ at $\lambda_s = 0$ (mod 1) (but not at $\lambda_s = 1/2$), which was found and discussed from a different point of view in [12].

Figure 2 also shows an even simpler approximation with only the first term of the Fourier expansion $\delta_1(t) - 1 \approx 2\cos(2\pi t)$ being retained; it represents the critical values K_m equally well, but does not reproduce the discontinuity in the function of angle α.

The simple relation (2.19) does not give a full picture for the entire family (2.2) either, as demonstrated by the example with a small value of $d = 0.01$ in Fig. 2 (circles). The separatrix splitting is thus seen to be nonsymmetric when $d \longrightarrow 0$ and $d \longrightarrow 1$. On the other hand, it follows from expression (2.2) for the force that the symmetry is preserved when both parameters of the mapping family change: $d \longrightarrow 1 - d$ and $K \longrightarrow -K$. Consequently, changing the sign of K also causes the behavior of the separatrix to change qualitatively. The symmetry is preserved only in the special case of $d = 1/2$, i.e., for the model of a symmetric piecewise linear mapping.

3. THE LIMIT $d \longrightarrow 0$: DISCONTINUITY IN FORCE

Let us first consider the limiting case $d = 0$, $K > 0$, where the force function $f(x)$ experiences a discontinuity (see Fig. 1). The limit differs qualitatively in that the two singularities of the potential at $d > 0$ now merge into one. Consequently, according to our theory, the separatrix splitting angle does not change sign; i.e., the separatrix splits up at any $K > 0$.

Figure 3 shows the results of numerical experiments both in the limit $d = 0$ itself (circles) and in its close vicinity $d = 0.001$ (triangles) and $d = 0.01$ (dots). The dependence $\alpha(K)$ proper is given, because the adiabaticity parameter $\lambda = \pi\sqrt{2d/K} = 0$ loses its meaning at $d = 0$. First, the passage to the limit $d \longrightarrow 0$ in model (2.2) is seen to be continuous with an empirical boundary [by maximum $\alpha(K)$] at

$$K \sim K_B \sim 7d. \tag{3.1}$$

The physical reason why relation (2.18) becomes inapplicable for $K \gtrsim K_B$ is that in deriving it, we ignored the change in velocity between the two singularities through the action of the first of them and the change in transit time Δ between them [see Eqs. (2.13) and (2.15)]. In the previous variables, the transition between the two modes is also shown in Fig. 2 for $d = 0.01$ (circles). We emphasize that there is a deviation from (2.18) only for $K \gtrsim K_B$ and that it is not repeated periodically as dependence (2.18) (see the circle in the upper left corner of Fig. 2). Thus, in the limit $d = 0$, the function of splitting angle actually does not change sign, and, consequently, the separatrix always splits up.

The same method as for $d \neq 0$ (Sect. 2) may be used for a quantitative analysis. The only difference is that the force itself now has a discontinuity, $\Delta f(x) = -2$, and it will therefore suffice to integrate (2.11) by parts only once. We have

$$\begin{aligned}\Delta H_0 &\approx K\int_{-\infty}^{\infty} dt\, p_s f(x_s)(\delta_1(t) - 1)\\ &\approx K p_0 \Delta f \dot{\psi}(t_0) \approx -\sqrt{2}K^{3/2}\dot{\psi}(t_0).\end{aligned} \tag{3.2}$$

Here, $p_0 \approx \sqrt{K/2}$ as before and

$$\dot{\psi}(t) = \frac{1}{2} - t \quad (\mathrm{mod}\ 1) \tag{3.3}$$

[see (2.14)]. However, the simple expression (2.18) in the small-angle approximation is now no longer applicable, because the following singularity arises when differentiating with respect to t_0:

$$\frac{dH_0}{dt} = -Kp_0(\delta_1(t) - 1), \tag{3.4}$$

the derivative is taken at two values: $t = t_0 = 0$ and 1/2 when $\Delta H_0 = 0$ [the intersection of the separatrix branches, formula (3.3)]. Each of these values determines the inclination of the corresponding separatrix

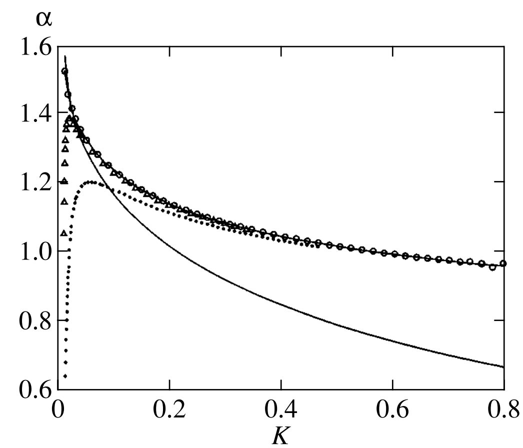

Fig. 3. A plot of separatrix splitting angle α versus parameter K for $d = 0.01$ (dots), 0.001 (triangles), and $d = 0$ (circles) as constructed from our numerical calculations. The lower and upper curves represent the approximate theory (3.6) and the exact theory (3.10), respectively.

branch with respect to the x axis (see Fig. 4). The singularity arises at $t = 0$ and corresponds to $\alpha_0 = \pi/2$ ($\tan\alpha_0 = \infty$). The angle of the other branch β is given by [cf. (2.18)]

$$\tan\beta = \frac{dp}{dx} \approx \frac{1}{2}\frac{dH_0}{p_0^2 dt_0} \approx \frac{K}{p_0} \approx \sqrt{2K}. \tag{3.5}$$

The factor 1/2 of the derivative emerges because ΔH_0 is calculated relative to the unperturbed separatrix (the solid broken line in Fig. 4), while the angle β (as α_0) is taken relative to the x axis. The signs of the angles are determined with the consideration that the branch with α_0 corresponds to moving forward in time, while the other branch corresponds to moving backward in time (see [11]). Finally, we obtain for the angle between the separatrix branches

$$\alpha(K) = \alpha_0 - \beta \approx \frac{\pi}{2} - \sqrt{2K}. \tag{3.6}$$

This simple dependence is indicated in Fig. 3 by the solid line. It agrees well with our numerical experiments for small K; however, the error increases with K, reaching about 40% at $K \approx 0.6$. At larger K, the entire simple pattern of the separatrix splitting for an individual resonance loses its meaning because many resonances overlap (see below).

The error at large K is attributable to the approximate use of the unperturbed separatrix [see (2.8) and (2.7)]

$$p_s(x) = \pm p_0(1 - 2|y|) \tag{3.7}$$

with an amplitude $p_0 \approx \sqrt{K/2}$ when calculating integral (3.2). An interesting feature of the system under consideration is that the unperturbed separatrix (two straight

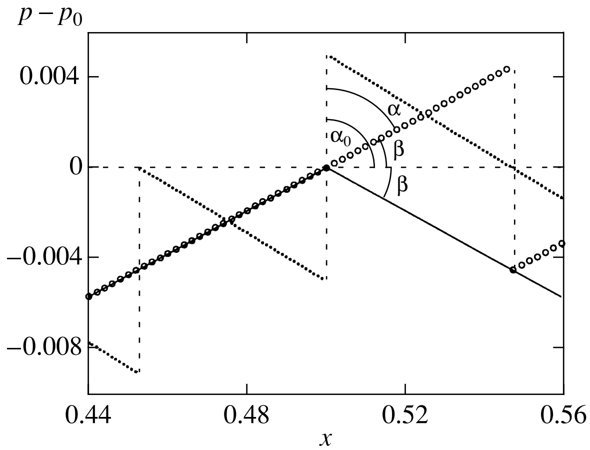

Fig. 4. An example of the separatrix splitting at $K = 0.005$ and $d = 0$: the solid line broken at $x = 0.5$ indicates the unperturbed separatrix (3.7); the separatrix branches are represented by dots (forward in time) and circles (backward in time); the breaks in the branches are connected by the dotted line showing a sequence of points; $p_0 \approx 0.04756$ is the ordinate of the point of intersection (3.8).

lines) retains its shape under the action of a perturbation (Fig. 4). This allows an exact value of $p_0(K)$ to be calculated for any K from the eigenvectors of initial mapping (2.1) at unstable fixed point $x = p = 0$. As a result, we obtain

$$p_0(K) = \frac{K}{\sqrt{2K + K^2} + K}. \tag{3.8}$$

However, when this expression is substituted into (3.5), the agreement even worsens rather than improving:

$$\tan\beta \approx \frac{K}{p_0(K)} > \sqrt{2K}. \tag{3.9}$$

The reason lies in another approximation of integral (3.2), retaining the contribution from the jump in force $\Delta f(x)$ alone. Again, because of the peculiar singularity of the unperturbed separatrix at $d = 0$, the angle β can be calculated exactly without any integration directly from the results in Fig. 4: $\tan\beta = 2p_0(K)$. We thus obtain

$$\alpha(K) = \frac{\pi}{2} - \arctan\left(\frac{2}{1 + \sqrt{1 + 2/K}}\right). \tag{3.10}$$

This expression represents the most accurate result of our theory (the upper solid line in Fig. 3), which is in excellent agreement with the numerical experiment (circles) up to the point at which the resonances begin to overlap. Since mapping (2.1) is periodic not only in x but also in p (and with the same period of 1), there is an infinite set of integer resonances at $p(0) = n$ and $H_0 = n^2/2$, where n is any integer, positive, negative, or zero. The latter special case is considered in this paper. The separatrices of adjacent integer resonances begin to overlap at $p_0 = 1/2$, which destroys their structure. Formally, this occurs only in the limit $K \longrightarrow \infty$ (3.8). Actually, however, this destruction begins much earlier because of the overlapping with intermediate fractional resonances (see [11]). Note also that, formally, the splitting angle is always $\alpha > \pi/4 \approx 0.785$ [see (3.10)]; however, actually and for the same reason, the regular dependence $\alpha(K)$ abruptly cuts off even at $\alpha \approx 0.96 \approx 55°$, $K = K_{cr} \approx 0.8$, and $p_0(K_{cr}) \approx 1/3$ (Fig. 3). For $K > K_{cr}$, the separatrix branches become so unstable that the splitting angle cannot be reliably measured. Interestingly, the deviation of the ordinate of the point at which the separatrix branches intersect $p_0(K)$, according to (3.8), in this range (up to $K = 1.24$) does not exceed 1%. However, this is enough for a strong and irregular distortion of the separatrix branches.

Thus, even at relatively large $K < 0.8$, the separatrix splitting angle is far from reaching zero, let alone changing sign; hence, the separatrix always splits up.

In [13], the limit $d \longrightarrow 0$ was also considered briefly but only for $K < 0$ (in our notation). The inverse limit, $d \longrightarrow 1$, loses its meaning in [13], because the family of mappings is defined there in such a way that in this case, the force $f(x) \longrightarrow 0$ vanishes. In our case, dependence (2.19) is preserved, at least up to $d = 0.999$ (Fig. 2). It should be noted, however, that the pattern of motion qualitatively changes in the limit itself ($d = 1$), because the motion along the unperturbed separatrix represents simple harmonic oscillations [see (2.7) and Fig. 1]

4. CONCLUSION: HOW TYPICAL IS THE CONSERVATION OF THE SEPARATRIX?

We have proposed and tested a simple theory of a new unexpected phenomenon—the conservation of the separatrix of nonlinear resonance in strong chaos on most of the phase plane of a dynamical system [9–13]. The mechanism of this phenomenon is based on a simple idea of the interference (in particular, cancellation [8]) of several singularities in the Hamiltonian of a dynamical system, which govern the separatrix splitting for nonlinear resonance. Numerical experiments and theoretical analysis were carried out for the family of 2D mappings (2.1) in the simplest case of two singularities, which also included the first example of a symmetric piecewise linear mapping [9, 11, 13]. Our study not only confirmed and explained this mechanism, but also allowed a simple theory to be developed to calculate both special values of $K = K_m$ and the dependence of the separatrix splitting angle $\alpha(K)$ over wide ranges of K and d [see (2.19), (3.6), and (3.10)].

We separately considered the passage to the limit $d \longrightarrow 0$, in which the separatrix splits up at any $K > 0$ (Sect. 3). In the opposite case $d \longrightarrow 1$, relation (2.19) remains valid, at least up to $d = 0.999$ (Fig. 2). It should be noted, however, that the pattern of motion qualitatively changes in the limit itself ($d = 1$). First, the motion along the unperturbed separatrix represents simple harmonic oscillations with a finite period, $T_1 = \pi\sqrt{2/K}$ [see (2.7) and Fig. 1]. In addition, the trajectory inside the resonance ($H_0 < 0$) does not reach the singularity of the potential at point $x = 0$ (mod 1) at all. Finally, our preliminary numerical experiments in this limit strongly suggest that the measure of the chaotic component decreases rapidly with decreasing K. It would be of great interest to continue the analysis of this special dynamical system.

We have considered only the integer resonances with $p(0) = n$, where n is any integer. The fractional resonances with $p(0) \approx n/q$ are known (see, e.g., [2, 3, 11]) to have the same structure in appropriate variables. Therefore, one might expect such a mechanism and its simple theory to be also applicable to fractional resonances. If confirmed, we hope in the immediate future, this would allow the conservation of the separatrix for fractional resonances predicted in [13] and revealed by numerical experiments [10–12] to be explained. Although the set of all fractional resonances on the p axis is dense everywhere, the set of all special values of $K = K_{qn}$ at which the separatrix is conserved is not dense [13]. However, its mean density is large enough, and one might expect the strong (although incomplete) suppression of global diffusion at any K. This hypothesis is additionally confirmed by the large number of periodic invariant curves predicted in [13], both ordinary ones with an irrational rotation number ν and new ones with a rational $\nu \neq 0$. We surmise that the emergence of the latter can be interpreted as the suppression of the resonances themselves together with their separatrices. One such strange case for $K = 1/4$ with $\nu = 1/3$ was observed in [11], but its further analysis was postponed to the future.

Of interest and importance is the following question: How typical are the conservation of the separatrix in general and its specific mechanism in particular? It is well known that a large number of examples and even whole families of the so-called completely integrable nonlinear dynamical systems have been "constructed" to date (see, e.g., [16]). There is absolutely no chaos in such systems. However, they are definitely not typical but, in a sense, form a set of measure zero in the space of all possible dynamical systems. From this viewpoint, the new phenomenon of the separatrix conservation in a chaotic system seems more typical, despite the very limited number of examples at present.

The condition for the existence of several singularities in the potential that we used as the basis for our study is neither necessary nor sufficient for the separatrix conservation per se. On the one hand, a preliminary analysis of other examples shows that the presence of several singularities in the force does not yet guarantee the separatrix conservation. For instance, if we simply extend force (2.2) with $d = 0$ by another period, so that two singularities will formally appear, the separatrix will break up as before at any K. In the case under consideration, this takes place because the potential assumes the form of two conjugate wells with an unstable fixed point exactly at the boundary between them. As a result, the unperturbed separatrix always proves to be localized at one of them (depending on initial conditions) with the only singularity.

On the other hand, for an analytic potential, the singularities that determine the separatrix splitting can be located not on the real time axis but in the complex plane. Such a situation appears to have been actually observed in a completely different problem of charged-particle confinement in Cohen's long magnetic trap [17] (see also [18]).

Finally, the passage of the separatrix splitting angle through zero depending on the system parameter and, hence, the separatrix conservation at certain values of this parameter are also possible in principle for a special form of the potential with no singularities whatsoever. This all undoubtedly deserves a further study.

In conclusion, note that even though the new effect of the separatrix conservation in chaos is not universal (a favorite term in current studies of dynamic chaos), nevertheless, we hope that the criterion for the interference of singularities and the theory developed on its basis (which can be easily generalized to an arbitrary number of singularities) can significantly help in studies of a wide class of Hamiltonian dynamical systems.

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REFERENCES

1. V. I. Arnold and A. Avez, *Ergodic Problems of Classical Mechanics* (Benjamin, New York, 1968; RCD, Izhevsk, 1999).
2. B. V. Chirikov, Phys. Rep. **52**, 263 (1979).
3. A. Lichtenberg and M. Lieberman, *Regular and Chaotic Dynamics* (Springer-Verlag, New York, 1992; Mir, Moscow, 1984).
4. G. M. Zaslavsky and R. Z. Sagdeev, *An Introduction to the Nonlinear Physics* (Nauka, Moscow, 1988).
5. B. V. Chirikov, Chaos, Solitons and Fractals **1**, 79 (1991).
6. J. Moser, *Stable and Random Motions in Dynamical Systems* (Princeton Univ. Press, Princeton, 1973).
7. B. V. Chirikov, E. Keil, and A. Sessler, J. Stat. Phys. **3**, 307 (1971).
8. M. Hénon and J. Wisdom, Physica D (Amsterdam) **8**, 157 (1983).
9. L. V. Ovsyannikov, private communication (1999).
10. V. V. Vecheslavov, Zh. Éksp. Teor. Fiz. **119**, 853 (2001) [JETP **92**, 744 (2001)].

11. V. V. Vecheslavov, Preprint No. 2000-27, IYaF RAN (Budker Institute of Nuclear Physics, Siberian Division, Russian Academy of Sciences, Novosibirsk, 2000); nlin.CD/0005048.
12. V. V. Vecheslavov, Preprint No. 99-69, IYaF RAN (Budker Institute of Nuclear Physics, Siberian Division, Russian Academy of Sciences, Novosibirsk, 1999).
13. S. Bullett, Commun. Math. Phys. **107**, 241 (1986).
14. M. Wojtkowski, Commun. Math. Phys. **80**, 453 (1981); Ergod. Theory Dyn. Syst. **2**, 525 (1982).
15. V. V. Vecheslavov and B. V. Chirikov, Zh. Éksp. Teor. Fiz. **114**, 1516 (1998) [JETP **87**, 823 (1998)].
16. B. G. Konopelchenko, *Nonlinear Integrable Equations* (Springer-Verlag, Berlin, 1987).
17. R. Cohen, in *Intrinsic Stochasticity in Plasmas*, Ed. by G. Laval and D. Gresillon (Editions de Physique, Orsay, 1979).
18. B. V. Chirikov, in *Reviews of Plasma Physics*, Ed. by B. B. Kadomtsev (Énergoatomizdat, Moscow, 1984; Consultants Bureau, New York, 1987), Vol. 13.

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