Spectral Decomposition and Community Detection

Jérôme Galtier and Mikaila Toko-Worou

Spectral Properties of Complex Networks, Trente, July 23-27 2012

Graph decomposition

Let $G = (V, E)$ be a graph. We aim at decomposing G into some parts while :

- minimizing the number of edges in-between the parts,

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The modularity Q (Newman, 2003) mesures the quality of the partitioning using a comparison with a random model. The **quality** is simply the ratio of edges that are internal (inside a cluster).

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Note that we have $\sum_{i \in \{1,\ldots,p\}} k_i = 1$ and $\sum_{i \in \{1,\ldots,p\}} |\delta\mathcal{C}_i| = 2|\delta\mathcal{C}|$.

What is NCut

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The **minimum NCut** consists in minimizing :

$$\sum_{i=1}^{i=p} \frac{|\delta \mathcal{C}_i|}{k_i}.$$

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The modularity $Q(C)$ is $\rho - \rho'$

Formulas for the two criterions

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The modularity *Q*

$$Q(\mathcal{C}) = 1 - \frac{|\delta \mathcal{C}|}{|E|} - \sum_{i=1}^{i=p} {k_i}^2.$$

Formula for a bipartition (p=2)

We have $k_1 + k_2 = 1$ and we derive the following expressions in $k_1 k_2$:

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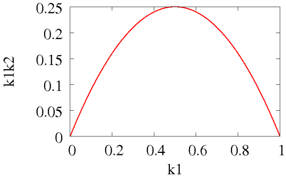

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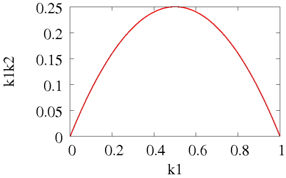

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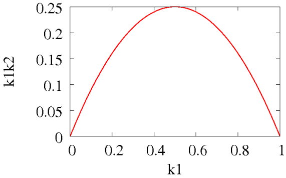

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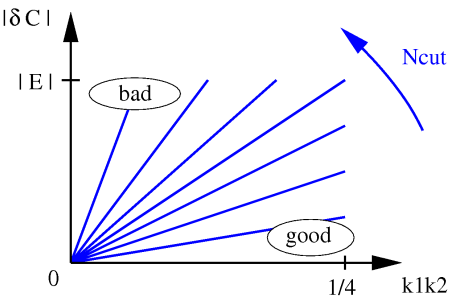
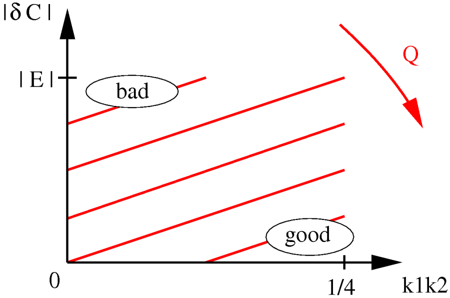
Modularity

$$Q(\mathcal{C}) = 2k_1 k_2 - \frac{|\delta \mathcal{C}|}{|E|}.$$

Insight for a bipartition



minimizing *Ncut*



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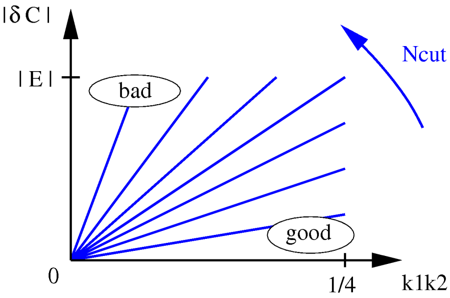

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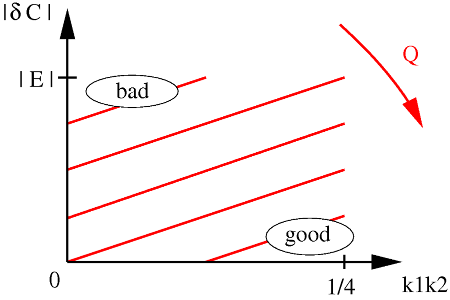

maximizing *Q*

→ the two criterias are extremely similar !

Further analysis for the NCut using matrix form

ξ is the n indicator vector ($\xi_u = 1$ if node u is in C_1 and -1, otherwise)

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$$Ncut(\mathcal{C}) = 2|\delta\mathcal{C}|\left(\frac{1}{k_1}+\frac{1}{k_2}\right) = \sum_{i,j,\xi_i\xi_j<0} -A_{ij}\xi_i\xi_j\left(\frac{1}{k_1}+\frac{1}{k_2}\right)$$

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It gives

$$Ncut(\mathcal{C}) = \frac{[(\mathbf{1}+\xi)-b(\mathbf{1}-\xi)]^t(D-A)[(\mathbf{1}+\xi)-b(\mathbf{1}-\xi)]}{b\mathbf{1}^t D\mathbf{1}},$$

$$\text{with } b = \frac{k_1}{1-k_1}.$$

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By setting $Y = (\mathbf{1}+\xi) - b(\mathbf{1}-\xi)$, we have $Y^TD\mathbf{1} = 0$ and $b\mathbf{1}^TD\mathbf{1} = Y^TDY$.

$\min_\xi Ncut(\xi)$ st : $\xi \in \{-1,1\}^n$	$\Leftrightarrow$	$\min_Y \frac{Y^T(D-A)Y}{Y^TDY}$ st : $Y^TD\mathbf{1} = 0$ and $Yu = \{-b,1\}\ \forall u \in V$, depending on $u \in C_1$, or $u \in C_2$.

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- the solution is the eigenvector corresponding to the **second smallest eigenvalue**.

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What about modularity ?

Denote $X = {}^t YY$ and suppose Y is a $1 \times n$ vector defined as follows (note that since $X_{u,u} = 1$, we have $y_u^2 = 1$) :

$$\begin{cases} y_u = 1 & \text{if } u \in \mathcal{C}_1 \\ y_u = -1 & \text{if } u \in \mathcal{C}_2 \end{cases}$$

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$$(A - \frac{1}{2|E|} DJD) \cdot X = -4|\delta\mathcal{C}| + 8|E| k_1 k_2.$$

both spectral relaxations put together

Let J the square matrix filled with 1's.

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NCut	*Modularity*
$\max(D^{-1/2}AD^{-1/2} - I) \cdot X$	$\max(A - \frac{1}{2\|E\|}DJD) \cdot X$
s.t. $X \cdot (D^{1/2}JD^{1/2}) = 0$	s.t. $X_{v,v} = 1 \ \forall v \in V$
$X \cdot I = \|V\|$	
$X \succeq 0$	$X \succeq 0$

cutting with eigenvalues : numerical results for Q

	karate	Arxiv
CNM	0.38	0.772
PL	0.42	0.757
WT	0.42	0.767
Louvain	0.42	0.813
SpecMod	0.42	0.772
SpecMod with refinement	0.42	0.801

tools when eigenvalues are not available

A matrix X is semidefinite positive if and only if there exists a matrix Y such that

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Let Y_v, $v \in V$, be the set of columns of Y. Each of them is a vector of $\mathbb{R}^n$.

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- $$X \cdot I = \sum_{v \in V} |Y_v|^2.$$

using KKT conditions !

we associate to each vertex u a point Y_u in $\mathbb{R}^d$ with d smaller than n.

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problem : $\boxed{\text{maximize} \sum_{\{u,v\}\in E} W_{u,v} Y_u \cdot Y_v.}$

the algorithm proposed

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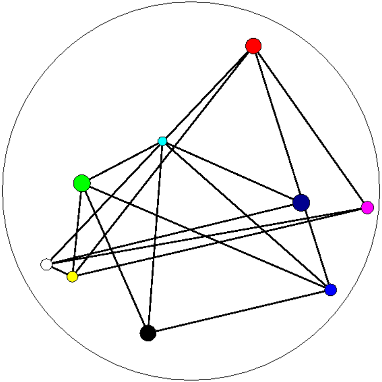
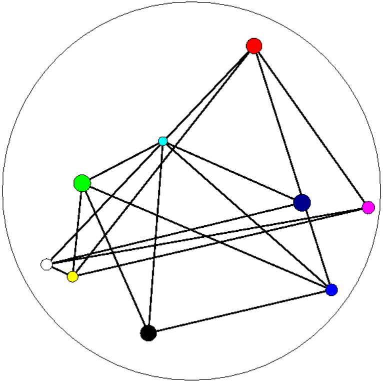
In practice, for $d = 3$, **3 min** are required to update 97.10^6 **nodes**.

compared methods

Ncut (using Lagrange relaxation)



Modularity



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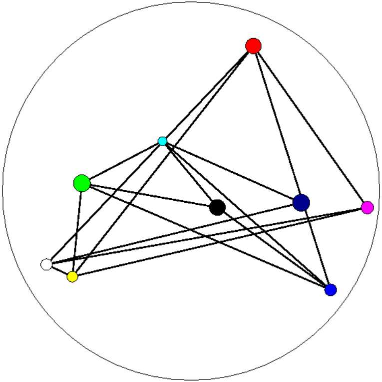

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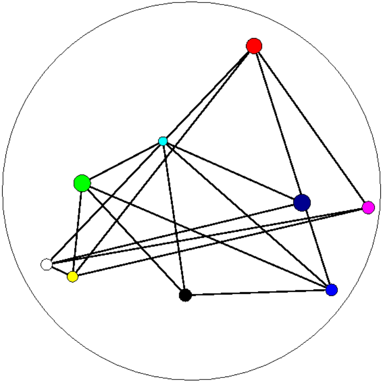

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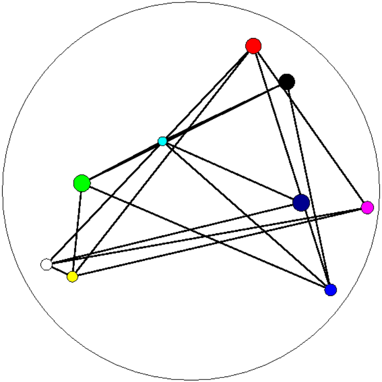

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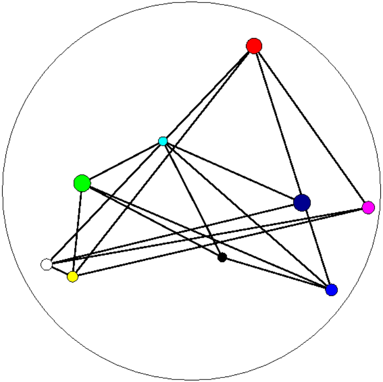

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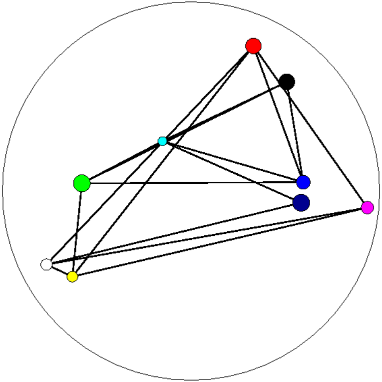

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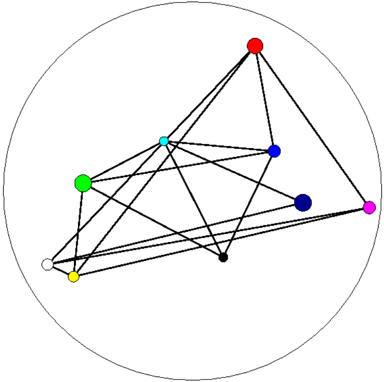

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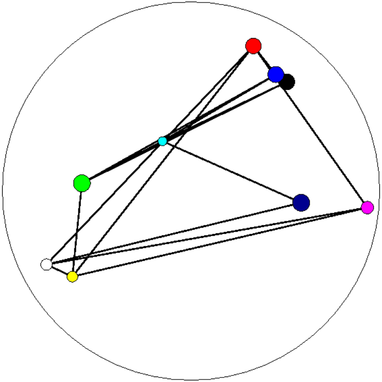

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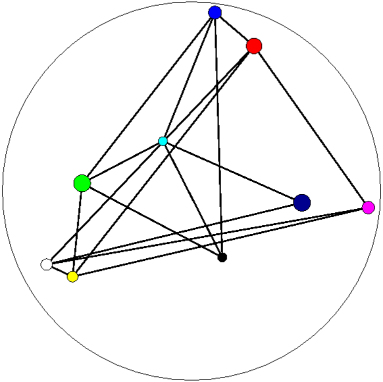

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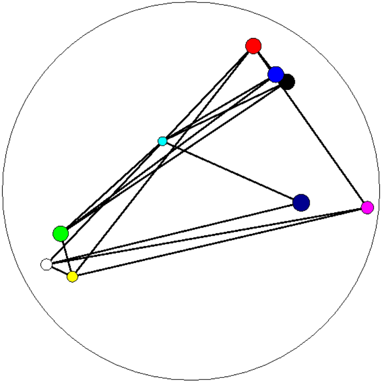

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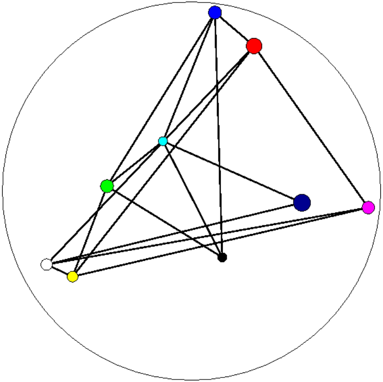

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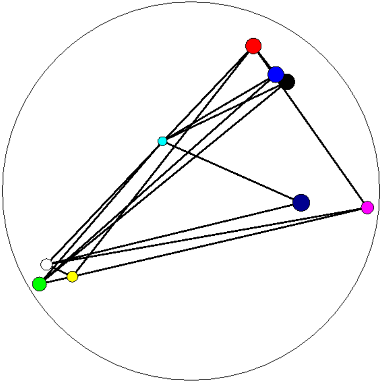

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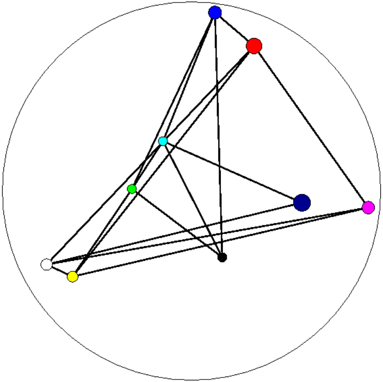

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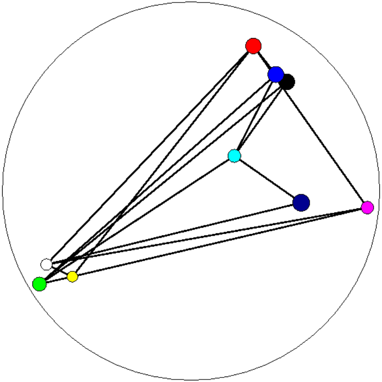

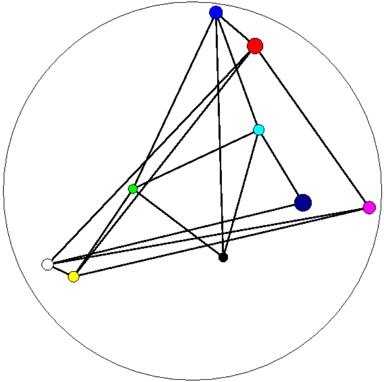

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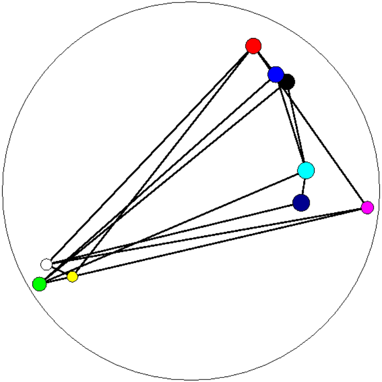

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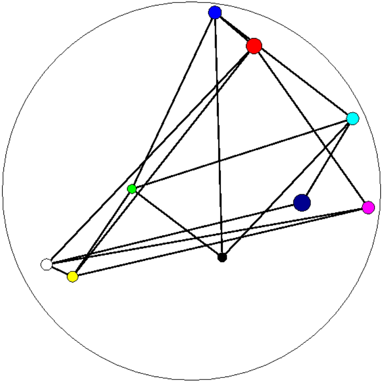

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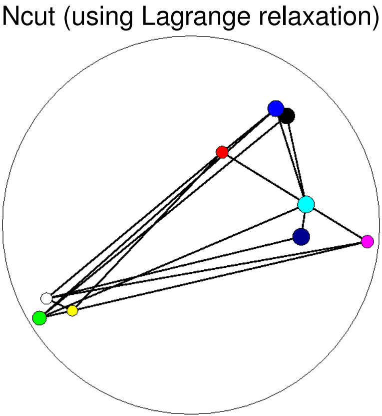

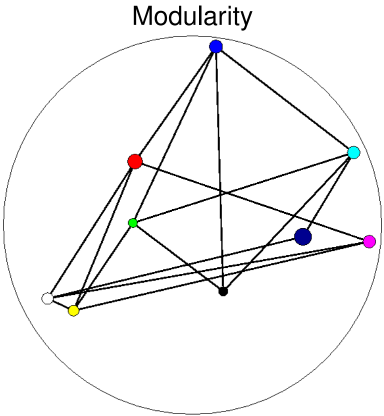

compared methods

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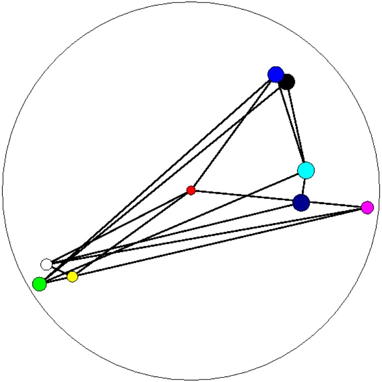

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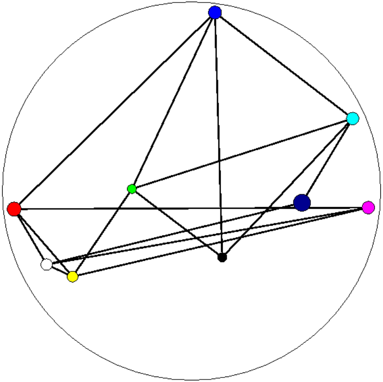

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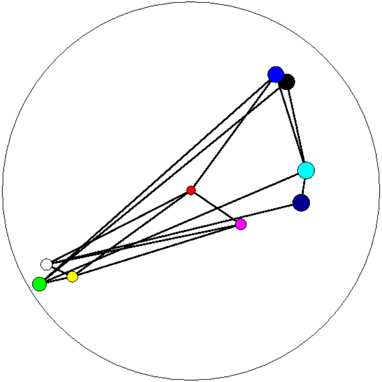

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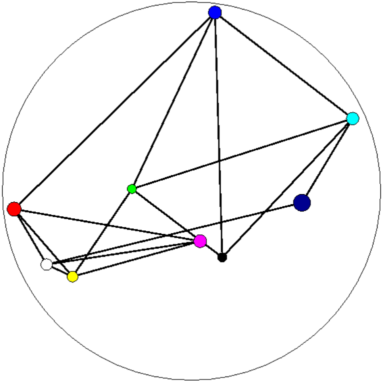

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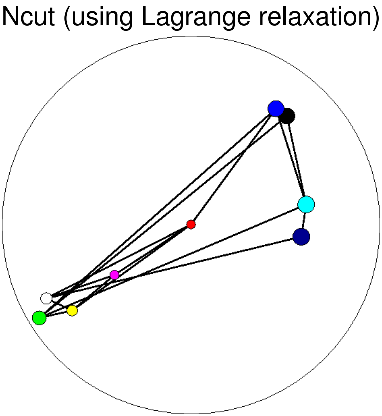

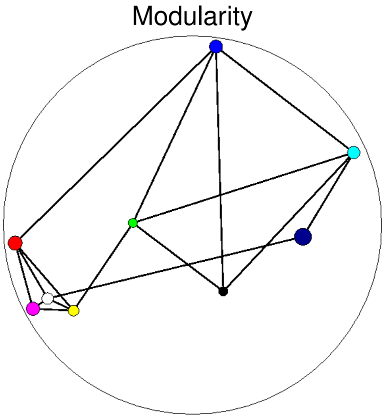

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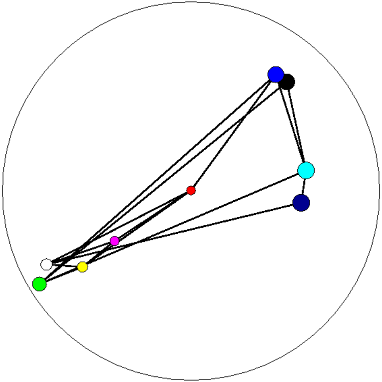

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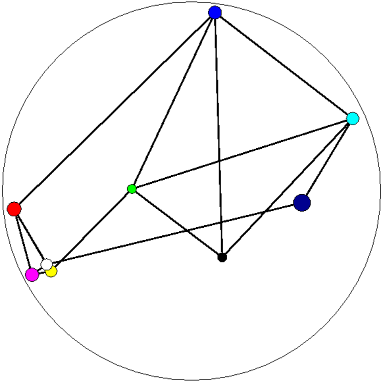

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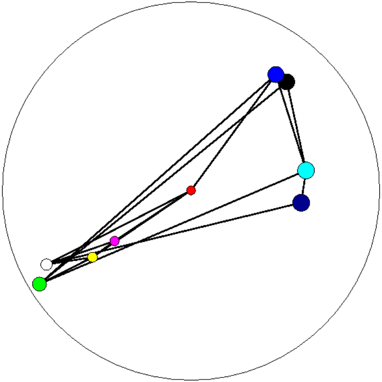

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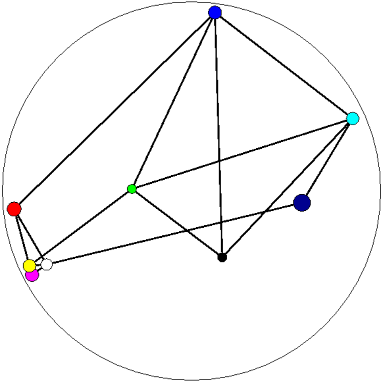

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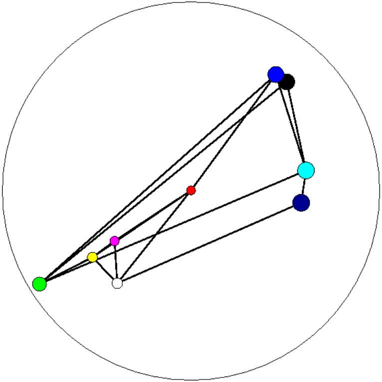

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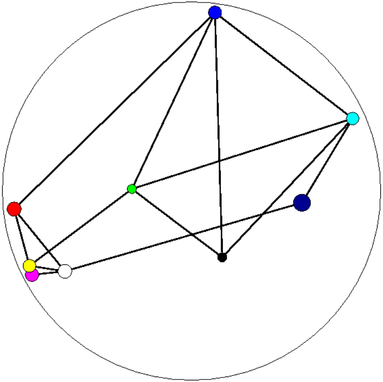

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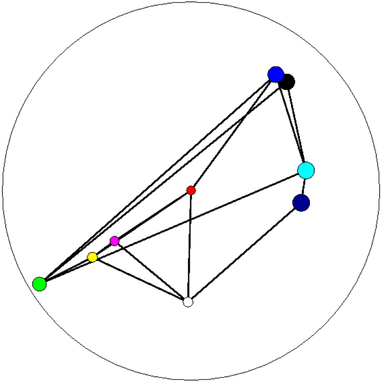

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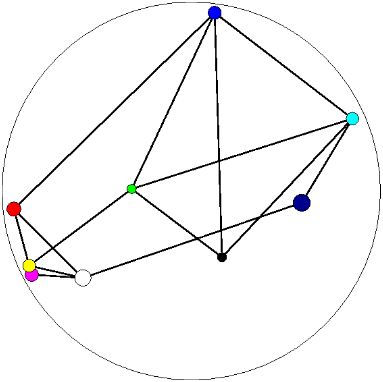

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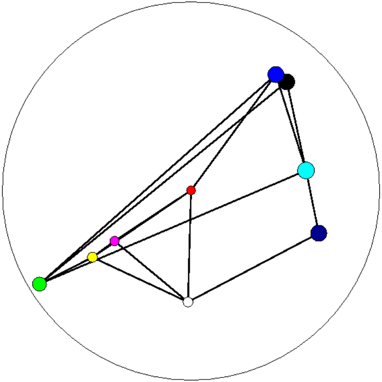

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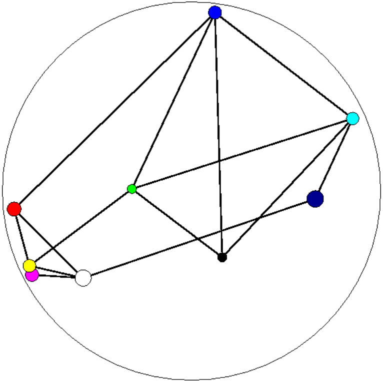

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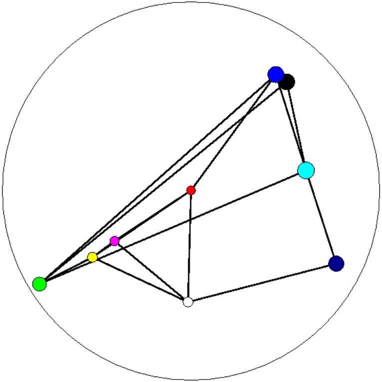

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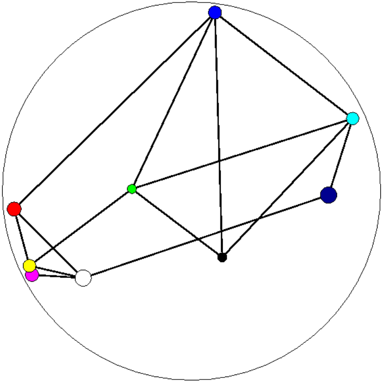

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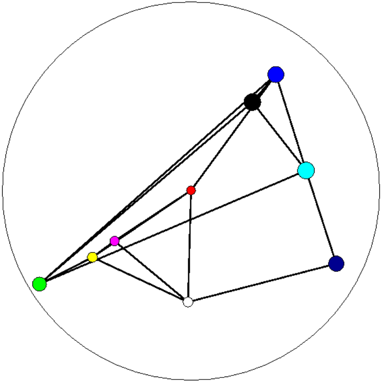

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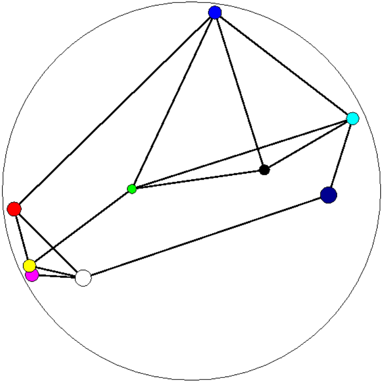

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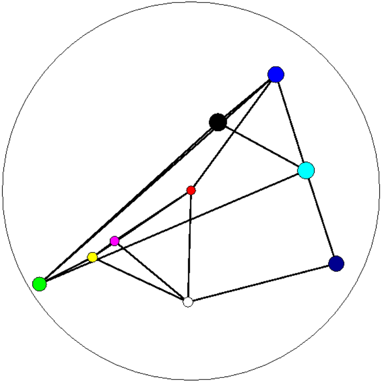

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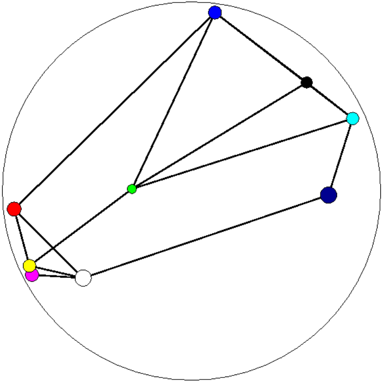

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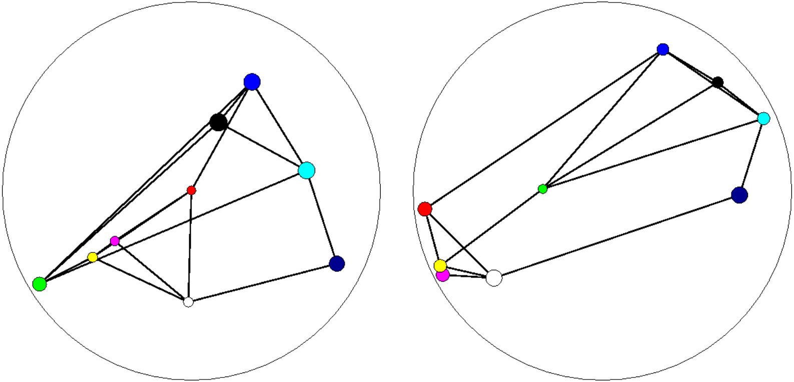
Ncut (using Lagrange relaxation) — Modularity



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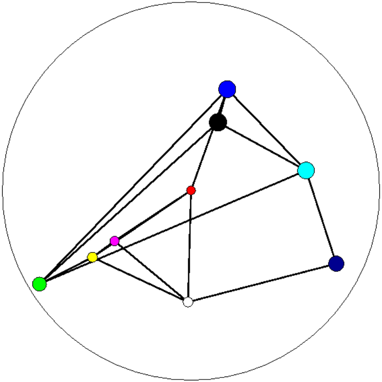

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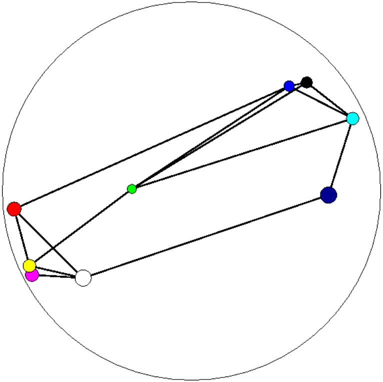

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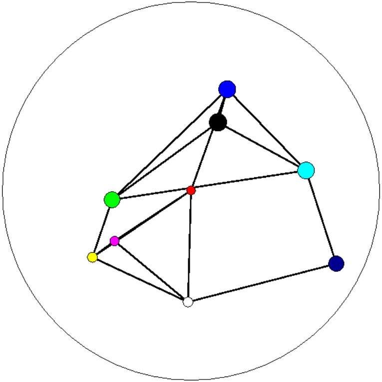

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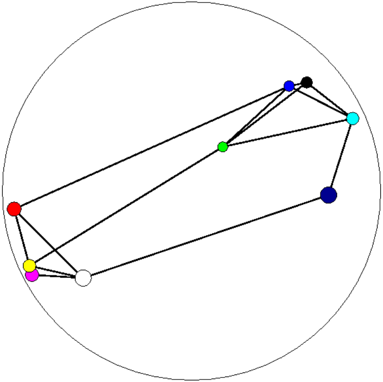

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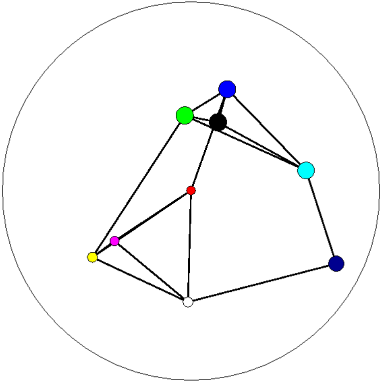

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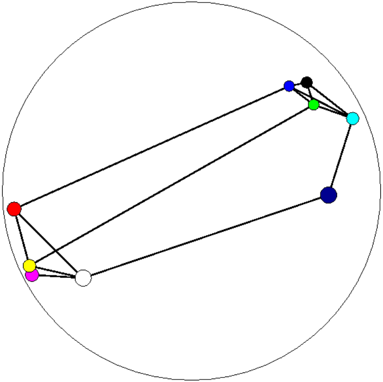

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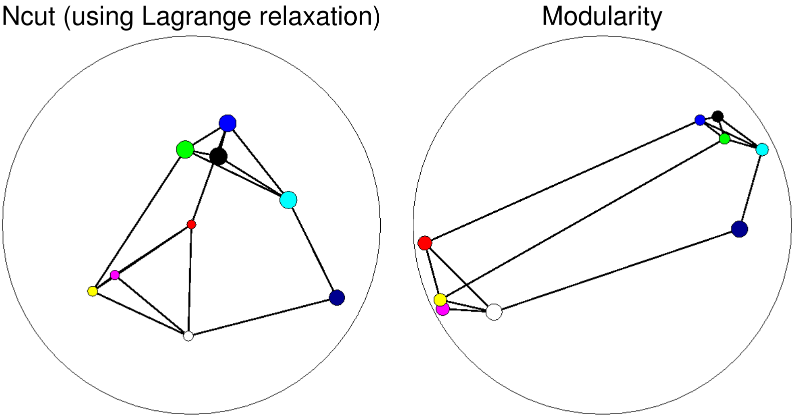

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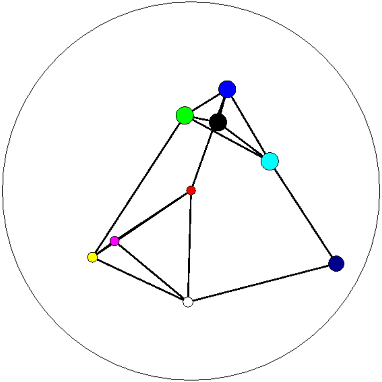

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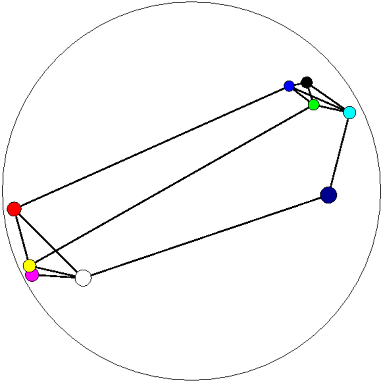

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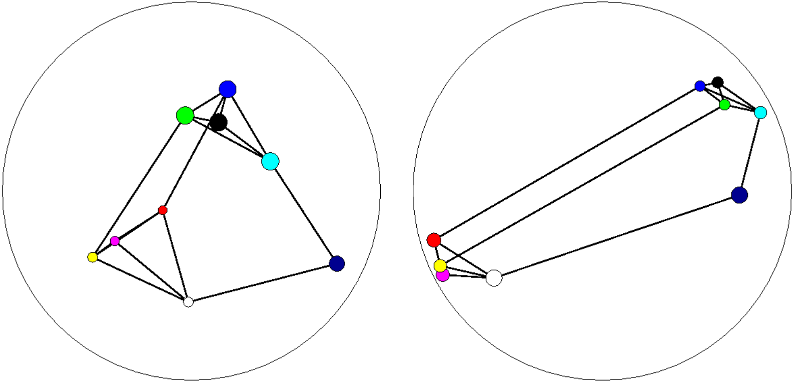

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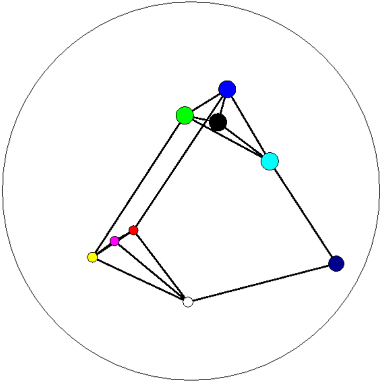

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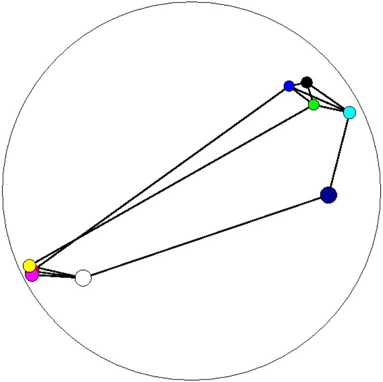

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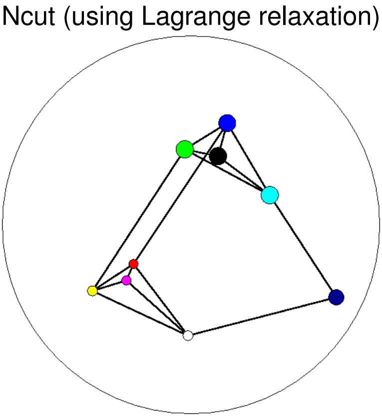

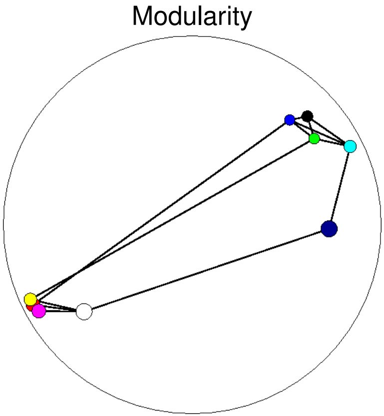

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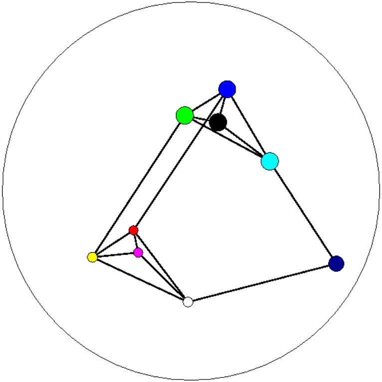

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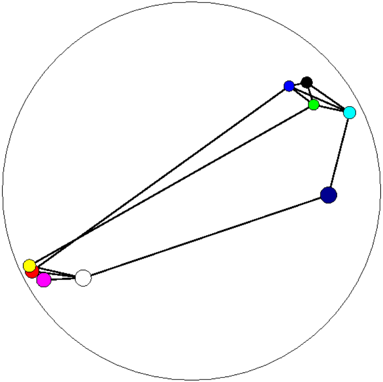

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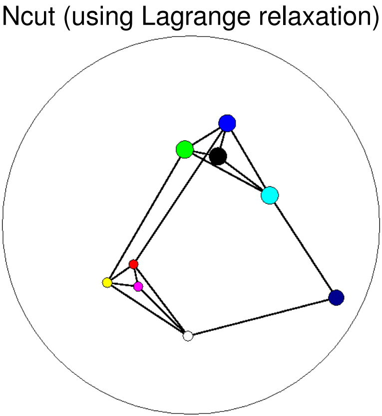

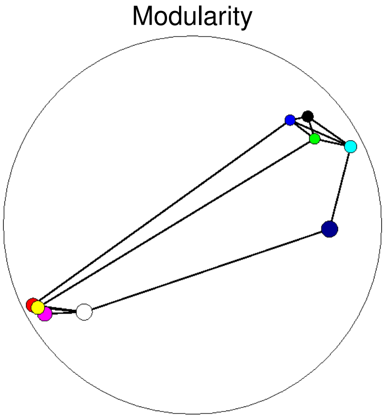

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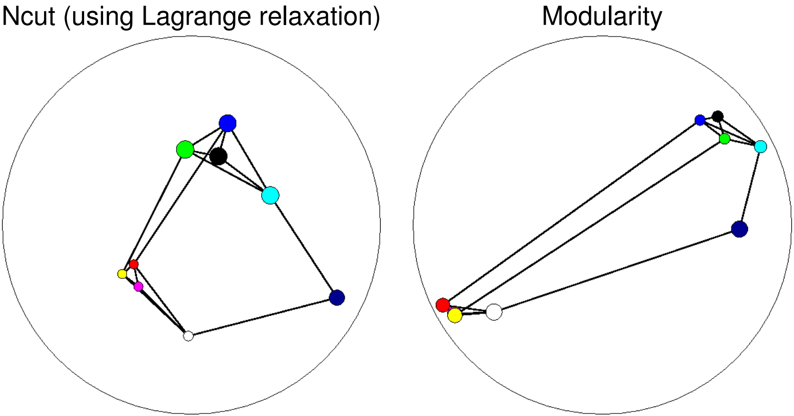

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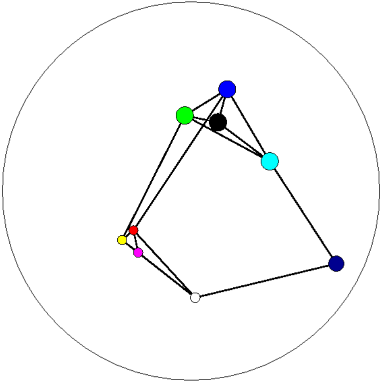

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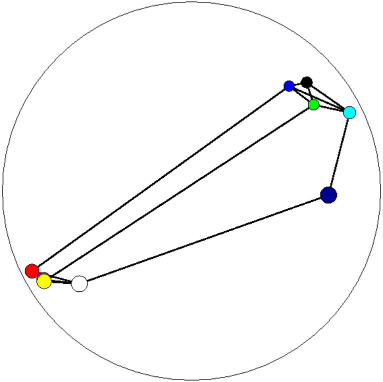

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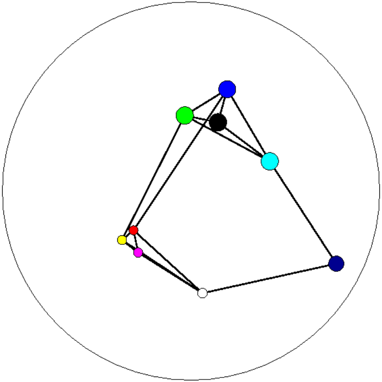

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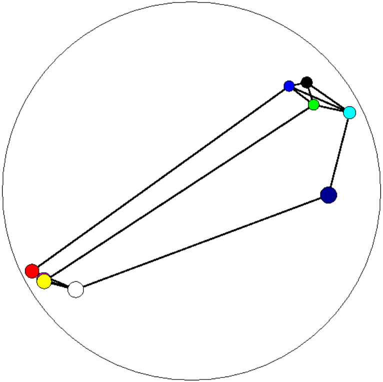

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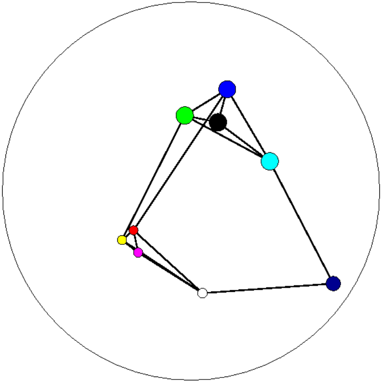

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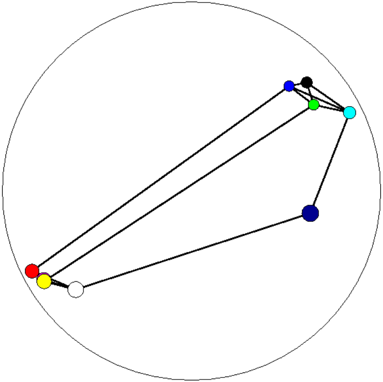

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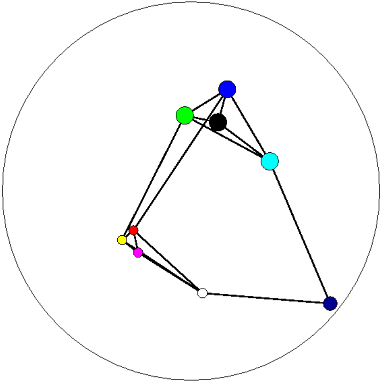

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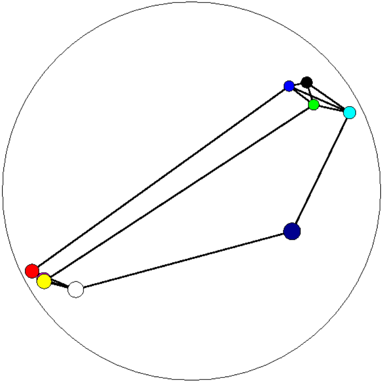

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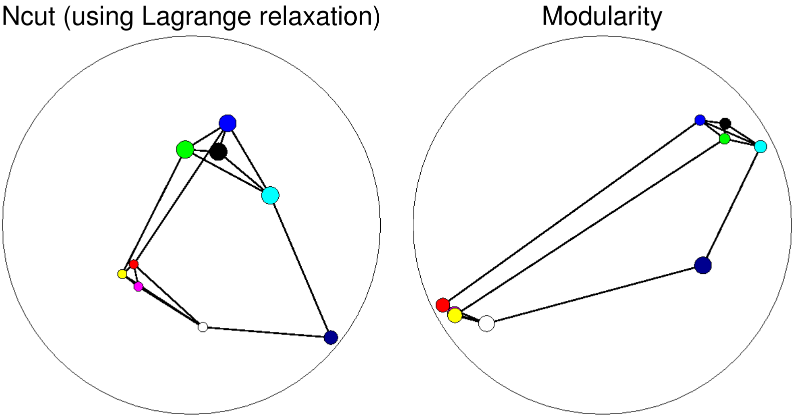

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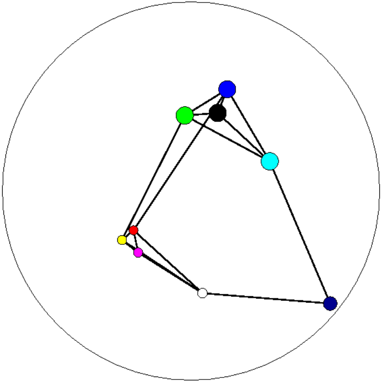

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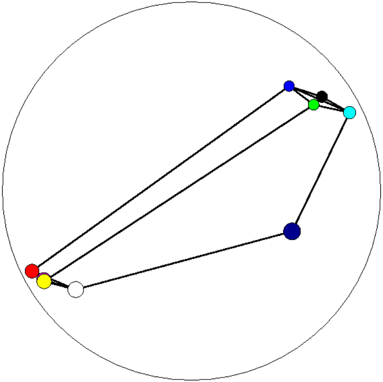

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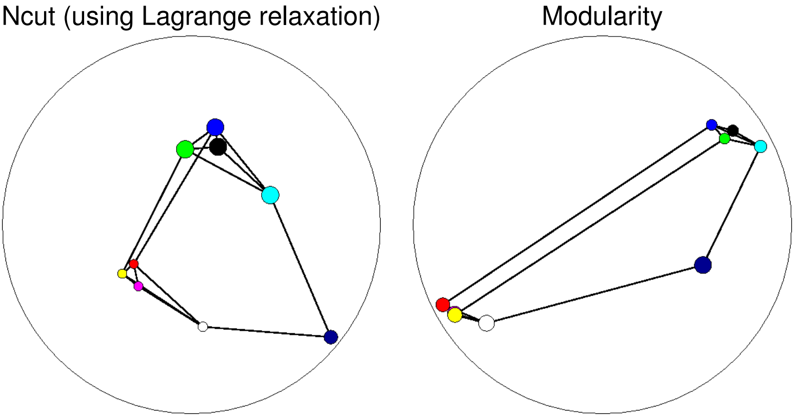

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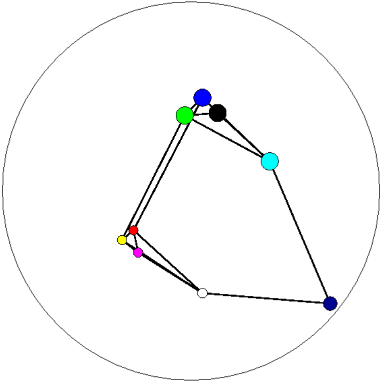

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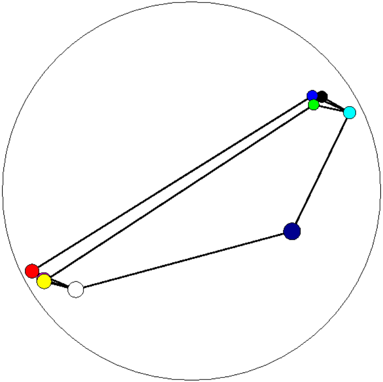

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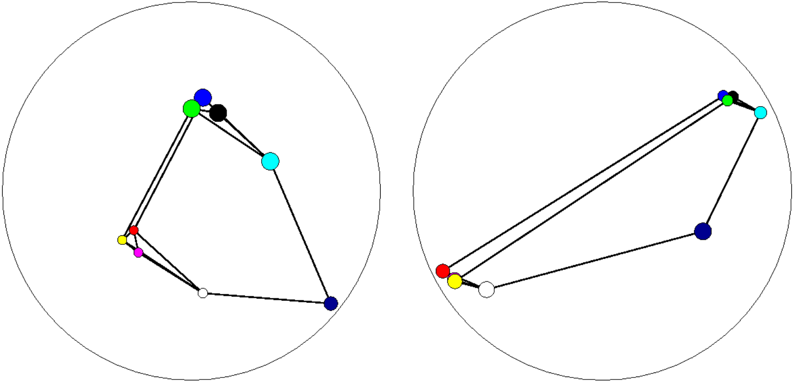
Ncut (using Lagrange relaxation)   Modularity



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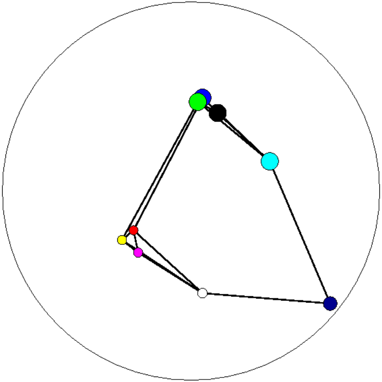

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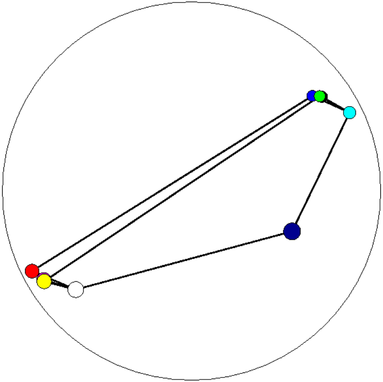

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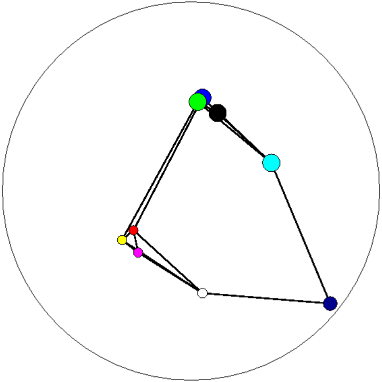

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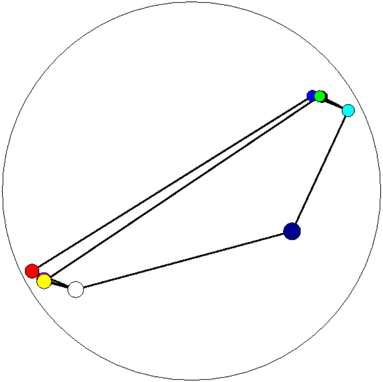

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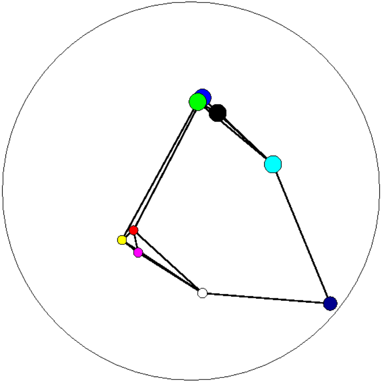

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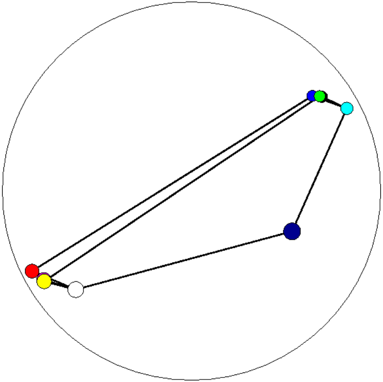

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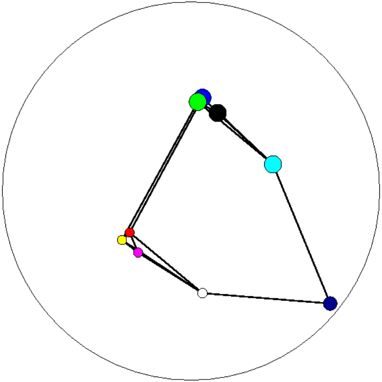

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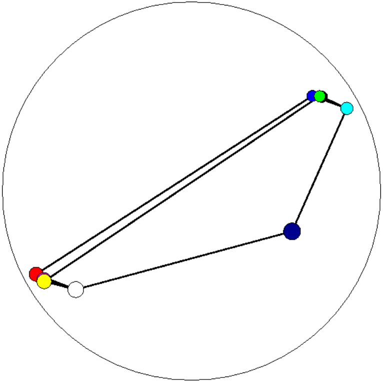

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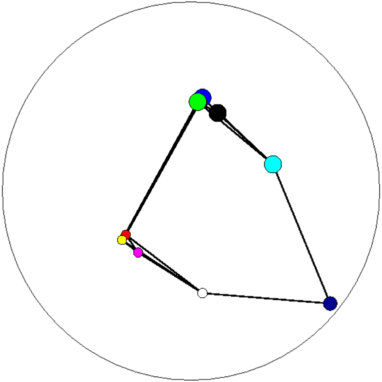

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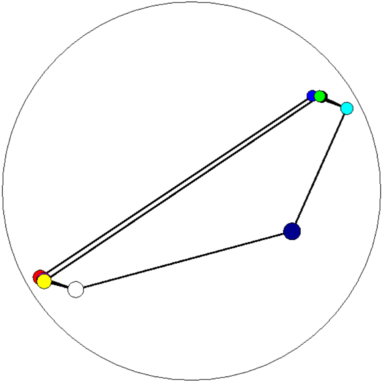

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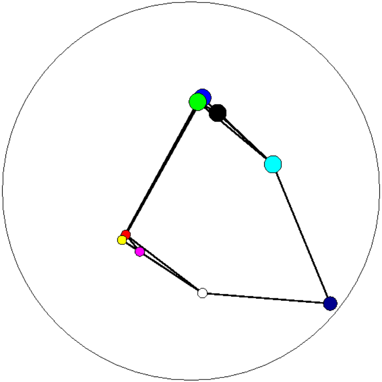

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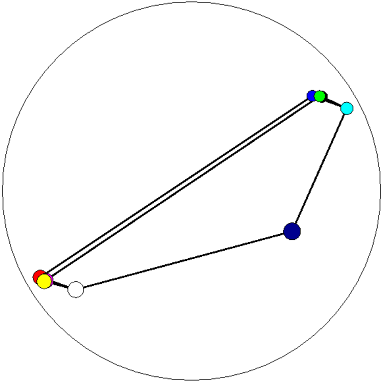

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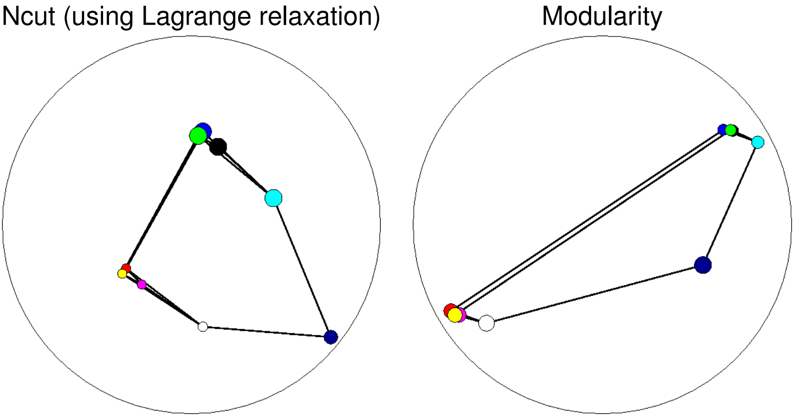
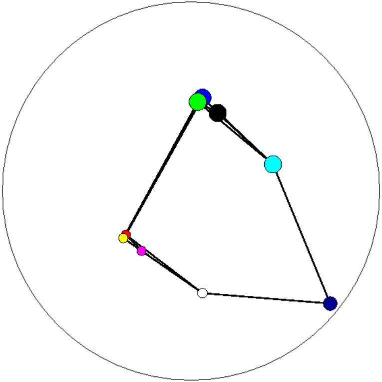
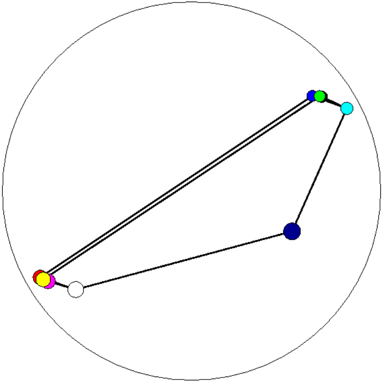


compared methods

Ncut (using Lagrange relaxation)



Modularity



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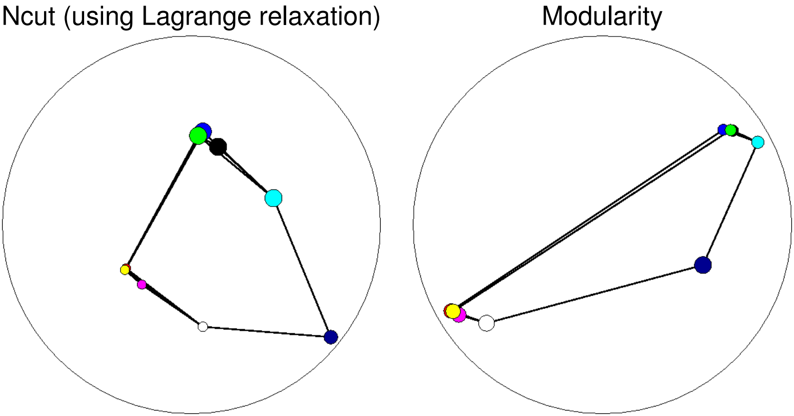

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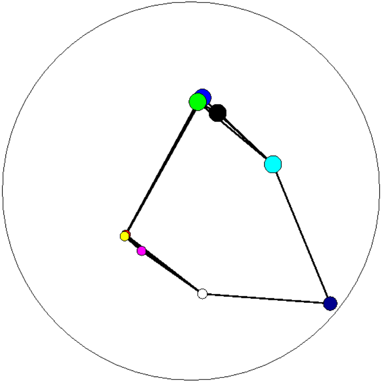

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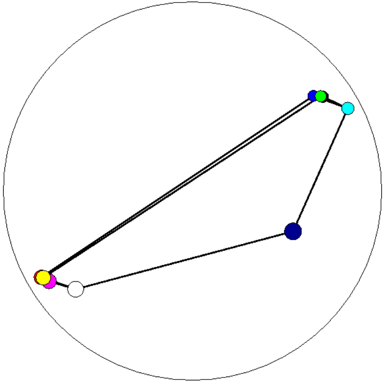

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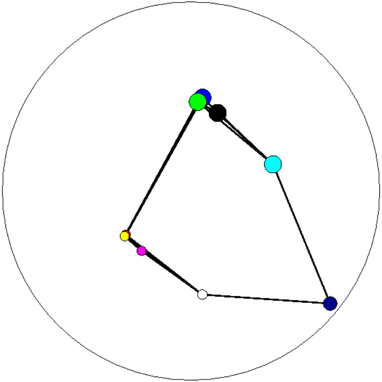

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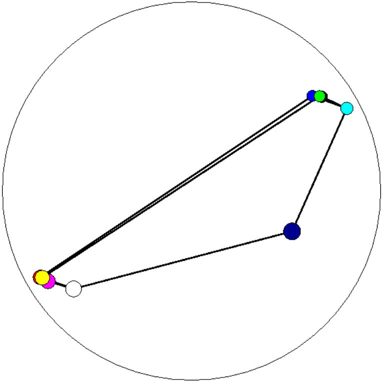

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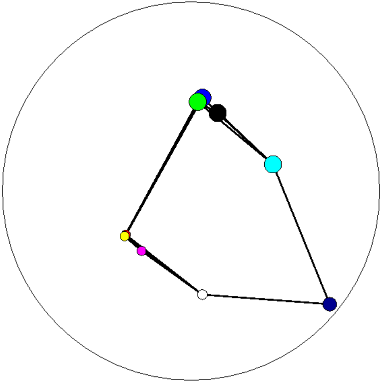

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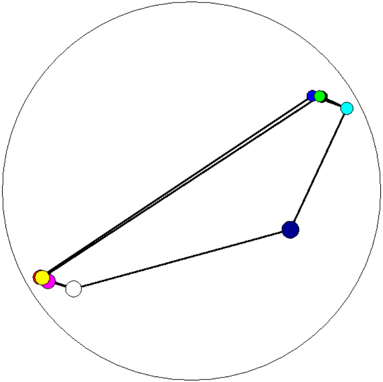

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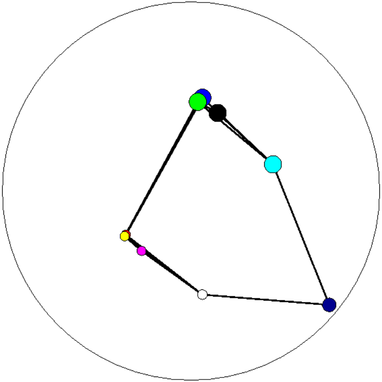

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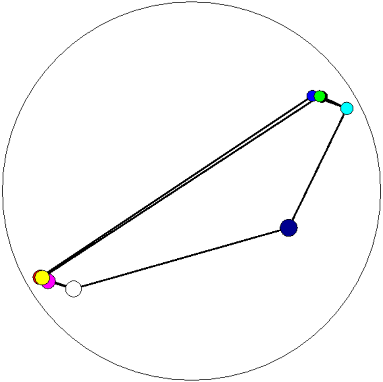

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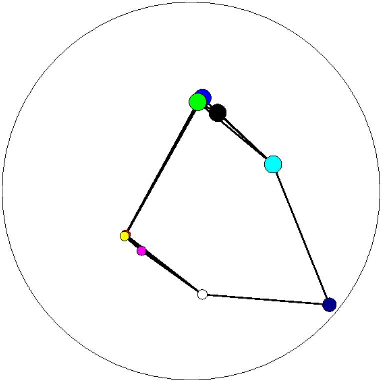

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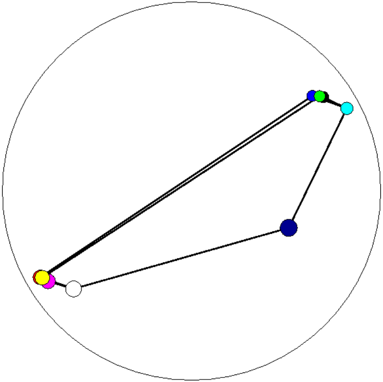

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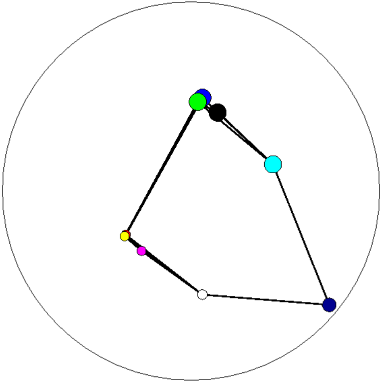

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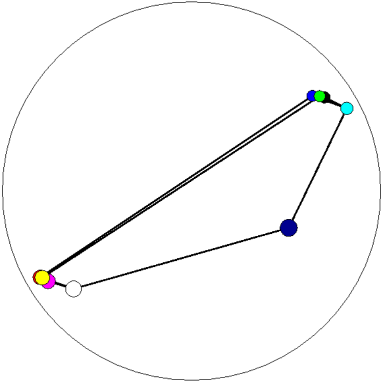

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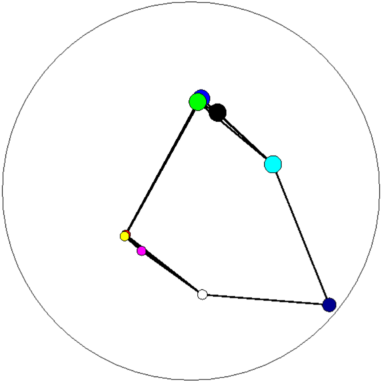

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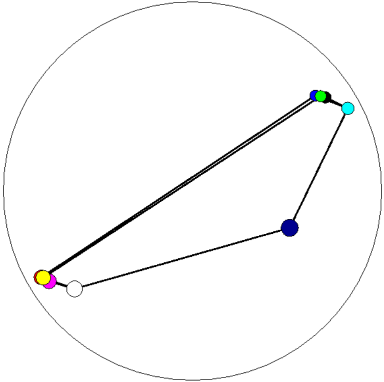

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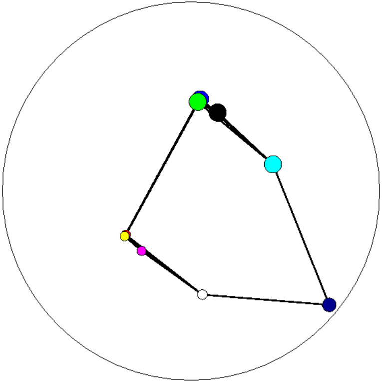

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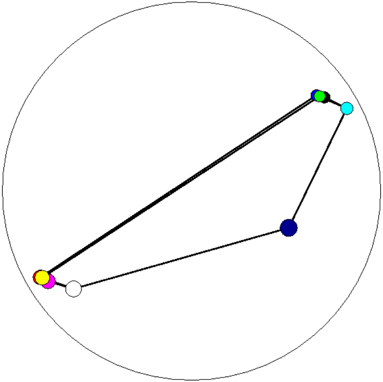

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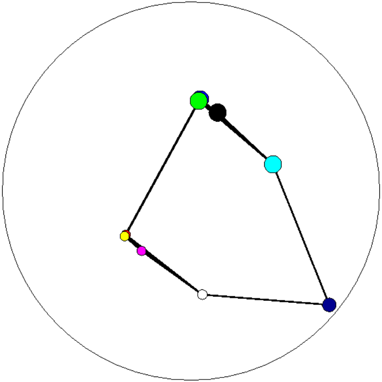

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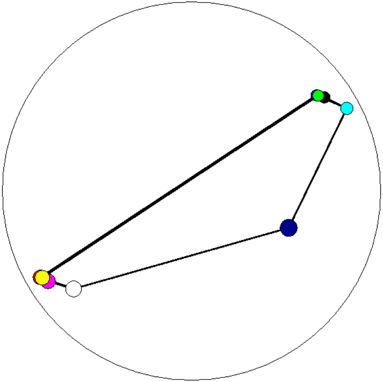

→ following Goemans Williamson for Max-Cut, we cut with a random hyperplane begin

thinking further in terms of graphs

eigenvalues are simply great to split graphs !

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however social networks present often strongly connected components in terms of link density that should be detected to understand **communities**

how to detect them ?

the answer comes from matroid theory : let us analyse the strength of these graphs

what is the strength of a graph ?

Given a graph $G = (V, E)$, we compute

$$\sigma(G) = \min_{\mathcal{C} \text{ partition}} \frac{|\delta \mathcal{C}|}{p-1},$$

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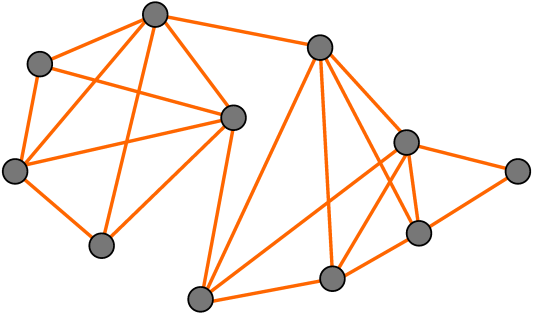
and the dual equivalent (the densest subgraph is terms of edges)

$$\gamma(G) = \max_{H \subseteq V,\ |H| \neq 1} \frac{|E(H)|}{|H| - 1},$$

Strength of graphs : intuition



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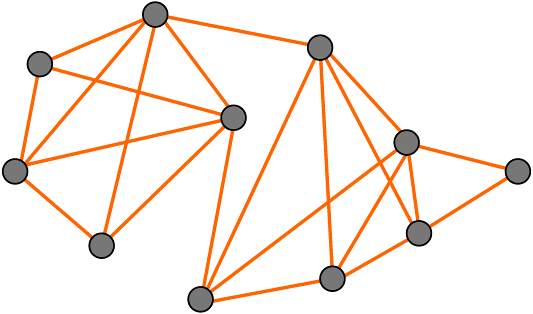

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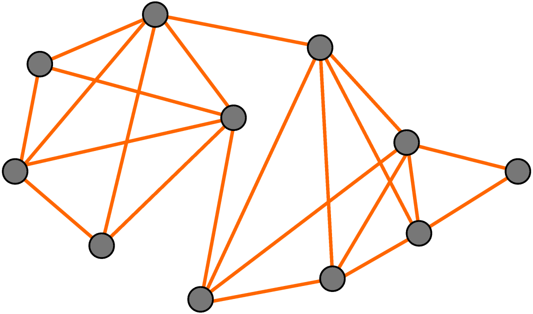

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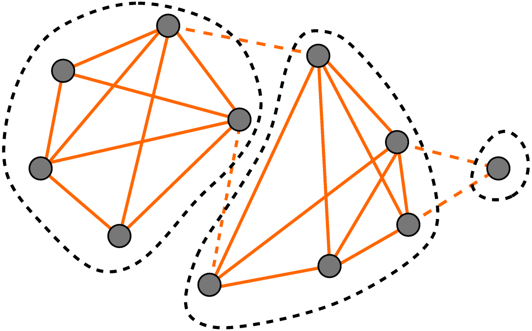

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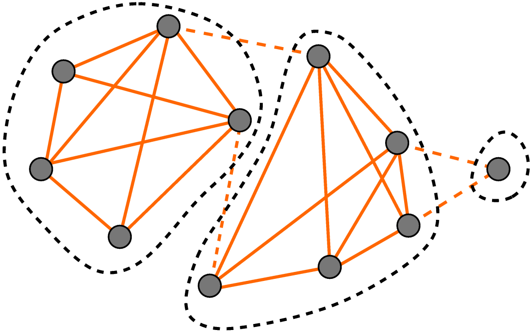

→ each sub-community that is not a singleton is then redivided and has provably a better strength.

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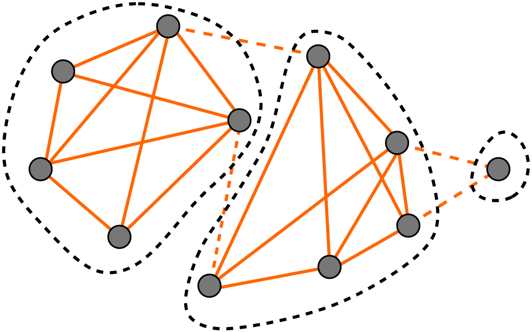

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and so $\boxed{\Lambda_1 \geq \gamma(G)}$.

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a word on the bibliography

Strength of graph is linked to *graph partitionning* and serves as the underground algorithm to approximate the *minimum cut* of a graph in almost linear time (Karger 2000).

Many algorithms use the maximum flow, which runs with best complexity $MF(n,m) = O(\min(\sqrt{m}, n^{2/3}) m \log(n^2/m+2))$ (Goldberg & Rao, 1998).

1984	Cunningham	$O(nm\ MF(n,n^2))$	Exact
1988	Gabow &	$O(\sqrt{\frac{m}{n}}(m+n\log n)\log\frac{m}{n})$	Integer
	Westermann	$O(nm\log\frac{m}{n})$	Integer
1991	Gusfield	$O(n^3 m)$	Exact
1991	Plotkin et ali	$O(m^2\sigma(G)\log(n)^2/n/\varepsilon^2)$	Within $1+\varepsilon$
1993	Trubin	$O(n\ MF(n,m))$	Exact
2008	Galtier	$O(m\log(n)^3/\varepsilon^2)$	Within $1+\varepsilon$
2011	Toko-Worou & Galtier	$O(m\gamma\log(n)^2/\varepsilon^2)$	γ within $1+\varepsilon$

A word on the linear approximation

The algorithm as basis takes a pushing flow scheme.

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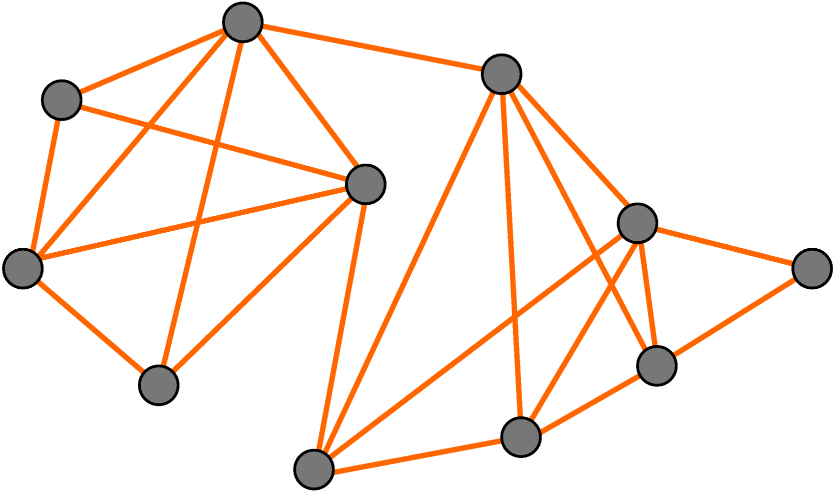

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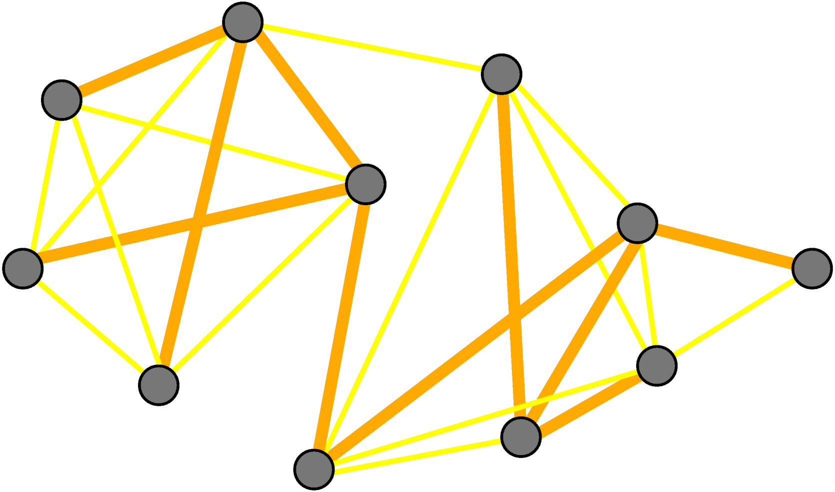

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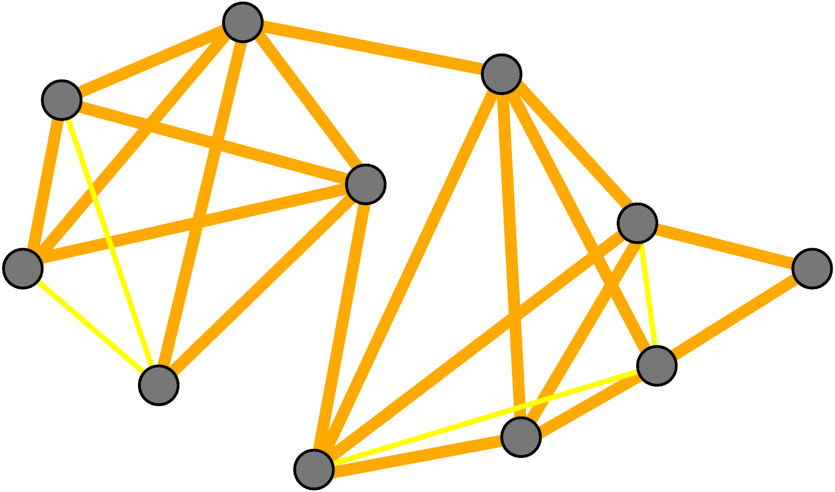

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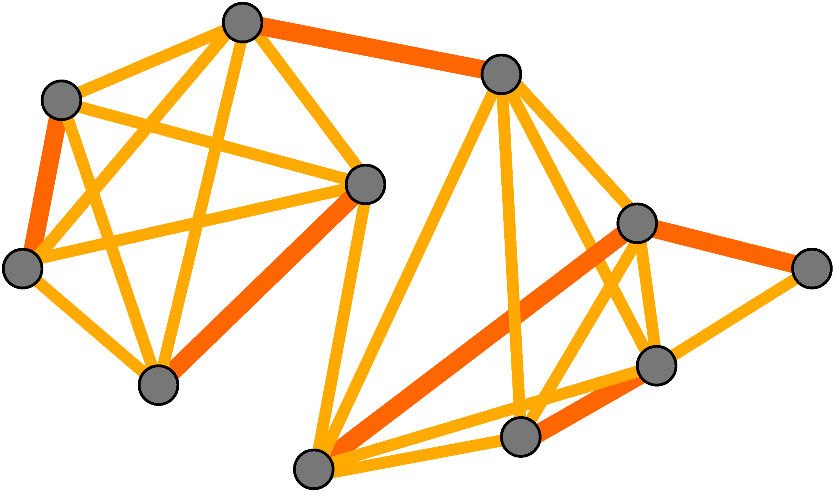

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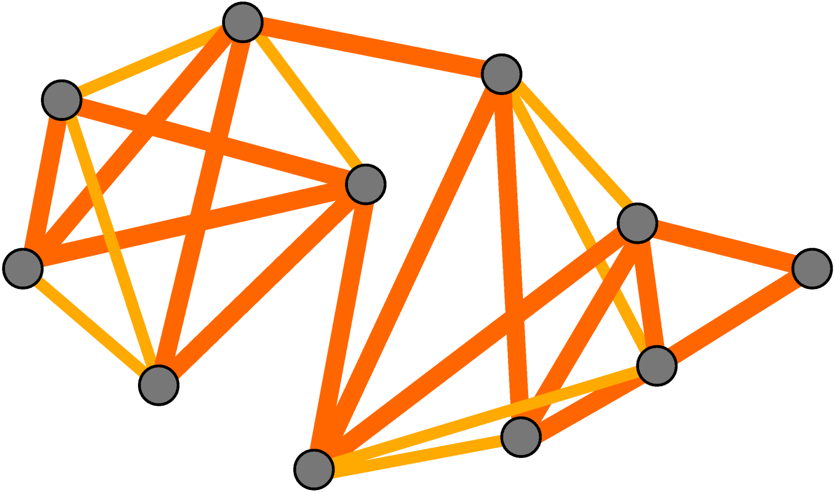

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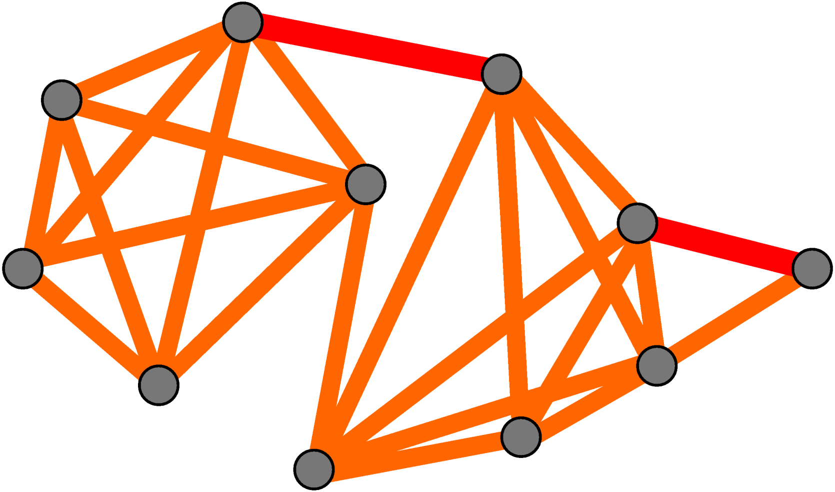

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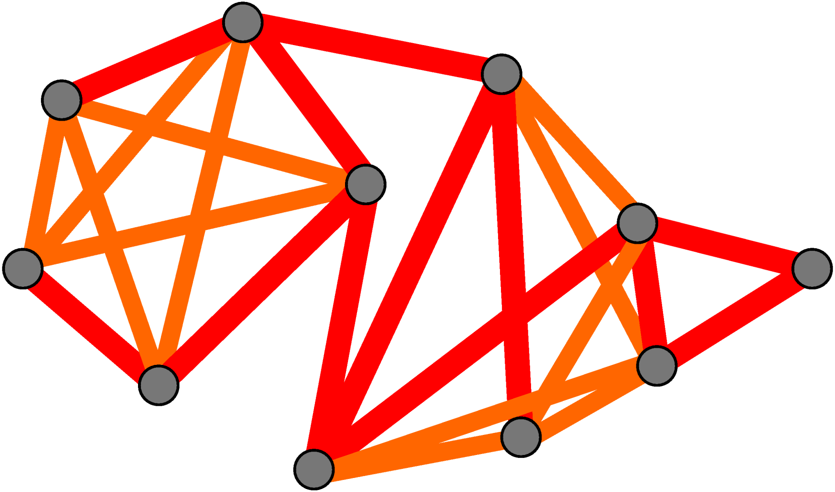

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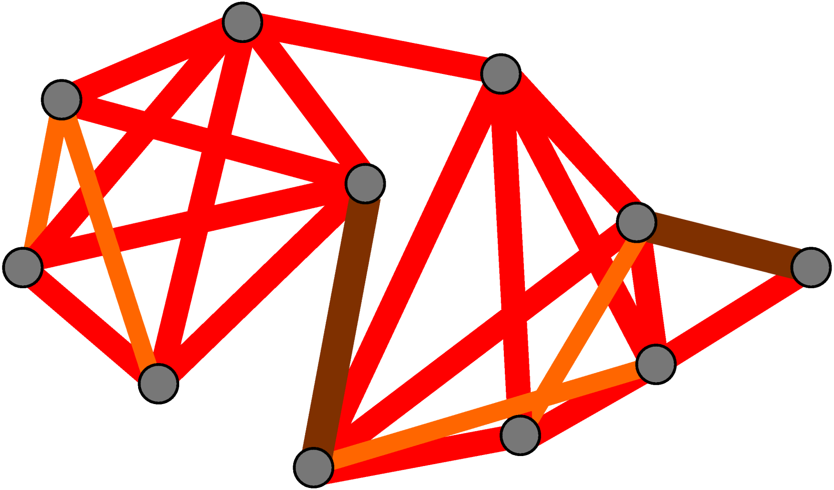

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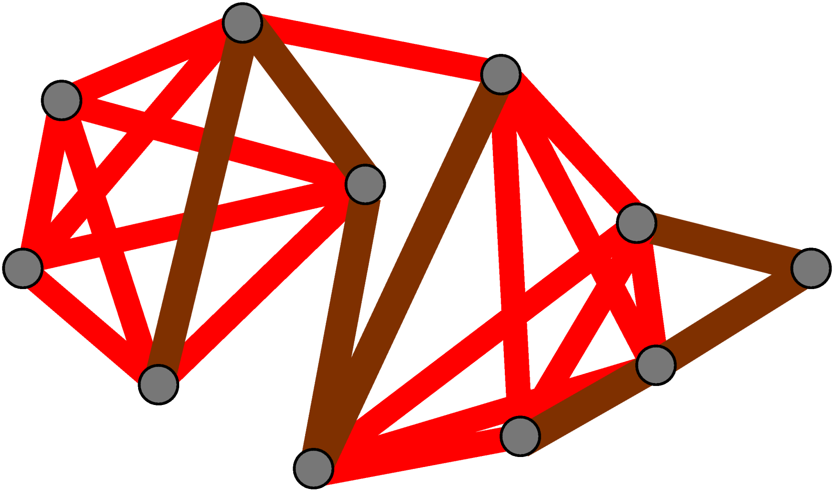

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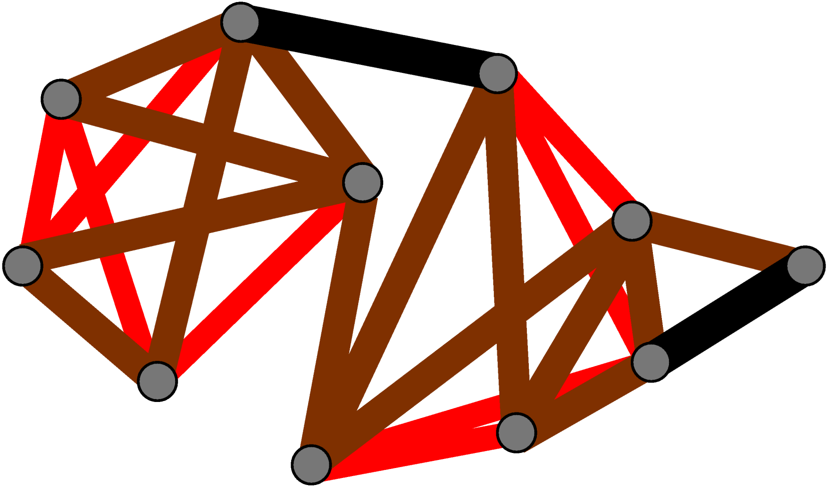

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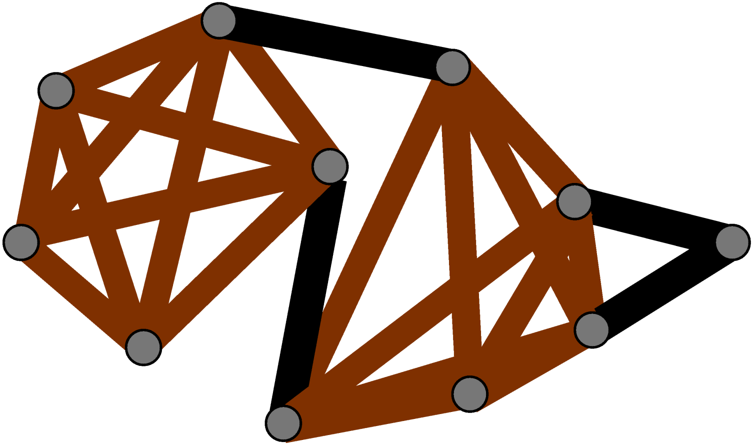
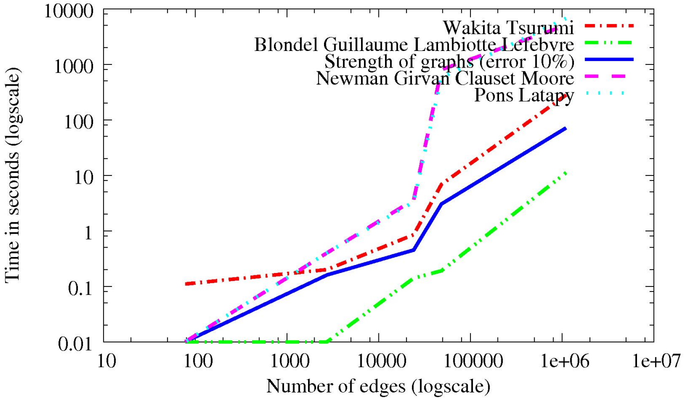
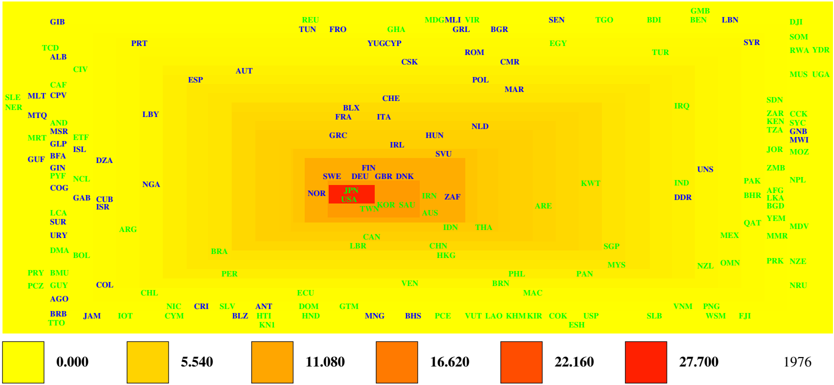


computational linearity of the approximation

The algorithm is almost linear with the number of links between documents. Here compared with popular heuristics and datasets :



Back to bilateral exchanges



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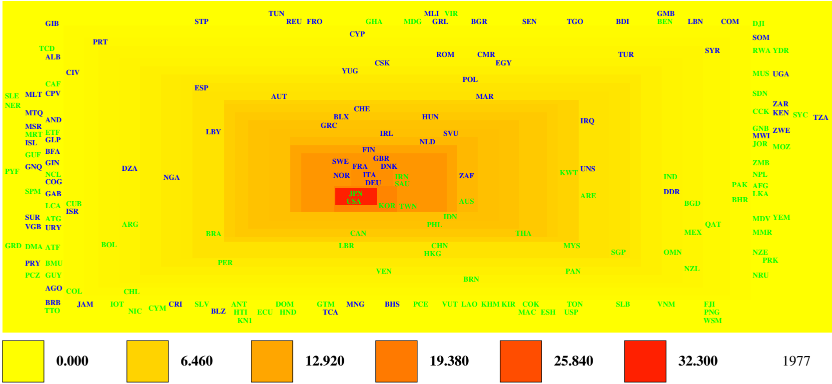

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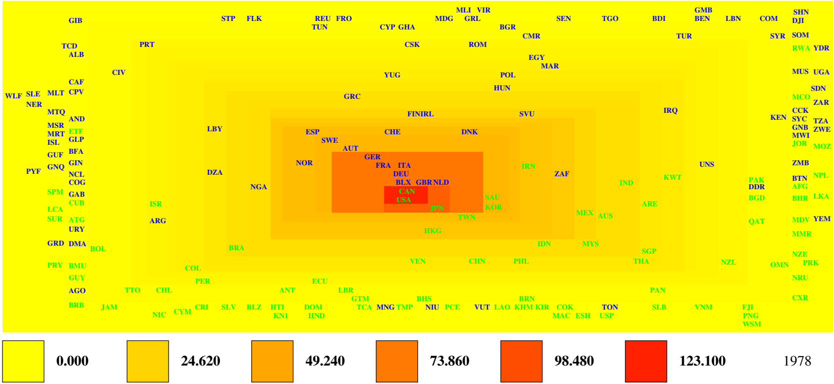

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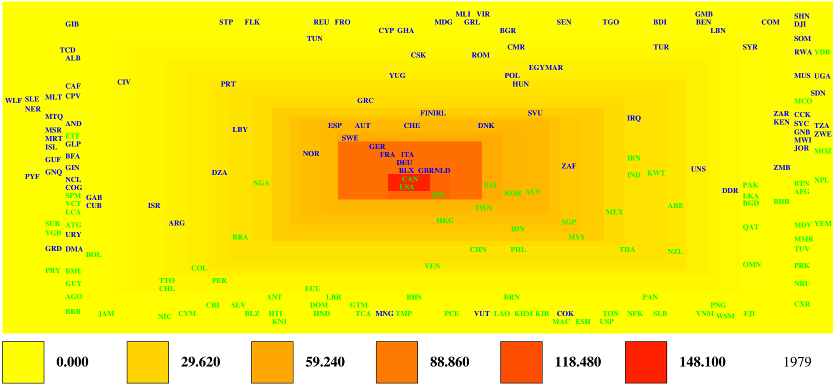

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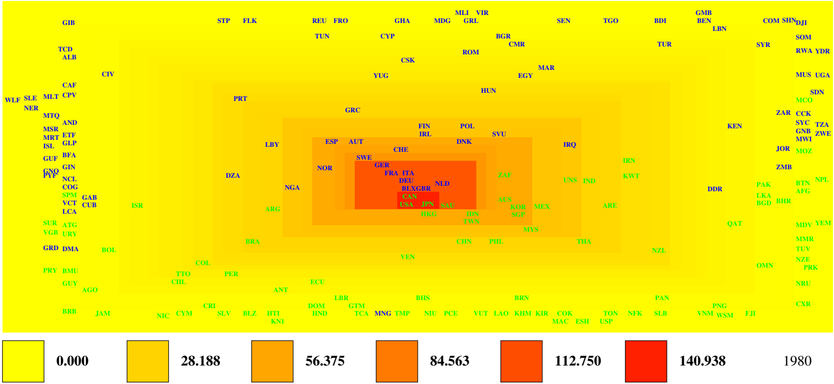

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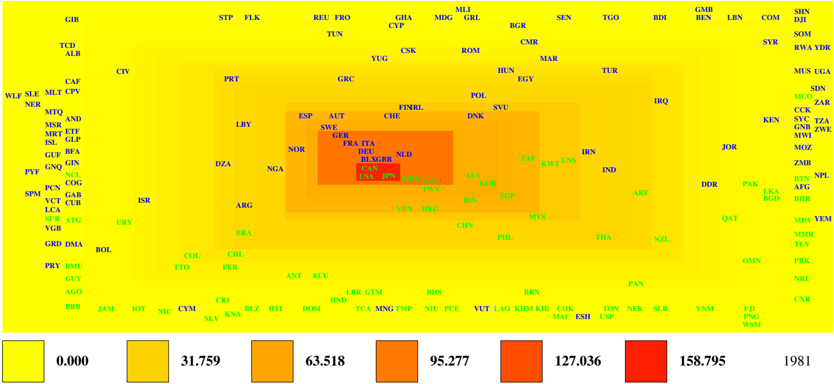

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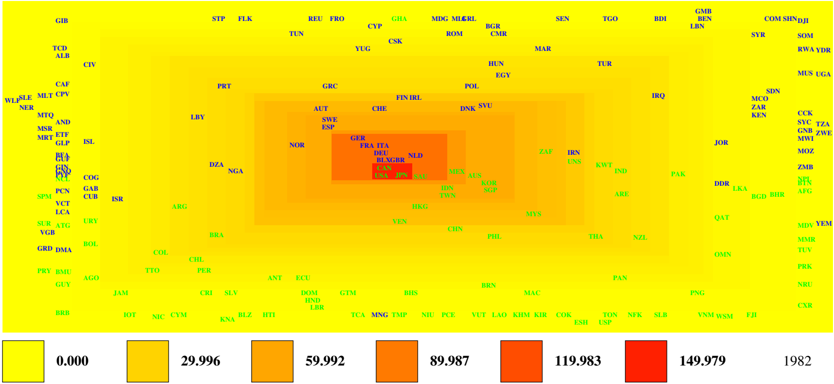

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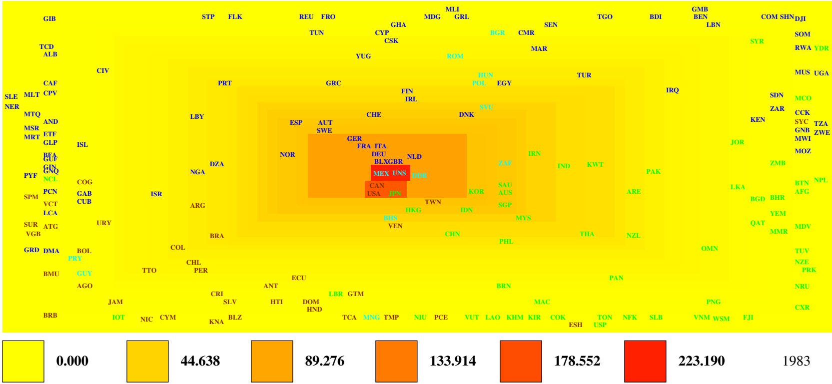

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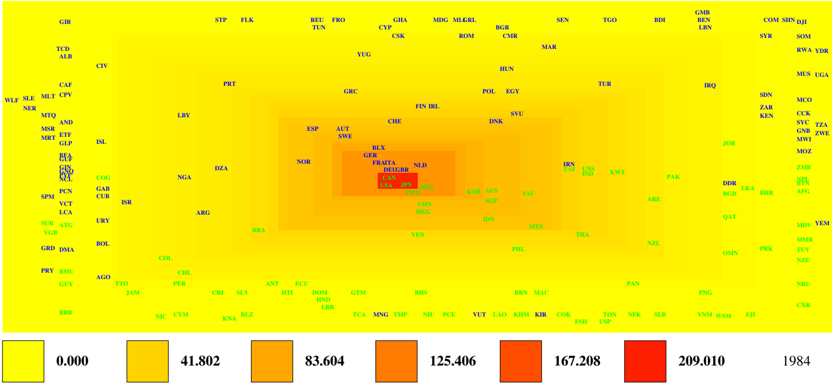

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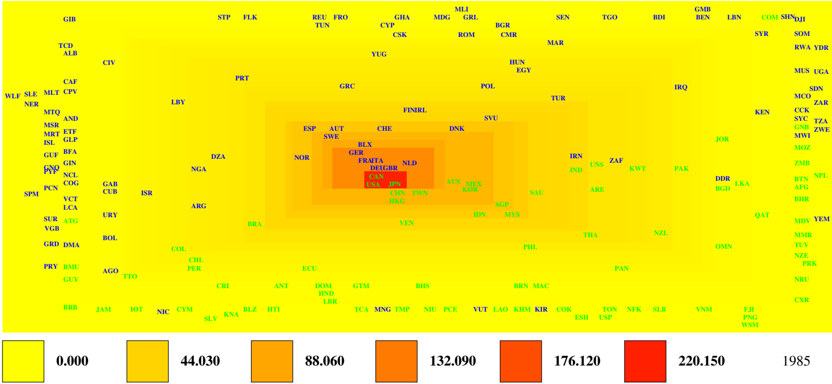

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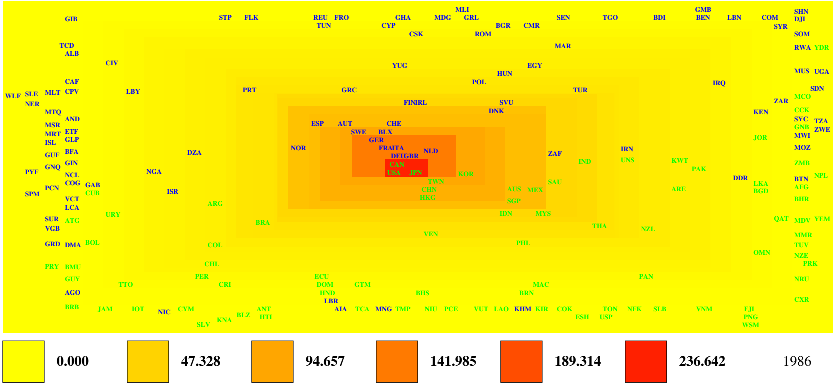

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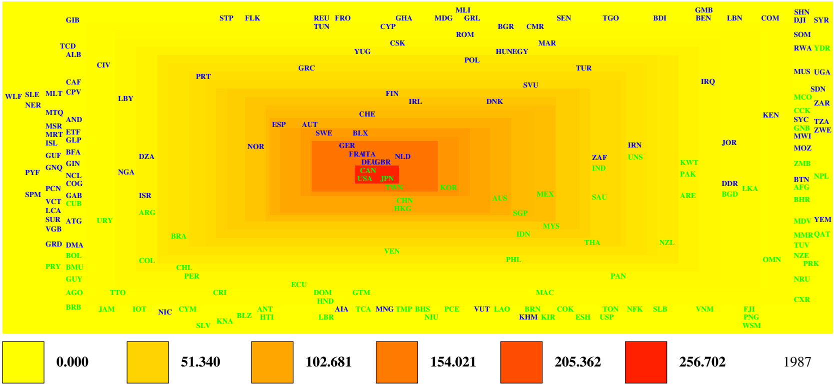

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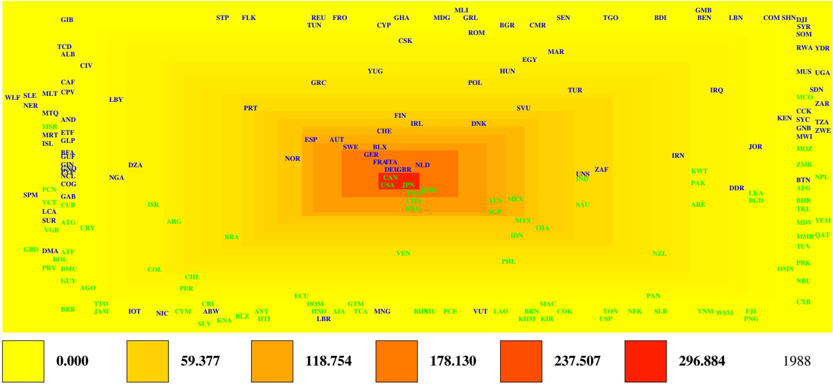

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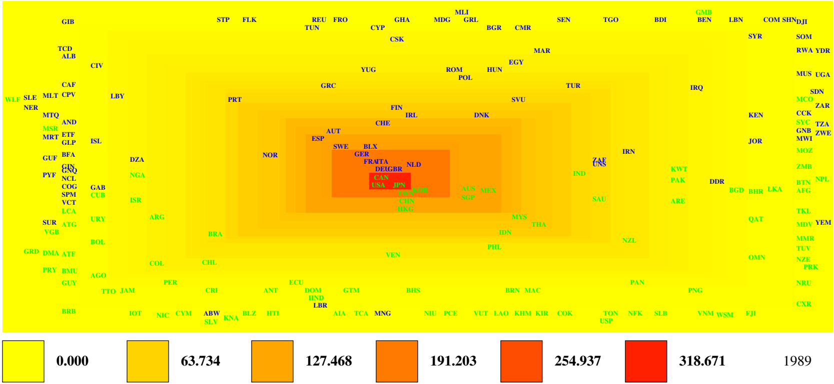

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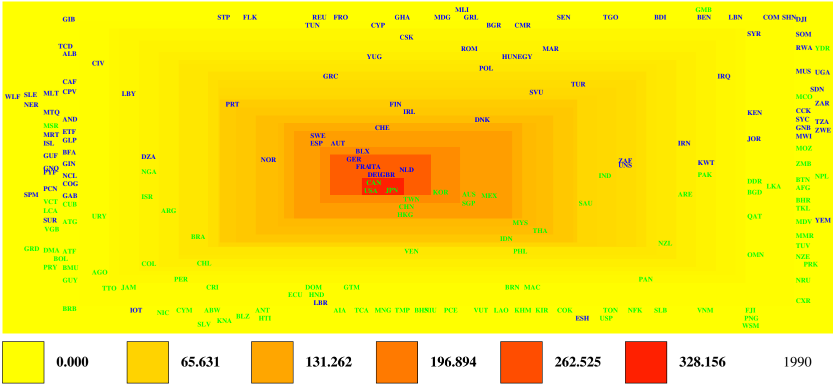

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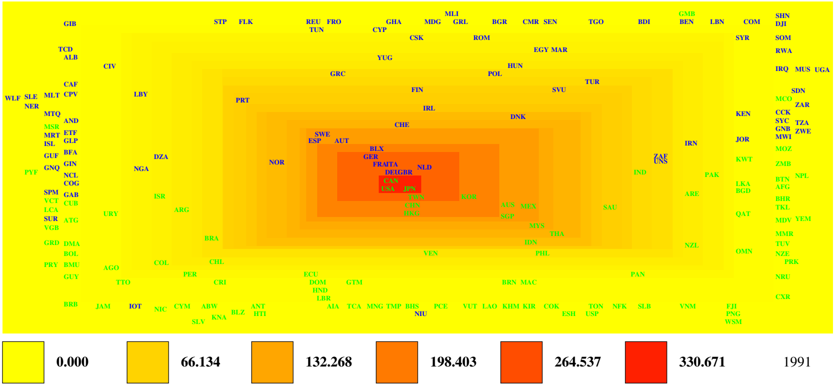

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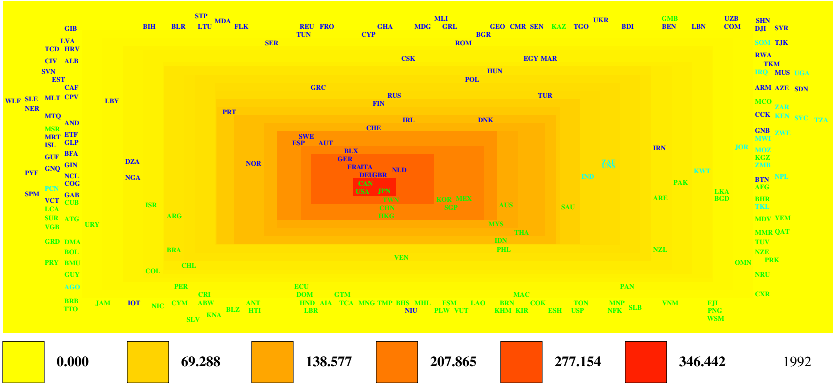

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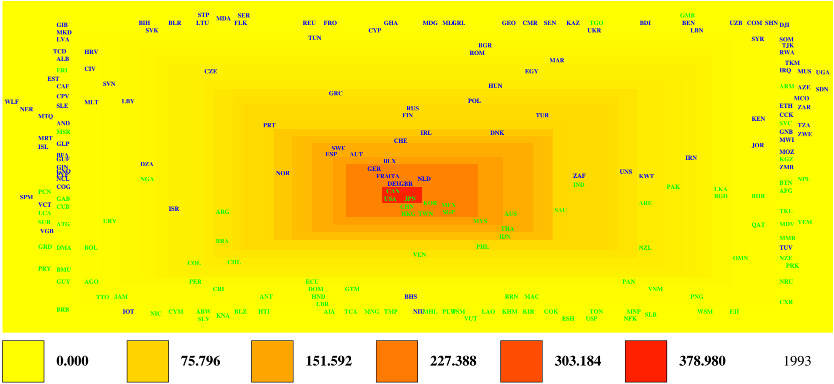

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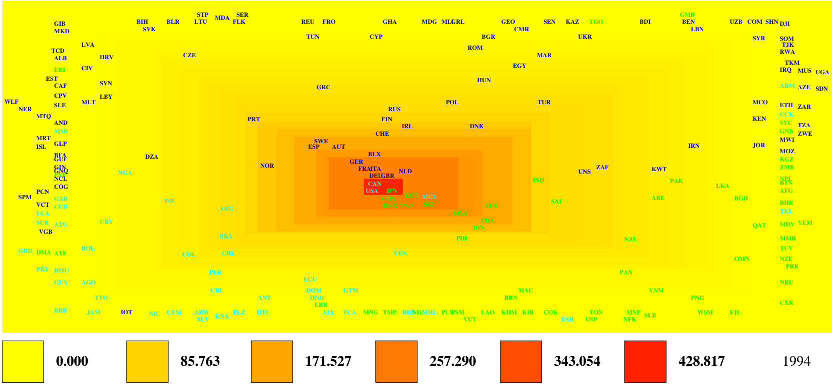

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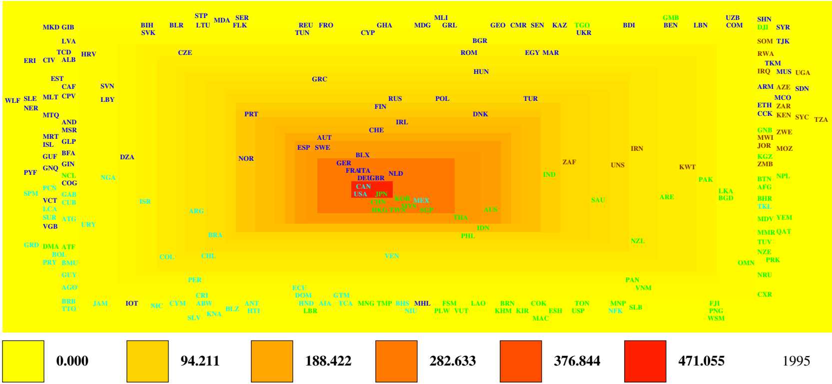

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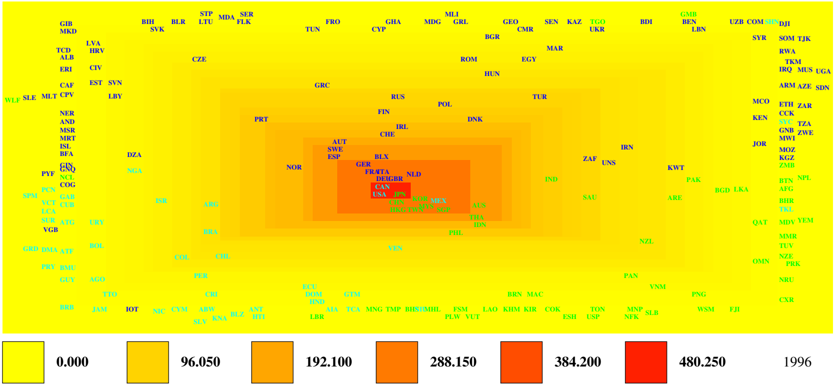

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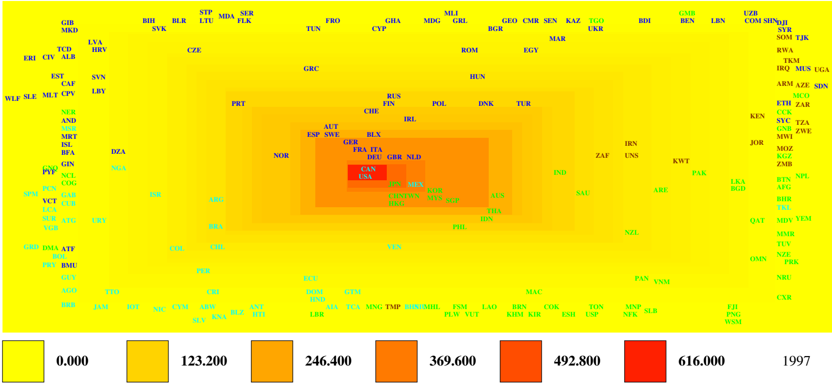

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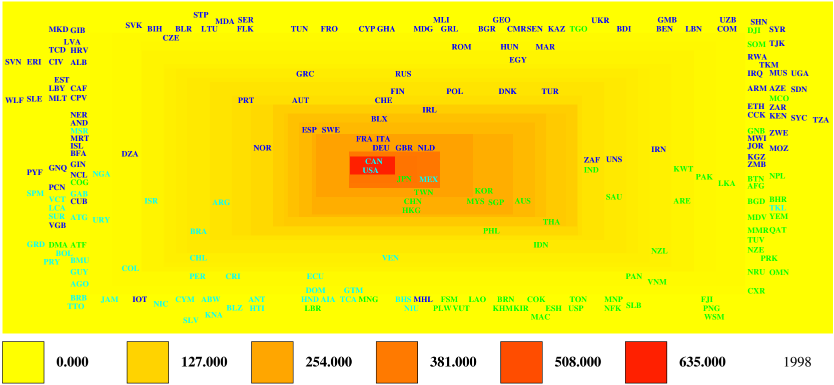

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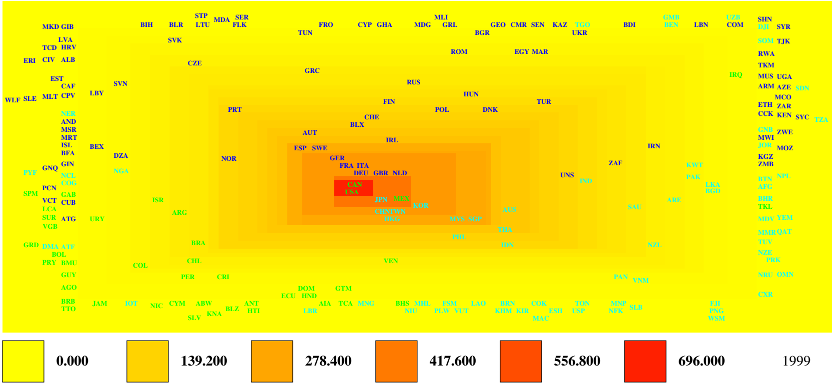
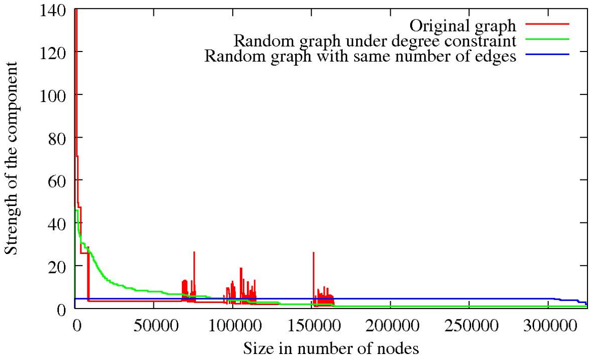


Diagram of the web



Conlusions

strength and modularity give nice analysis of graph communities

- first eigenvector of A is a relaxation of $\gamma(G)$
- second eigenvector of A is related (also with relaxations) to separation (*Cut*, *NCut* and Q)

and we have discussed of easy algorithms to compute them. . .

Questions

- what about directivity ?

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Thanks for four attention ! ! !

(and may the strength be with us)

Brute analysis of the complexity

Each edge cannot be updated more that $\frac{\log(\delta)}{\log(1+\varepsilon)} = O(\frac{\log(n)}{\varepsilon^2})$,
Each step updates $n-1$ edges and runs in $O(m\log(n))$,
$\rightarrow$ the computation takes less than $O(\frac{m^2\log(n)^2}{n\varepsilon^2})$.

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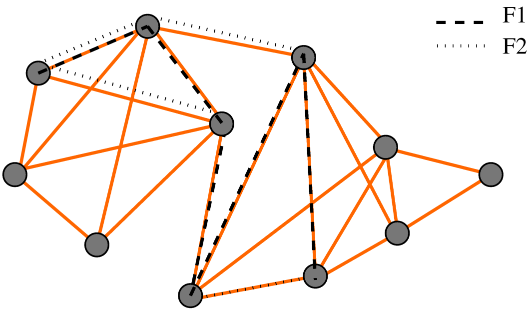
order on forests

A forest F_1 is more connecting than a forest F_2 ($F_1 \succeq F_2$) if the endpoints of any path of F_2 are connected in F_1.



augment and connecting order

Let $e \in E$. We say that e is independent of forest F if there is no path in F between endpoints of e. Otherwise it is dependent.

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Remark : Suppose $F_1 \succeq F_2$ and e is independent of F_1, then e is independent of F_2.

edge addition on ordered forests

idea : order the forests to add edges

$F_1 \succeq F_2 \succeq \cdots \succeq F_p$

take $e \in E$.

augment the first F_i such that e is independent to F_i.

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