Quantum chaos applications: from simple models to quantum computers and Google matrix

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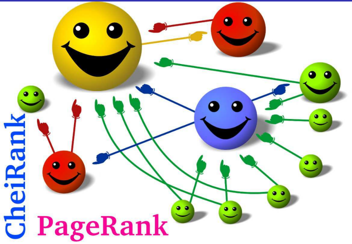
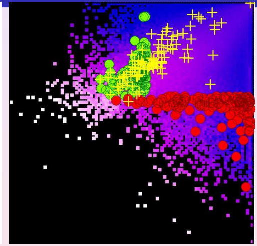
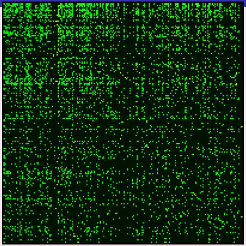


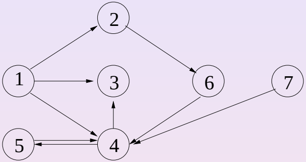




L1: Simple models of classical and quantum chaos
L2: Anderson localization in presence of nonlinearity and interactions
L3: Quantum chaos in many-body systems and quantum computers
L4: Google matrix and directed networks

S.Brin and L.Page, Comp. Networks ISDN Systems **30**, 107 (1998)

Markov chains (1906) and Directed networks



Weighted adjacency matrix

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For a directed network with N nodes the adjacency matrix $\mathbf{A}$ is defined as $A_{ij} = 1$ if there is a link from node j to node i and $A_{ij} = 0$ otherwise. The weighted adjacency matrix is

$$S_{ij} = A_{ij} / \sum_k A_{kj}$$

In addition the elements of columns with only zeros elements are replaced by $1/N$.

How Google works

Google Matrix and Computation of PageRank

$\mathbf{p} = \mathbf{Sp} \Rightarrow \mathbf{p}$= stationary vector of **S**; can be computed by iteration of **S**.

To remove convergence problems:

- Replace columns of 0 (dangling nodes) by $\frac{1}{N}$:

  In our example, $\mathbf{S} = \begin{pmatrix} 0 & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{7} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{7} & 0 & 1 & 1 & 1 \\ 0 & 0 & \frac{1}{7} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{7} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 \end{pmatrix}$.

- To remove degeneracies of $\lambda = 1$, replace **S** by **Google matrix**

  $\mathbf{G} = \alpha \mathbf{S} + (1-\alpha)\frac{\mathbf{E}}{N}$; $Gp = \lambda p$ => Perron-Frobenius operator

- α models a random surfer with a random jump after approximately 6 clicks (usually $\alpha = 0.85$); PageRank vector => p at $\lambda = 1$ ($\sum_j p_j = 1$).

- CheiRank: $\mathbf{S}^*$ with inverted link directions
  Fogaras (2003) ... Chepelianskii arXiv:1003.5455 (2010) ...

Real directed networks

Real networks are characterized by:

- **small world property**: average distance between 2 nodes $\sim \log N$
- **scale-free property**: distribution of the number of outgoing or incoming links $P(k) \sim k^{-\gamma}$

Can be explained by a twofold mechanism:

- Constant growth: new nodes appear regularly and are attached to the network
- Preferential attachment: nodes are preferentially linked to already highly connected vertices.

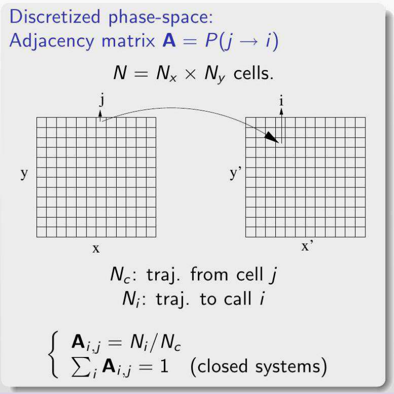
PageRank vector for large WWW:

- $p_j \sim 1/j^{\beta}$, wher j is the ordered index
- number of nodes N_n with PageRank p scales as $N_n \sim 1/p^{\nu}$ with numerical values $\nu = 1 + 1/\beta \approx 2.1$ and $\beta \approx 0.9$.
- WWW at present: $\sim 10^{11}$ web pages

Donato *et al.* EPJB **38**, 239 (2004)

Ulam networks

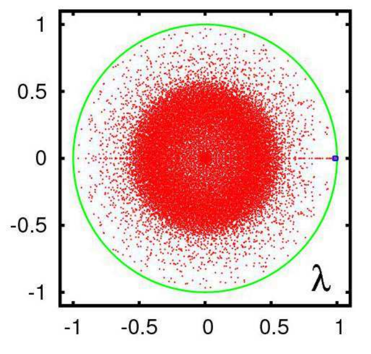
Ulam conjecture (method) for discrete approximant of Perron-Frobenius operator of dynamical systems

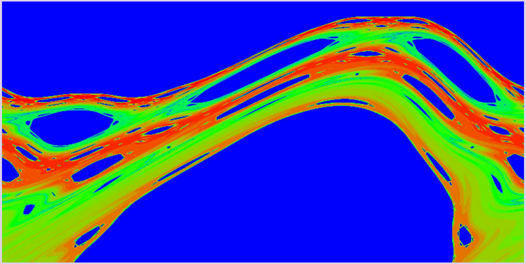


S.M.Ulam, *A Collection of mathematical problems*, Interscience, **8**, 73 N.Y. (1960)
A rigorous prove for hyperbolic maps:
T.-Y.Li J.Approx. Theory **17**, 177 (1976)
Related works:
Z. Kovacs and T. Tel, Phys. Rev. A 40, 4641 (1989)
M.Blank, G.Keller, and C.Liverani, Nonlinearity **15**, 1905 (2002)
D.Terhesiu and G.Froyland, Nonlinearity **21**, 1953 (2008)

Links to Markov chains: $\infty\infty\infty\infty\infty\infty\infty\infty\infty\infty$
Contre-example: Hamiltonian systems with invariant curves, e.g. the Chirikov standard map: noise, induced by coarse-graining, destroys the KAM curves and gives homogeneous ergodic eigenvector at $\lambda = 1$

Ulam method for the Chirikov standard map





Left: spectrum $G\psi = \lambda\psi$, $M \times M/2$ cells; $M = 280$, $N_d = 16609$, exact and Arnoldi method for matrix diagonalization; generalized Ulam method of one trajectory.

Right: modulus of eigenstate of $\lambda_2 = 0.99878...$, $M = 1600$, $N_d = 494964$.

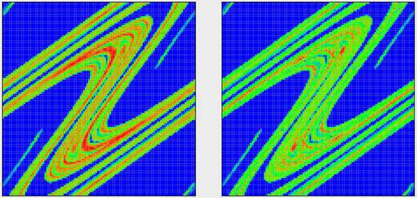
Here $K = K_G$

(Frahm, DS (2010))

Ulam method for dissipative systems

Scattering

$$\begin{cases} \bar{y} &= y + K \sin(x + y/2) \\ \bar{x} &= x + (y + \bar{y})/2 \pmod{2\pi} \end{cases}$$

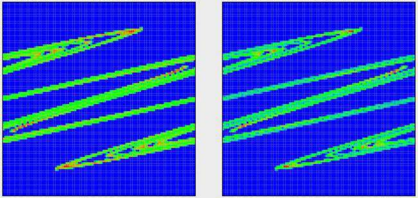


$N = 110 \times 110,\ K = 7, a = 2$

$\lambda_1 = 0.756 \quad \lambda_3 = -0.01 + i0.513$

Dissipation

$$\begin{cases} \bar{y} &= \eta y + K \sin x \\ \bar{x} &= x + \bar{y} \pmod{2\pi} \end{cases}$$



$N = 110 \times 110,\ K = 7, \eta = 0.3$

$\lambda_1 = 1 \quad \lambda_3 = -0.258 + i0.445$

(Ermann, DS (2010))

Fractal Weyl law

invented for open quantum systems, quantum chaotic scattering:
the number of Gamow eigenstates N_γ, that have escape rates γ in a finite bandwidth $0 \le \gamma \le \gamma_b$, scales as

$$N_\gamma \propto \hbar^{-\nu}, \quad \nu = d/2$$

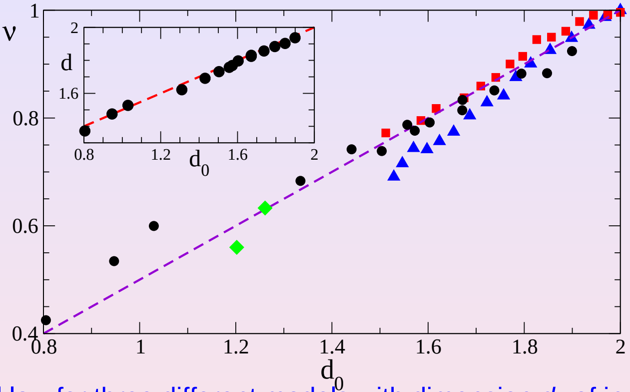
where d is a fractal dimension of a strange invariant set formed by obits non-escaping in the future and in the past

References:
J.Sjostrand, Duke Math. J. **60**, 1 (1990)
M.Zworski, Not. Am. Math. Soc. **46**, 319 (1999)
W.T.Lu, S.Sridhar and M.Zworski, Phys. Rev. Lett. **91**, 154101 (2003)
S.Nonnenmacher and M.Zworski, Commun. Math. Phys. **269**, 311 (2007)

Quantum Chirikov standard map with absorption
F.Borgonovi, I.Guarneri, DLS, Phys. Rev. A **43**, 4517 (1991)
DLS, Phys. Rev. E **77**, 015202(R) (2008)

Perron-Frobenius operators?

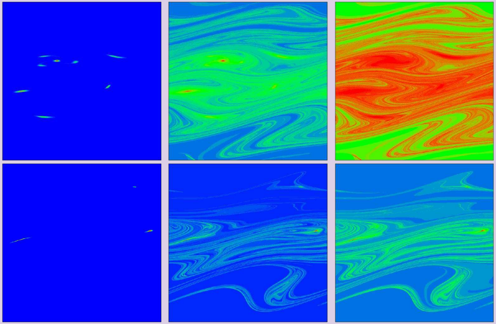
Fractal Weyl law for Ulam networks



Fractal Weyl law for three different models with dimension d_0 of invariant set. The fractal Weyl exponent ν is shown as a function of fractal dimension d_0 of the strange repeller in model 1 and strange attractor in model 2 and Henon map; dashed line shows the theory dependence $\nu = d_0/2$. Inset shows relation between the fractal dimension d of trajectories nonescaping in future and the fractal inv-set dimension d_0 for model 1; dashed line is $d = d_0/2 + 1$. (Ermann, DS (2010))

Google matrix of dynamical attractors

Weak point of AB model => large gap, no sensitivity to α



PageRank p_j for the Google matrix generated by the Chirikov typical map at $T = 10$, $k = 0.22$, $\eta = 0.99$ (set $T10$, top row) and $T = 20$, $k = 0.3$, $\eta = 0.97$ (set $T20$, bottom row) with $\alpha = 1, 0.95, 0.85$ (left to right). The phase space region $0 \le x < 2\pi; -\pi \le p < \pi$ is divided on $N = 3.6 \cdot 10^5$ cells.

Chirikov typical map (1969) with dissipation

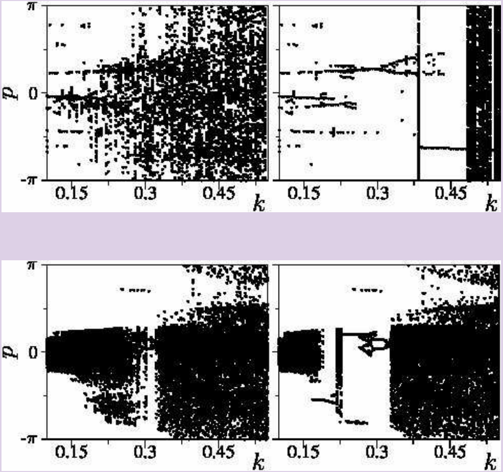
$\bar{p} = \eta p + k \sin(x + \theta_t)$, $\bar{x} = x + \bar{p}$

$\theta_t = \theta_{t+T}$ are random phases periodically repeated after T iterations, chaos border $k_c \approx 2.5/T^{3/2}$, Kolmogorov-Sinai entropy $h \approx 0.29k^{2/3}$;

grid of $N = N_x \times N_p$ cells with $N_c \sim 10^4$ trajectories which generates links (transition probabilities) from one cell to another; effective noise of cell size;

maximum $N = 22500; 1.44 \cdot 10^6$

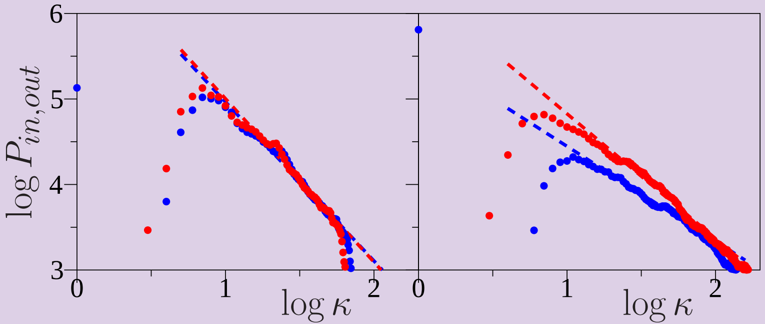
Bifurcation diagram



<= Bifurcation diagram showing values of p vs. map parameter k for the set $T10$. The values of p, obtained from 10 trajectories with initial random positions in the phase space region, are shown for integer moments of time $100 < t/T \leq 110$ (left) and $10^4 < t/T \leq 10^4 + 100$ (right).

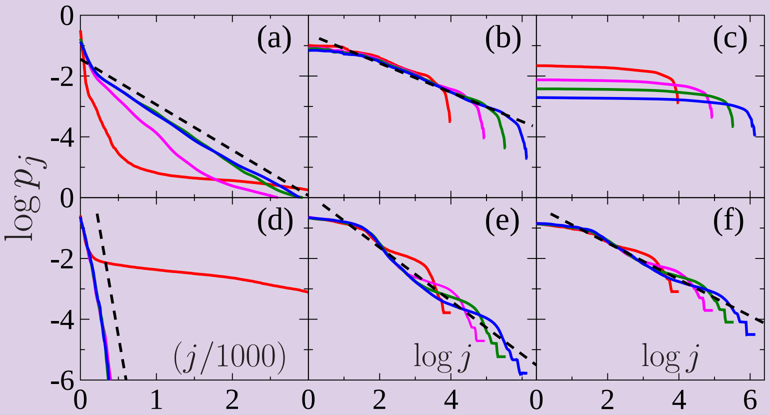
<= Same for set $T20$.

Distribution of links



Differential distribution of number of nodes with *ingoing* $P_{in}(\kappa)$ and *outgoing* $P_{out}(\kappa)$ links κ for sets $T10$ (left) and $T20$ (right). The straight dashed lines give the algebraic fit $P(\kappa) \sim \kappa^{-\mu}$ with the exponent $\mu = 1.86, 1.11$ ($T10, T20$) for *ingoing* and $\mu = 1.91, 1.46$ ($T10, T20$) *outgoing* links. Here $N = 1.44 \cdot 10^6$ and $P(\kappa)$ gives a number of nodes at a given integer number of links κ for this matrix size. Blue point at $\kappa = 0$ shows that in the whole matrix there is a significant number of nodes with zero *ingoing* links. Typical number of nodes $\kappa \sim \exp(hT)$.

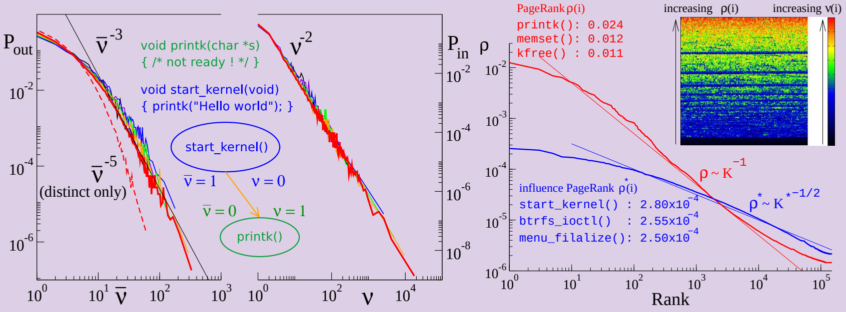
PageRank distribution



Differential distribution of number of nodes with PageRank distribution p_j for $N = 10^4$, $9 \cdot 10^4$, $3.6 \cdot 10^5$ and $1.44 \cdot 10^6$ curves, the dashed straight lines show fits $p_j \sim 1/j^{\beta}$ with β: 0.48 (b), 0.88 (e), 0.60 (f). Dashed lines in panels (a),(d) show an exponential Boltzmann decay (see text, lines are shifted in j for clarity). In panels (a),(d) the curves at large N become superimposed. Panel order as in color Fig. above.

Linux Kernel Network

Procedure call network for Linux

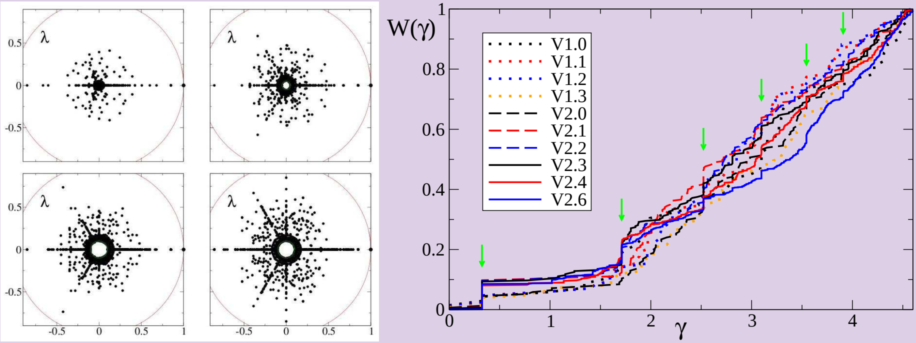


Links distribution (left); PageRank and inverse PageRank (CheiRank) distribution (right) for Linux versions up to 2.6.32 with $N = 285509$ ($\rho \sim 1/j^{\beta}, \beta = 1/(\nu - 1)$).

(Chepelianskii arxiv:1003.5455)

Fractal Weyl law for Linux Network

Sjöstrand Duke Math J. 60, 1 (1990), Zworski *et al.* PRL 91, 154101 (2003) → quantum chaotic scattering;
Ermann, DS EPJB 75, 299 (2010)→ Perron-Frobenius operators

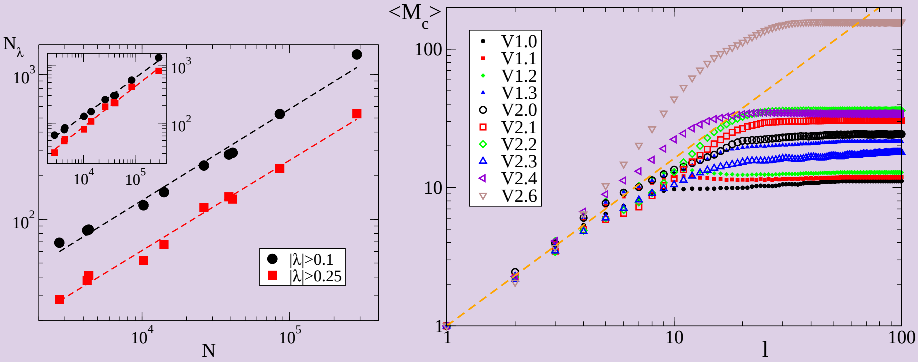


Spectrum of Google matrix (left); integrated density of states for relaxation rate $\gamma = -2 \ln |\lambda|$ (right) for Linux versions, $\alpha = 0.85$.

(Ermann, Chepelianskii, DS (2011))

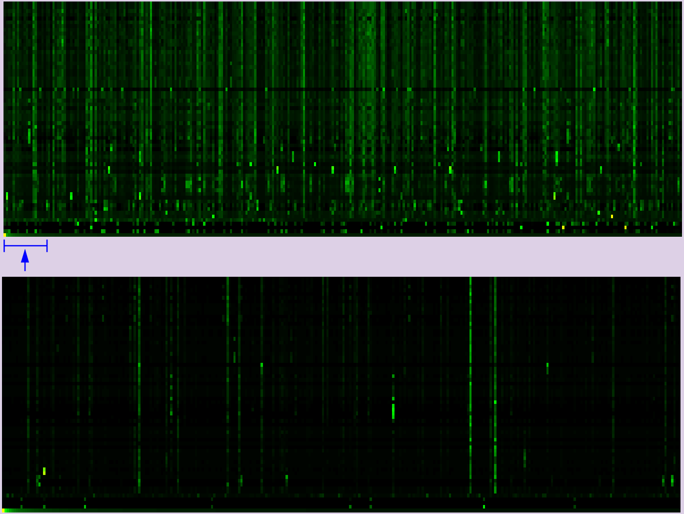
Fractal Weyl law for Linux Network

Number of states $N_\lambda \sim N^\nu, \quad \nu = d/2 \quad (N \sim 1/\hbar^{d/2})$



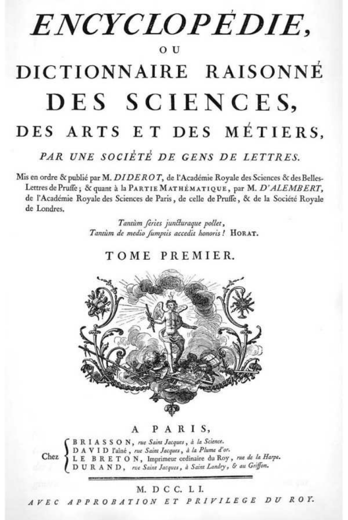
Number of states N_λ with $|\lambda| > 0.1; 0.25$ vs. N, lines show $N_\lambda \sim N^\nu$ with $\nu \approx 0.65$ (left); average mass $< M_c >$ (number of nodes) as a functon of network distance l, line shows the power law for fractal dimension $< M_c > \sim l^d$ with $d \approx 1.3$ (right).

Fractal Weyl law for Linux Network



Coarse-grained probability distribution $|\psi_i(j)|^2$ for the eigenstates of the Google matrix of Linux Kernel version 2.6.32.

From Encyclopédie (1751) to Wikipedia (2009)



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OU
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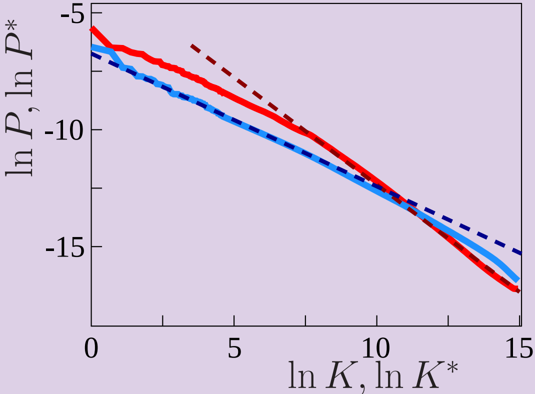
Jorge Luis Borges *The Library of Babel, Ficciones*

Wikipedia ranking of human knowledge

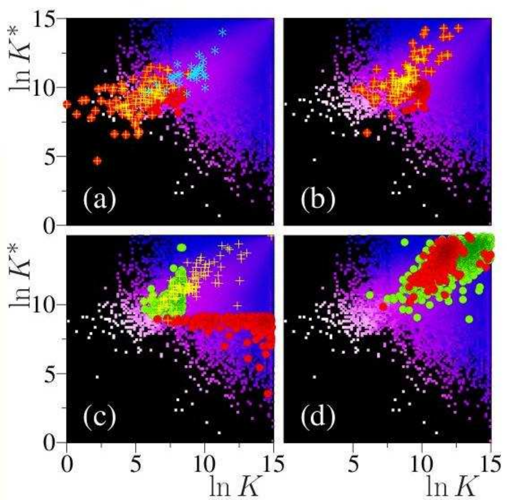
Wikipedia English articles $N = 3282257$ dated Aug 18, 2009



Dependence of probability of PagRank P (red) and CheiRank P^* (blue) on corresponding rank indexes K, K^*; lines show slopes $\beta = 1/(\nu - 1)$ with $\beta = 0.92; 0.57$ respectively for $\nu = 2.09; 2.76$.

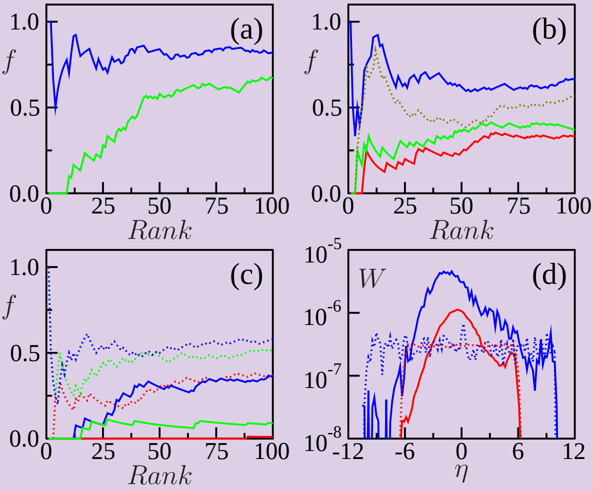
[Zhirov, Zhirov, DS (2010)]

Two-dimensional ranking of Wikipedia articles



Density distribution in plane of PageRank and CheiRank indexes ($\ln K$, $\ln K^*$): (a)100 top countries from 2DRank (red), 100 top from SJR (yellow), 30 Dow-Jones companies (cyan); (b)100 top universities from 2DRank (red) and Shanghai (yellow); (c)100 top personalities from PageRank (green), CheiRank (red) and Hart book (yellow); (d)758 physicists (green) and 193 Nobel laureates (red).

Two-dimensional ranking of Wikipedia articles



Overlap fraction f with (a)SJR countries ranking, (b)Shanghai universities ranking; (c)Hart personalities ranking (K blue, K_2 green, K^* red, black curve from Google); (d)slice of probability in 2D plane.

Wikipedia ranking of universities, personalities

Universities:

PageRank: 1. Harvard, 2. Oxford, 3. Cambridge, 4. Columbia, 5. Yale, 6. MIT, 7. Stanford, 8. Berkeley, 9. Princeton, 10. Cornell.

2DRank: 1. Columbia, 2. U. of Florida, 3. Florida State U., 4. Berkeley, 5. Northwestern U., 6. Brown, 7. U. Southern CA, 8. Carnegie Mellon, 9. MIT, 10. U. Michigan.

CheiRank: 1. Columbia, 2. U. of Florida, 3. Florida State U., 4. Brooklyn College, 5. Amherst College, 6. U. of Western Ontario, 7. U. Sheffield, 8. Berkeley, 9. Northwestern U., 10. Northeastern U.

Personalities:

PageRank: 1. Napoleon I of France, 2. George W. Bush, 3. Elizabeth II of the United Kingdom, 4. William Shakespeare, 5. Carl Linnaeus, 6. Adolf Hitler, 7. Aristotle, 8. Bill Clinton, 9. Franklin D. Roosevelt, 10. Ronald Reagan.

2DRank: 1. Michael Jackson, 2. Frank Lloyd Wright, 3. David Bowie, 4. Hillary Rodham Clinton, 5. Charles Darwin, 6. Stephen King, 7. Richard Nixon, 8. Isaac Asimov, 9. Frank Sinatra, 10. Elvis Presley.

CheiRank: 1. Kasey S. Pipes, 2. Roger Calmel, 3. Yury G. Chernavsky, 4. Josh Billings (pitcher), 5. George Lyell, 6. Landon Donovan, 7. Marilyn C. Solvay, 8. Matt Kelley, 9. Johann Georg Hagen, 10. Chikage Oogi.

Wikipedia ranking of physicists

Physicists:

PageRank: 1. Aristotle, 2. Albert Einstein, 3. Isaac Newton, 4. Thomas Edison, 5. Benjamin Franklin, 6. Gottfried Leibniz, 7. Avicenna, 8. Carl Friedrich Gauss, 9. Galileo Galilei, 10. Nikola Tesla.

2DRank: 1. Albert Einstein, 2. Nikola Tesla, 3. Benjamin Franklin, 4. Avicenna, 5. Isaac Newton, 6. Thomas Edison, 7. Stephen Hawking, 8. Gottfried Leibniz, 9. Richard Feynman, 10. Aristotle.

CheiRank: 1. Hubert Reeves, 2. Shen Kuo, 3. Stephen Hawking, 4. Nikola Tesla, 5. Albert Einstein, 6. Arthur Stanley Eddington, 7. Richard Feynman, 8. John Joseph Montgomery, 9. Josiah Willard Gibbs, 10. Heinrich Hertz.

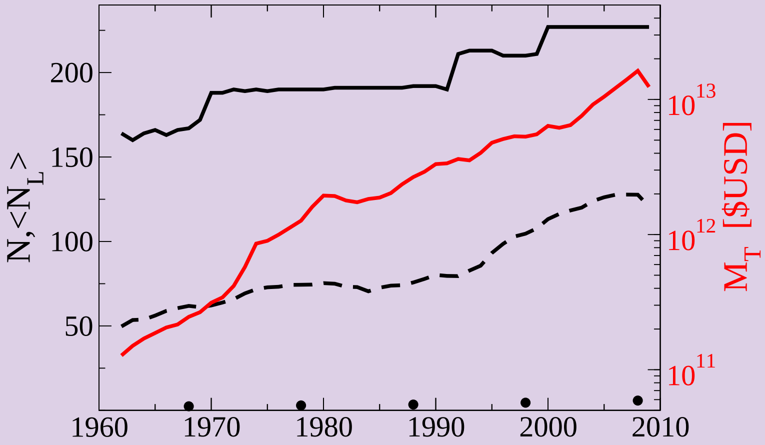
Nobel laureates:

PageRank: 1. Albert Einstein, 2. Enrico Fermi, 3. Richard Feynman, 4. Max Planck, 5. Guglielmo Marconi, 6. Werner Heisenberg, 7. Marie Curie, 8. Niels Bohr, 9. Paul Dirac, 10. J.J.Thomson.

2DRank: 1. Albert Einstein, 2. Richard Feynman, 3. Werner Heisenberg, 4. Enrico Fermi, 5. Max Born, 6. Marie Curie, 7. Wolfgang Pauli, 8. Max Planck, 9. Eugene Wigner, 10. Paul Dirac.

CheiRank: 1. Albert Einstein, 2. Richard Feynman, 3. Werner Heisenberg, 4. Brian David Josephson, 5. Abdus Salam, 6. C.V.Raman, 7. Peter Debye, 8. Enrico Fermi, 9. Wolfgang Pauli, 10. Steven Weinberg.

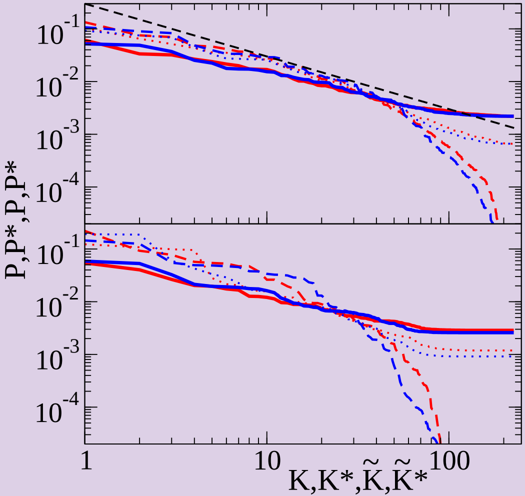
World trade network (WTN) of United Nations COMTRADE 1962-2009



Number of countries, links and mass volume in USD

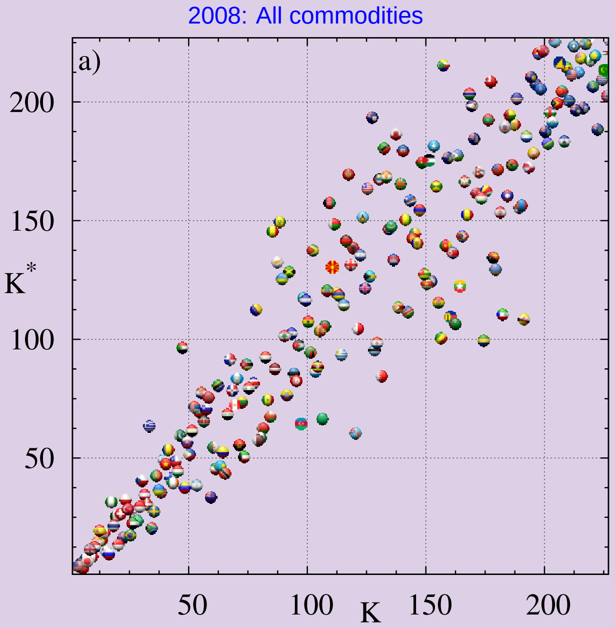
Leonardo Ermann, DS (2011)

PageRank, CheiRank of World Trade

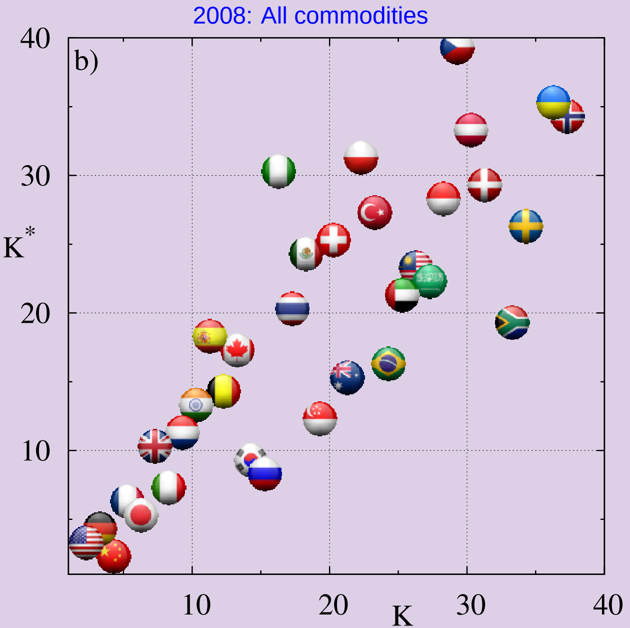


2008: Probabilities of PageRank $P(K)$ (red), CheiRank $P^*(K^*)$ (blue) for all commodities (top) and crude petroleum (bottom) (dashed curves are for ImportRank, ExportRank), 227 countries

Ranking of World Trade

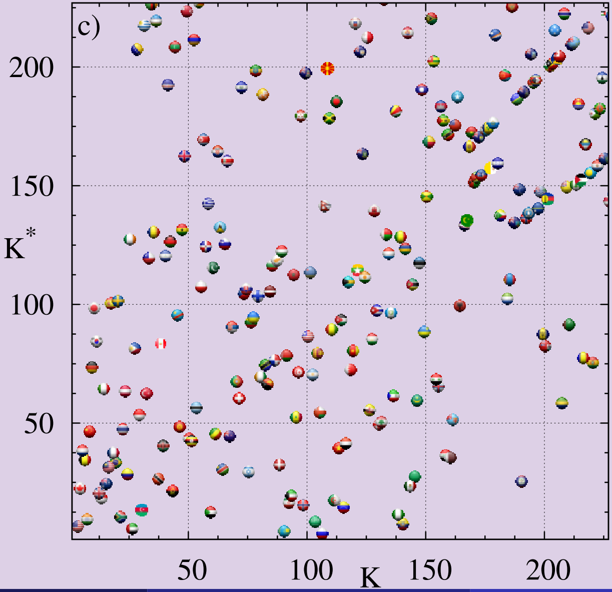


Ranking of World Trade

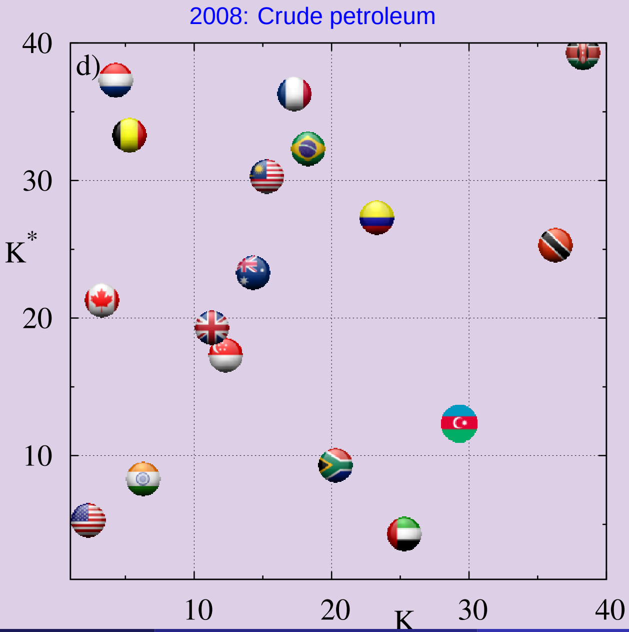


Ranking of World Trade

2008: Crude petroleum

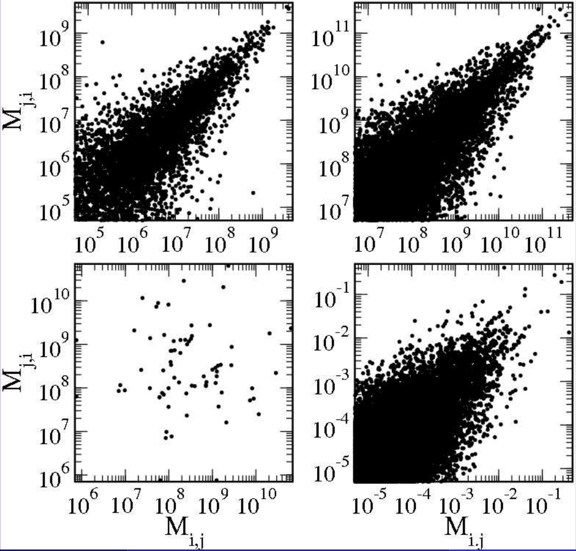


Ranking of World Trade



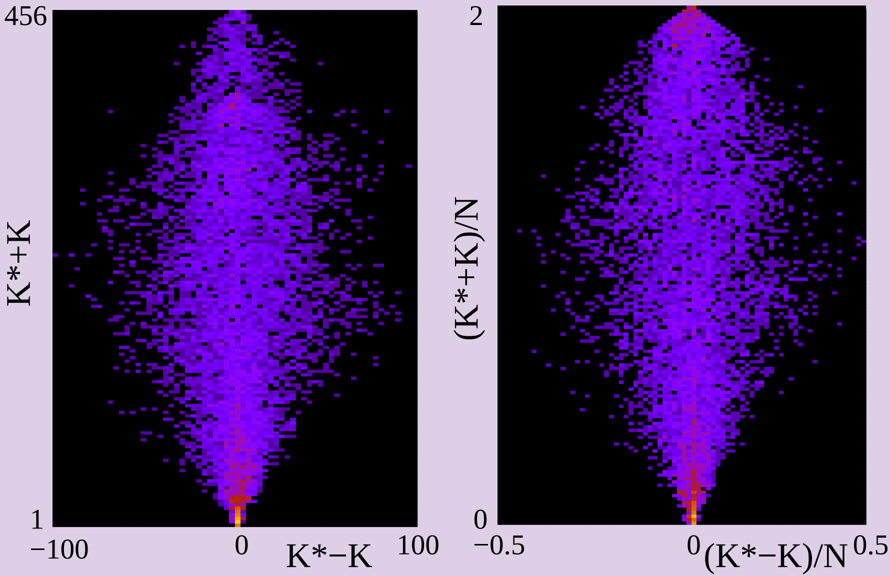
Mass flow on World Trade Network (WTN)

RMT model $M_{ij} = \epsilon_i \epsilon_j / ij$ (all commod. 1962/2008 left/right top; petroleum left bottom; model right bottom)



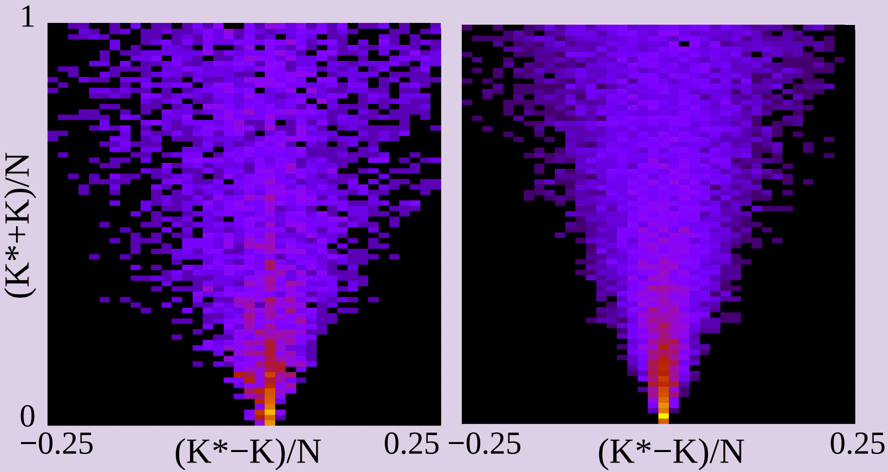
Global distribution for WTN

All commodities 1962-2009



Global distribution for WTN

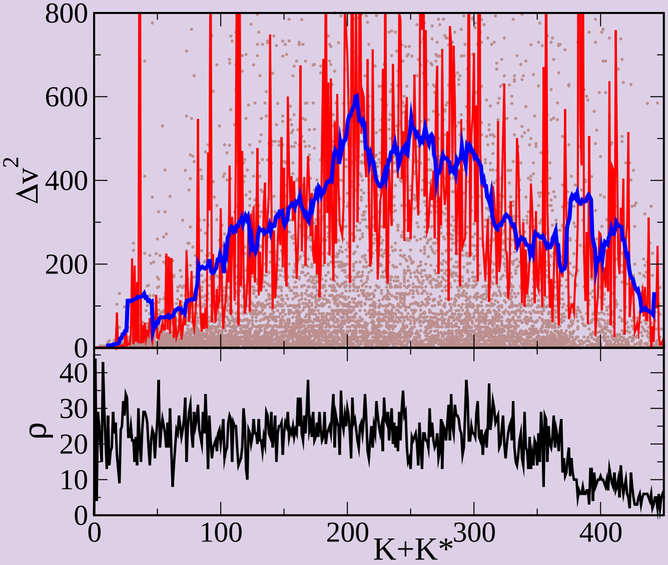
All commodities 1962-2009: left - zoom, right - RMT model



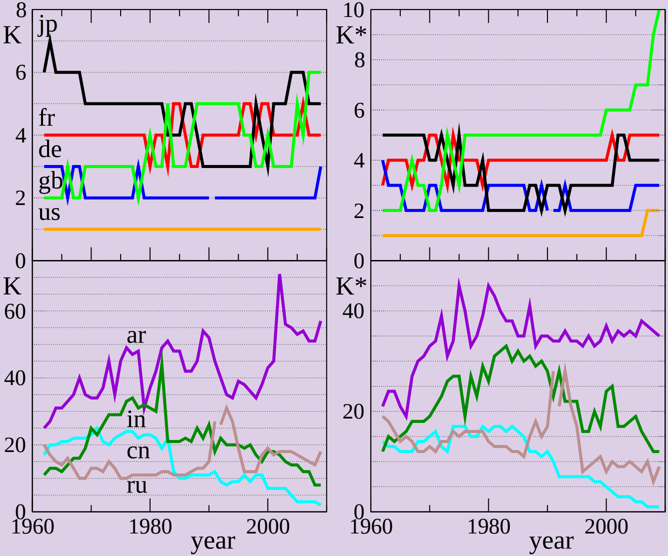
The poor stay poor and the rich stay rich

Velocity fluctuations for WTN

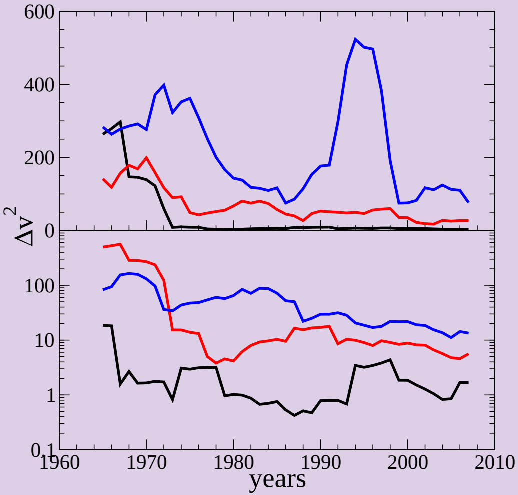
1962-2009: Rank velocity fluctuations $(\Delta v)^2 = (\Delta K)^2 + (\Delta K^*)^2$



Rank evolution in time



Rank evolution in time



Top: $1 \leq K + K^* \leq 40$; $41 \leq K + K^* \leq 80$; $81 \leq K + K^* \leq 120$;
Bottom: $1 \leq K + K^* \leq 20$; $21 \leq K + K^* \leq 40$; $41 \leq K + K^* \leq 60$

Rank table 2008 (74% of countries of G20)

Table 1. Top 20 ranking for *all commodities* – 2008.

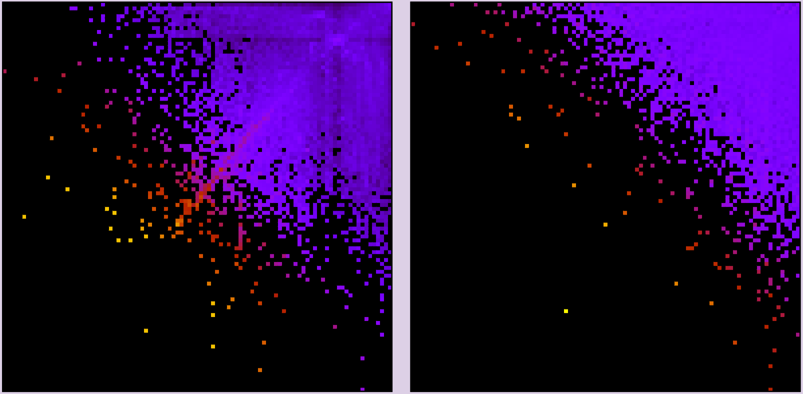
Ran	K	K^*	K_2	$\tilde{K}$	$\tilde{K}^*$
1	USA	China	USA	USA	China
2	Germany	USA	China	Germany	Germany
3	China	Germany	Germany	China	USA
4	France	Japan	Japan	France	Japan
5	Japan	France	France	Japan	France
6	UK	Italy	Italy	UK	Netherlands
7	Italy	Russian Fed.	UK	Netherlands	Italy
8	Netherlands	Rep. of Korea	Netherlands	Italy	Russian Fed.
9	India	UK	India	Belgium	UK
10	Spain	Netherlands	Rep. of Korea	Canada	Belgium
11	Belgium	Singapore	Belgium	Spain	Canada
12	Canada	India	Russian Fed.	Rep. of Korea	Rep. of Korea
13	Rep. of Korea	Belgium	Canada	Russian Fed.	Mexico
14	Russian Fed.	Australia	Spain	Mexico	Saudi Arabia
15	Nigeria	Brazil	Singapore	Singapore	Singapore
16	Thailand	Canada	Thailand	India	Spain
17	Mexico	Spain	Australia	Poland	Malaysia
18	Singapore	South Africa	Brazil	Switzerland	Brazil
19	Switzerland	Thailand	Mexico	Turkey	India
20	Australia	U. Arab Emir.	U. Arab Emir.	Brazil	Switzerland

Rank table 2008

Table 2. Top 20 ranking for *crude petroleum* – 2008.

Ran	K	K^*	K_2	$\tilde{K}$	$\tilde{K}^*$
1	USA	● Russian Fed.	USA	USA	● Saudi Arabia
2	Canada	● Kazakhstan	India	Japan	● Russian Fed.
3	Netherlands	U. Arab Emir.	Singapore	China	U. Arab Emir.
4	Belgium	USA	UK	Italy	● Nigeria
5	India	Ecuador	South Africa	Rep. of Korea	Iran
6	China	● Saudi Arabia	Canada	India	Venezuela
7	Germany	India	Australia	Germany	Norway
8	Japan	South Africa	U. Arab Emir.	Netherlands	● Canada
9	Rep. of Korea	● Nigeria	Colombia	France	Angola
10	UK	Sudan	Azerbaijan	UK	Iraq
11	Singapore	Azerbaijan	Malaysia	Spain	Libya
12	Italy	Venezuela	Brazil	Singapore	● Kazakhstan
13	Australia	Norway	Belgium	Canada	Kuwait
14	Malaysia	Iran	Trinidad and Tobago	Thailand	Azerbaijan
15	Spain	Algeria	France	Belgium	Algeria
16	France	Singapore	Netherlands	Brazil	Mexico
17	Brazil	Kuwait	Kenya	Turkey	UK
18	Sweden	UK	Angola	South Africa	Qatar
19	South Africa	Angola	China	Poland	Oman
20	Thailand	● Canada	Thailand	Australia	Netherlands

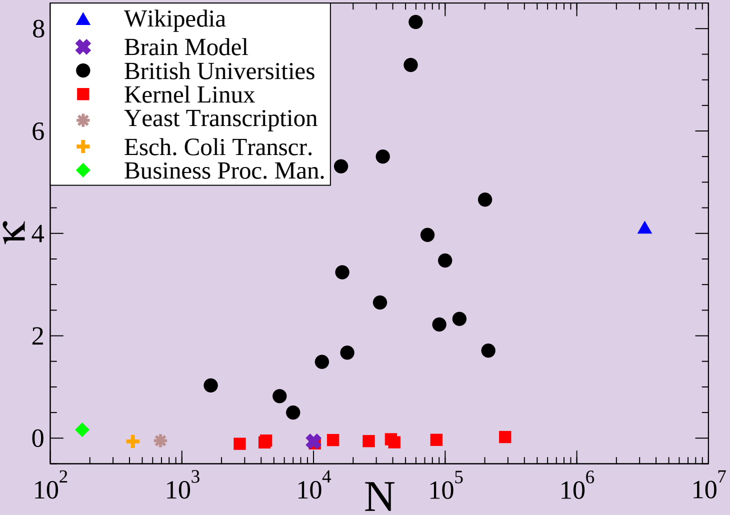
Towards two-dimensional search engine



Node density distribution in PageRank ($log_N K$) and CheRank ($log_N K^*$) plane for Univ. Cambridge (left) in 2006 and Linux v.2.4 (right) ($N \approx 200000$)

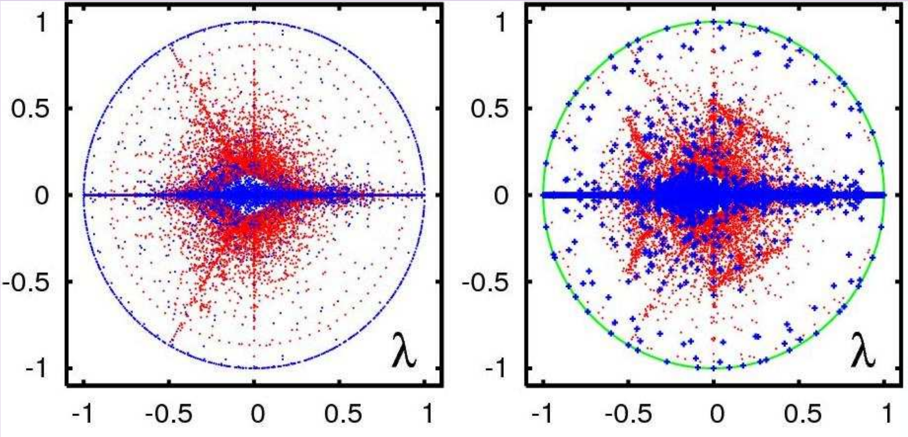
[Ermann, Chepelianskii, DS (2011)]

Correlator of PageRank and CheiRank



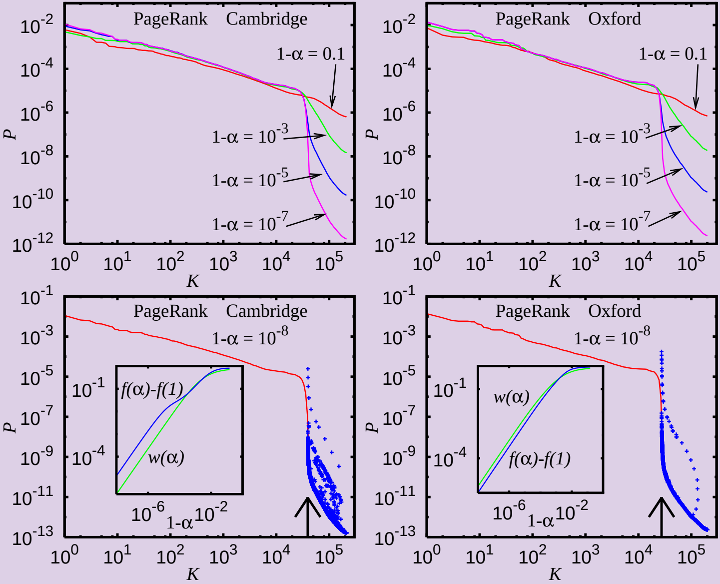
$\kappa = N \sum_i P(K(i))P(K^*(i)) - 1$

Absence of spectral gap in real WWW

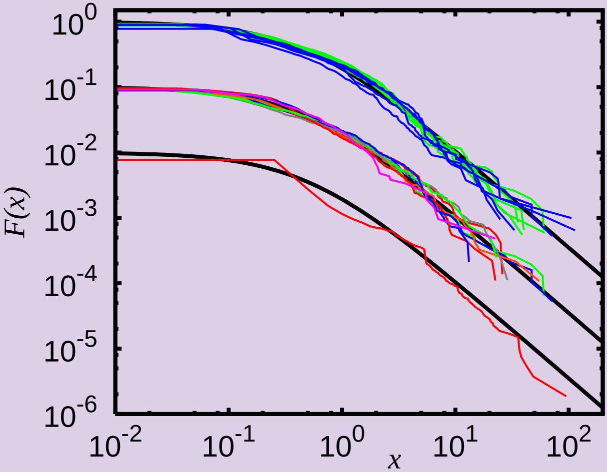


Arnoldi method: Spectrum of Google matrix for Univ. of Cambridge (left) and Oxford (right) in 2006 ($N \approx 200000$, $\alpha = 1$). [Frahm, Georgeot, DS (2011)]

Universal Emergence of PageRank

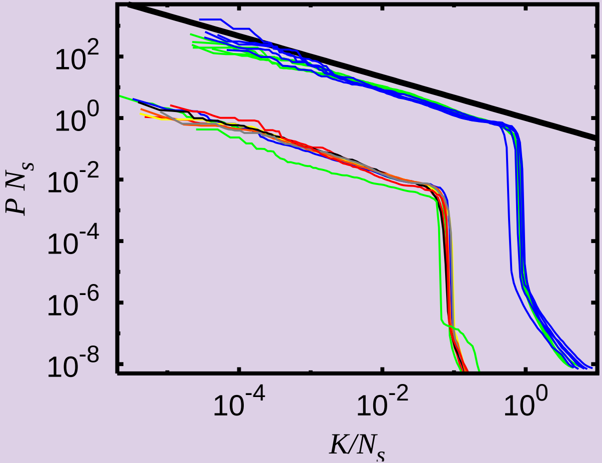


Invariant subspaces size distribution



$F(x)$ integrated number of invariant subspaces with size larger that d/d_0; $x = d/d_0$, d_0 is average size of subspaces (top: Cambridge, Oxford 2002-2006; middle: all others; bottom: Wikipedia CheiRank). Curve: $F(x) = 1/(1+2x)^{3/2}$.

PageRank at $\alpha \to 1$



Top: Cambridge, Oxford 2002-2006; bottom: all others ($\alpha = 1 - 10^{-8}$).

$P = \frac{1-\alpha}{1-\alpha S} \frac{1}{N} e$; $P = \sum_{\lambda_j=1} c_j \psi_j + \sum_{\lambda_j \neq 1} \frac{1-\alpha}{(1-\alpha)+\alpha(1-\lambda_j)} c_j \psi_j$

Google Matrix Applications

practically to everything ....

more data at
http://www.quantware.ups-tlse.fr/QWLIB/2drankwikipedia/ .../tradecheirank/

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