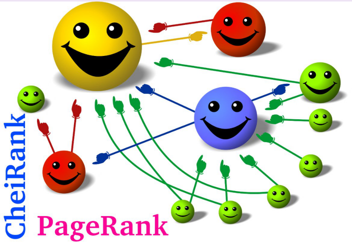
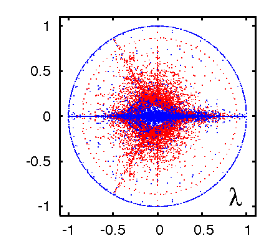
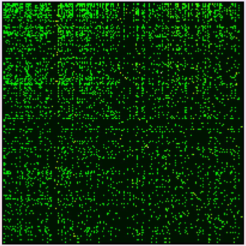
Google matrix of social and brain networks

Dima Shepelyansky (CNRS, Toulouse)
www.quantware.ups-tlse.fr/dima

Collaboration: L.Ermann, K.Frahm, B.Georgeot, O.Zhirov + A.Chepelianskii
support → EC FET Open grant NADINE







1945: Nuclear physics → Wigner (1955)→ Random Matrix Theory
1991: WWW, small world social networks → Markov (1906) → Google matrix

S.Brin and L.Page, Comp. Networks ISDN Systems **30**, 107 (1998)

WWW vision of brain network

John von Neumann ==> first parallels between architecture of the computer and the brain (1958)

WWW with 10^{11} pages vs. brain with 10^{10} neurons

PRL **94,** 018102 (2005) PHYSICAL REVIEW LETTERS week ending 14 JANUARY 2005

Scale-Free Brain Functional Networks

Victor M. Eguíluz,[1] Dante R. Chialvo,[2] Guillermo A. Cecchi,[3] Marwan Baliki,[2] and A. Vania Apkarian[2]
[1]*Instituto Mediterráneo de Estudios Avanzados, IMEDEA (CSIC-UIB), E07122 Palma de Mallorca, Spain*
[2]*Department of Physiology, Northwestern University, Chicago, Illinois, 60611, USA*
[3]*IBM T.J. Watson Research Center, 1101 Kitchawan Rd., Yorktown Heights, New York 10598, USA*
(Received 13 January 2004; published 6 January 2005)

Functional magnetic resonance imaging is used to extract *functional networks* connecting correlated human brain sites. Analysis of the resulting networks in different tasks shows that (a) the distribution of

doi:10.1093/brain/awr223 Brain 2011: 134; 2912–2928 | **2912**

BRAIN
A JOURNAL OF NEUROLOGY

Altered functional–structural coupling of large-scale brain networks in idiopathic generalized epilepsy

Zhiqiang Zhang,[1,*] Wei Liao,[2,*] Huafu Chen,[2] Dante Mantini,[3] Ju-Rong Ding,[2] Qiang Xu,[2] Zhengge Wang,[1] Cuiping Yuan,[1] Guanghui Chen,[4] Qing Jiao[1] and Guangming Lu[1]

Brain networks

NeuroImage 59 (2012) 3784–3804

Contents lists available at SciVerse ScienceDirect

NeuroImage

journal homepage: www.elsevier.com/locate/ynimg

Hierarchical topological network analysis of anatomical human brain connectivity and differences related to sex and kinship

Julio M. Duarte-Carvajalino [a], Neda Jahanshad [b,c], Christophe Lenglet [d], Katie L. McMahon [e], Greig I. de Zubicaray [f], Nicholas G. Martin [g], Margaret J. Wright [f,g], Paul M. Thompson [b], Guillermo Sapiro [a,*]

[a] Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis, MN, USA
[b] Laboratory of Neuro Imaging, Department of Neurology, UCLA School of Medicine, Los Angeles, CA, USA

REVIEWS

Nature Reviews Neuroscience | AOP, published online 13 April 2012; doi:10.1038/nrn3214

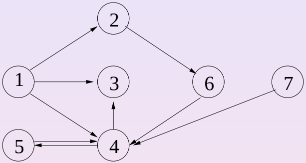
The economy of brain network organization

Ed Bullmore[1,2,3] and Olaf Sporns[4]

Abstract | The brain is expensive, incurring high material and metabolic costs for its size — relative to the size of the body — and many aspects of brain network organization can be

How Google works

Markov chains (1906) and Directed networks



Weighted adjacency matrix

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For a directed network with N nodes the adjacency matrix $\mathbf{A}$ is defined as $A_{ij} = 1$ if there is a link from node j to node i and $A_{ij} = 0$ otherwise. The weighted adjacency matrix is

$$S_{ij} = A_{ij} / \sum_k A_{kj}$$

In addition the elements of columns with only zeros elements are replaced by $1/N$.

How Google works

Google Matrix and Computation of PageRank

$\mathbf{P} = \mathbf{SP} \Rightarrow$ **P**= stationary vector of **S**; can be computed by iteration of **S**.

To remove convergence problems:

- Replace columns of 0 (dangling nodes) by $\frac{1}{N}$:

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{7} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{7} & 0 & 1 & 1 & 1 \\ 0 & 0 & \frac{1}{7} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{7} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 \end{pmatrix}; \mathbf{S}^* = \begin{pmatrix} \frac{1}{7} & 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & 0 & 0 & 1 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & \frac{1}{2} & 0 & 1 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{7} \end{pmatrix}.$$

- To remove degeneracies of $\lambda = 1$, replace **S** by **Google matrix**

  $\mathbf{G} = \alpha \mathbf{S} + (1 - \alpha)\frac{\mathbf{E}}{N}$; $GP = \lambda P$ => Perron-Frobenius operator

- α models a random surfer with a random jump after approximately 6 clicks (usually $\alpha = 0.85$); PageRank vector => P at $\lambda = 1$ ($\sum_j P_j = 1$).
- CheiRank vector P^*: $G^* = \alpha \mathbf{S}^* + (1 - \alpha)\frac{\mathbf{E}}{N}$, $G^* P^* = P^*$ ($\mathbf{S}^*$ with inverted link directions)

  Fogaras (2003) ... Chepelianskii arXiv:1003.5455 (2010) ...

Real directed networks

Real networks are characterized by:

- **small world property**: average distance between 2 nodes $\sim \log N$
- **scale-free property**: distribution of the number of ingoing or outgoing links $\rho(k) \sim k^{-\nu}$

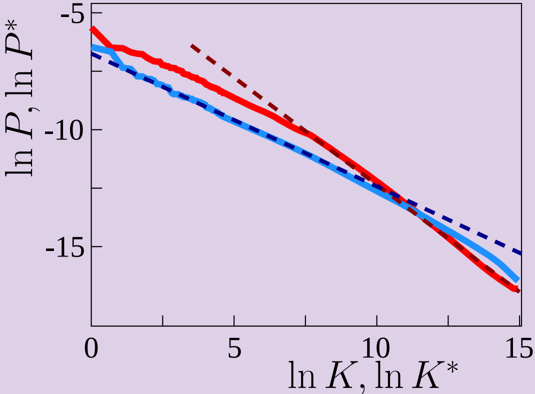
PageRank vector for large WWW:

- $P(K) \sim 1/K^{\beta}$, where K is the ordered rank index
- number of nodes N_n with PageRank P scales as $N_n \sim 1/P^{\nu}$ with numerical values $\nu = 1 + 1/\beta \approx 2.1$ and $\beta \approx 0.9$.
- PageRank $P(K)$ on average is proportional to the number of ingoing links
- CheiRank $P^*(K^*) \sim 1/K^{*\beta}$ on average is proportional to the number of outgoing links ($\nu \approx 2.7; \beta = 1/(\nu - 1) \approx 0.6$)
- WWW at present: $\sim 10^{11}$ web pages

Donato *et al.* EPJB **38**, 239 (2004)

Wikipedia ranking of human knowledge

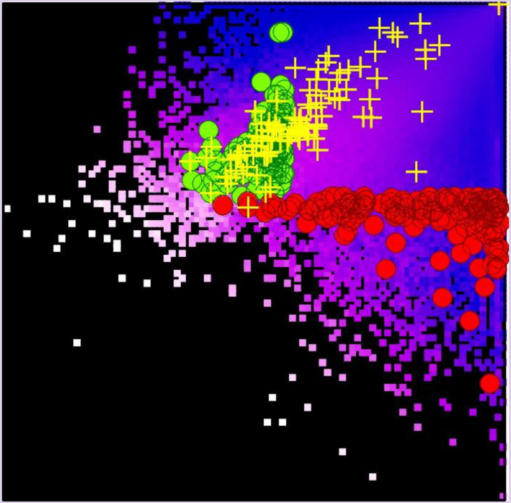
Wikipedia English articles $N = 3282257$ dated Aug 18, 2009



Dependence of probability of PagRank P (red) and CheiRank P^* (blue) on corresponding rank indexes K, K^*; lines show slopes $\beta = 1/(\nu - 1)$ with $\beta = 0.92; 0.57$ respectively for $\nu = 2.09; 2.76$.

[Zhirov, Zhirov, DS EPJB **77**, 523 (2010)]

Two-dimensional ranking of Wikipedia articles



Density distribution in plane of PageRank and CheiRank indexes $(\ln K, \ln K^*)$: 100 top personalities from PageRank (green), CheiRank (red) and Hart book (yellow)

Wikipedia ranking of universities, personalities

Universities:

PageRank: 1. Harvard, 2. Oxford, 3. Cambridge, 4. Columbia, 5. Yale, 6. MIT, 7. Stanford, 8. Berkeley, 9. Princeton, 10. Cornell.

2DRank: 1. Columbia, 2. U. of Florida, 3. Florida State U., 4. Berkeley, 5. Northwestern U., 6. Brown, 7. U. Southern CA, 8. Carnegie Mellon, 9. MIT, 10. U. Michigan.

CheiRank: 1. Columbia, 2. U. of Florida, 3. Florida State U., 4. Brooklyn College, 5. Amherst College, 6. U. of Western Ontario, 7. U. Sheffi eld, 8. Berkeley, 9. Northwestern U., 10. Northeastern U.

Personalities:

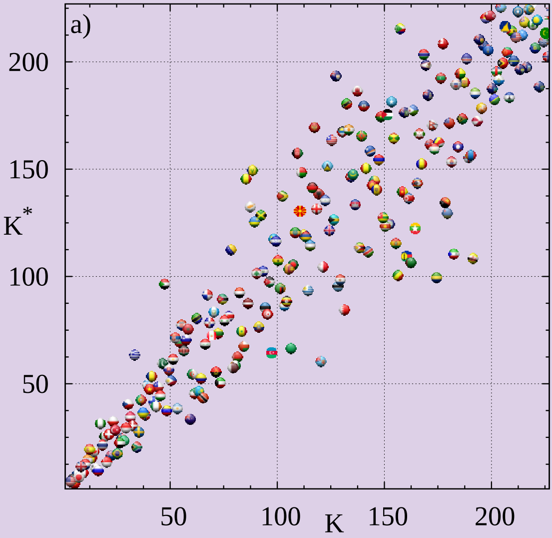
PageRank: 1. Napoleon I of France, 2. George W. Bush, 3. Elizabeth II of the United Kingdom, 4. William Shakespeare, 5. Carl Linnaeus, 6. Adolf Hitler, 7. Aristotle, 8. Bill Clinton, 9. Franklin D. Roosevelt, 10. Ronald Reagan.

2DRank: 1. Michael Jackson, 2. Frank Lloyd Wright, 3. David Bowie, 4. Hillary Rodham Clinton, 5. Charles Darwin, 6. Stephen King, 7. Richard Nixon, 8. Isaac Asimov, 9. Frank Sinatra, 10. Elvis Presley.

CheiRank: 1. Kasey S. Pipes, 2. Roger Calmel, 3. Yury G. Chernavsky, 4. Josh Billings (pitcher), 5. George Lyell, 6. Landon Donovan, 7. Marilyn C. Solvay, 8. Matt Kelley, 9. Johann Georg Hagen, 10. Chikage Oogi.

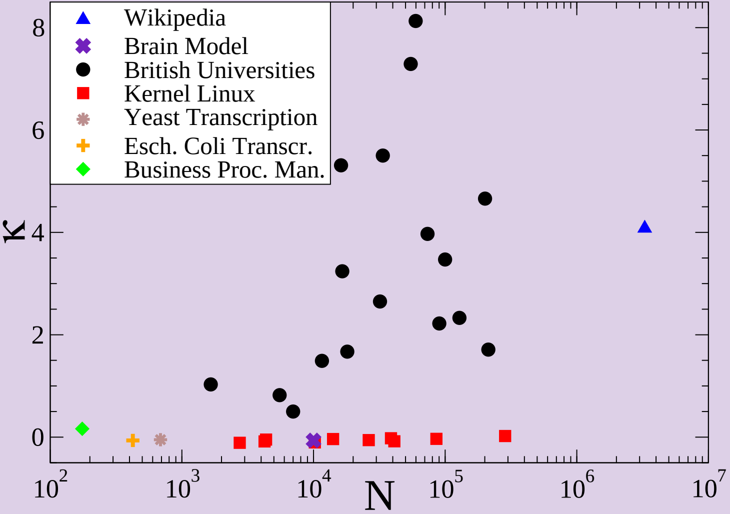
Ranking of World Trade

UN COMTRADE database 2008: All commodities



Ermann, DS arxiv:1103.5027 (2011)

Correlator of PageRank and CheiRank



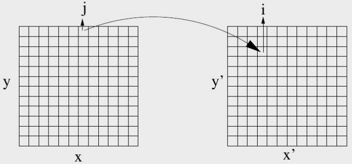
$\kappa = N \sum_i P(K(i))P^*(K^*(i)) - 1$

Ulam networks

Ulam conjecture (method) for discrete approximant of Perron-Frobenius operator of dynamical systems

Discretized phase-space:
Adjacency matrix $\mathbf{A} = P(j \to i)$

$N = N_x \times N_y$ cells.



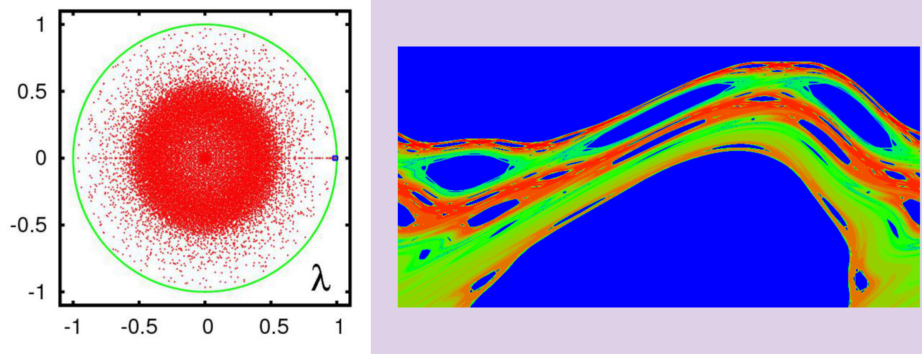
N_c: traj. from cell j
N_i: traj. to call i

$$\begin{cases} \mathbf{A}_{i,j} = N_i/N_c \\ \sum_i \mathbf{A}_{i,j} = 1 \quad \text{(closed systems)} \end{cases}$$

S.M.Ulam, *A Collection of mathematical problems*, Interscience, **8**, 73 N.Y. (1960)
A rigorous prove for hyperbolic maps:
T.-Y.Li J.Approx. Theory **17**, 177 (1976)
Related works:
Z. Kovacs and T. Tel, Phys. Rev. A 40, 4641 (1989)
M.Blank, G.Keller, and C.Liverani, Nonlinearity **15**, 1905 (2002)
D.Terhesiu and G.Froyland, Nonlinearity **21**, 1953 (2008)

Links to Markov chains: $\infty\infty\infty\infty\infty\infty\infty\infty\infty\infty$
Contre-example: Hamiltonian systems with invariant curves, e.g. the Chirikov standard map: noise, induced by coarse-graining, destroys the KAM curves and gives homogeneous ergodic eigenvector at $\lambda = 1$

Ulam method for the Chirikov standard map

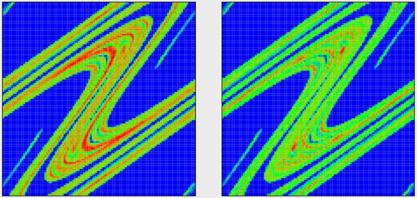


Left: spectrum $G\psi = \lambda\psi$, $M \times M/2$ cells; $M = 280$, $N_d = 16609$, exact and Arnoldi method for matrix diagonalization; generalized Ulam method of one trajectory.
Right: modulus of eigenstate of $\lambda_2 = 0.99878...$, $M = 1600$, $N_d = 494964$.
Here $K = K_G$
(Frahm, DS (2010))

Ulam method for dissipative systems

Scattering

$$\begin{cases} \bar{y} = y + K \sin(x + y/2) \\ \bar{x} = x + (y + \bar{y})/2 \pmod{2\pi} \end{cases}$$

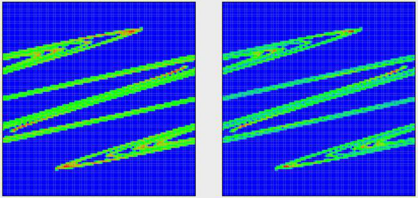


$N = 110 \times 110,\ K = 7, a = 2$

$\lambda_1 = 0.756 \quad \lambda_3 = -0.01 + i0.513$

Dissipation

$$\begin{cases} \bar{y} = \eta y + K \sin x \\ \bar{x} = x + \bar{y} \pmod{2\pi} \end{cases}$$



$N = 110 \times 110,\ K = 7, \eta = 0.3$

$\lambda_1 = 1 \quad \lambda_3 = -0.258 + i0.445$

(Ermann, DS (2010))

Fractal Weyl law

invented for open quantum systems, quantum chaotic scattering:
the number of Gamow eigenstates N_γ, that have escape rates γ in a finite bandwidth $0 \leq \gamma \leq \gamma_b$, scales as

$$N_\gamma \propto \hbar^{-\nu}, \quad \nu = d/2$$

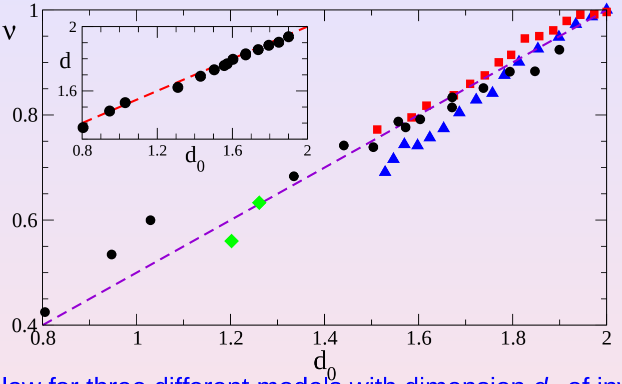
where d is a fractal dimension of a strange invariant set formed by obits non-escaping in the future and in the past

References:
J.Sjostrand, Duke Math. J. **60**, 1 (1990)
M.Zworski, Not. Am. Math. Soc. **46**, 319 (1999)
W.T.Lu, S.Sridhar and M.Zworski, Phys. Rev. Lett. **91**, 154101 (2003)
S.Nonnenmacher and M.Zworski, Commun. Math. Phys. **269**, 311 (2007)

Quantum Chirikov standard map with absorption
F.Borgonovi, I.Guarneri, DLS, Phys. Rev. A **43**, 4517 (1991)
DLS, Phys. Rev. E **77**, 015202(R) (2008)

Perron-Frobenius operators?

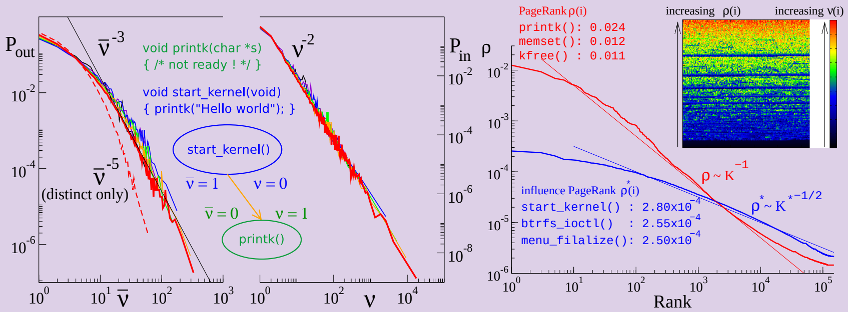
Fractal Weyl law for Ulam networks



Fractal Weyl law for three different models with dimension d_0 of invariant set. The fractal Weyl exponent ν is shown as a function of fractal dimension d_0 of the strange repeller in model 1 and strange attractor in model 2 and Henon map; dashed line shows the theory dependence $\nu = d_0/2$. Inset shows relation between the fractal dimension d of trajectories nonescaping in future and the fractal inv-set dimension d_0 for model 1; dashed line is $d = d_0/2 + 1$. (Ermann, DS (2010))

Linux Kernel Network

Procedure call network for Linux

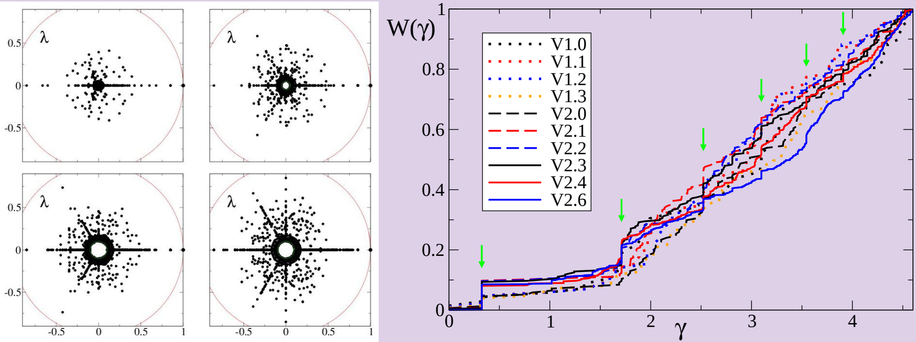


Links distribution (left); PageRank and inverse PageRank (CheiRank) distribution (right) for Linux versions up to 2.6.32 with $N = 285509$ ($\rho \sim 1/j^{\beta}, \beta = 1/(\nu - 1)$).

(Chepelianskii arxiv:1003.5455)

Fractal Weyl law for Linux Network

Sjöstrand Duke Math J. 60, 1 (1990), Zworski *et al.* PRL 91, 154101 (2003) → quantum chaotic scattering;
Ermann, DS EPJB 75, 299 (2010)→ Perron-Frobenius operators

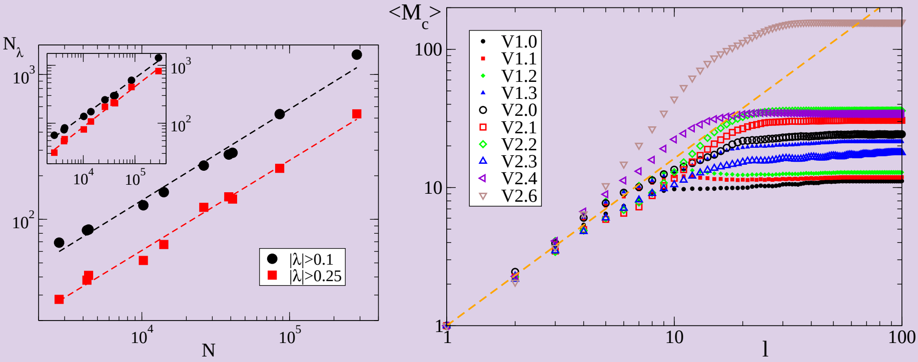


Spectrum of Google matrix (left); integrated density of states for relaxation rate $\gamma = -2\ln|\lambda|$ (right) for Linux versions, $\alpha = 0.85$.

(Ermann, Chepelianskii, DS (2011))

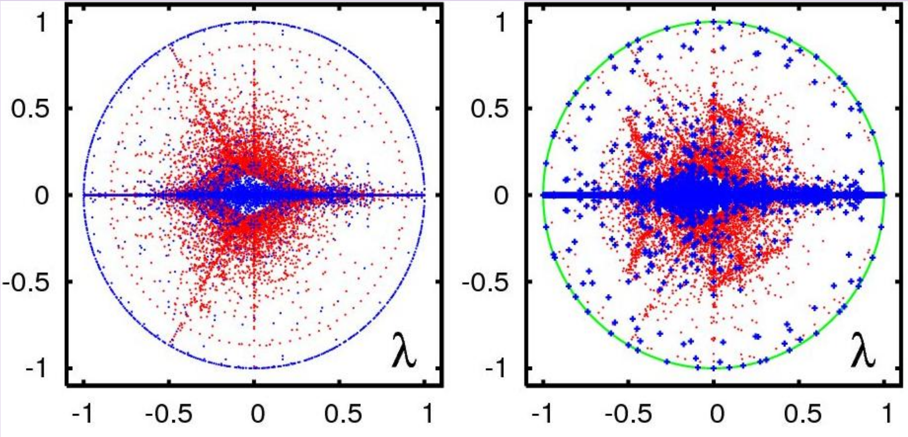
Fractal Weyl law for Linux Network

Number of states $N_\lambda \sim N^\nu, \quad \nu = d/2 \quad (N \sim 1/\hbar^{d/2})$



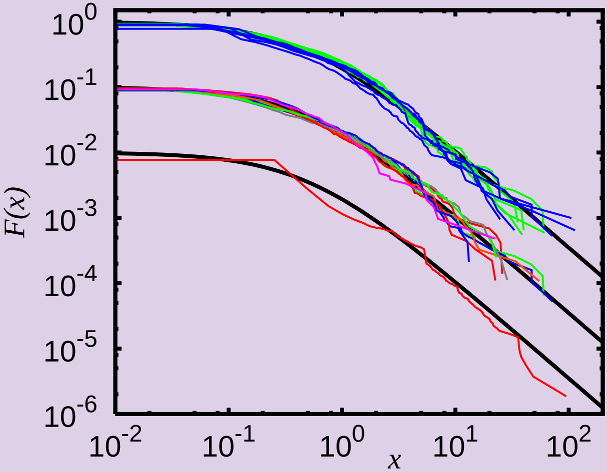
Number of states N_λ with $|\lambda| > 0.1; 0.25$ vs. N, lines show $N_\lambda \sim N^\nu$ with $\nu \approx 0.65$ (left); average mass $< M_c >$ (number of nodes) as a functon of network distance l, line shows the power law for fractal dimension $< M_c > \sim l^d$ with $d \approx 1.3$ (right).

Spectrum of UK University networks



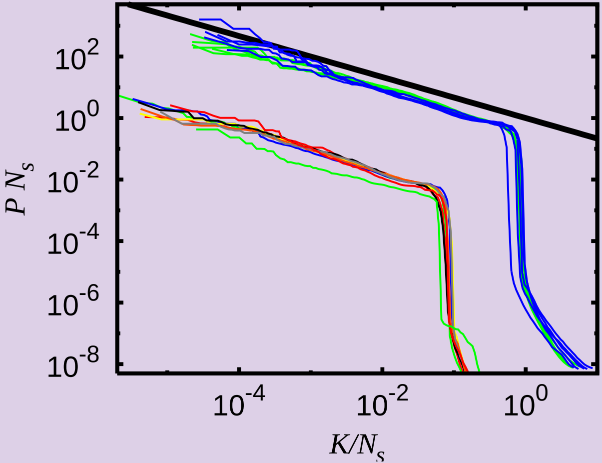
Arnoldi method: Spectrum of Google matrix for Univ. of Cambridge (left) and Oxford (right) in 2006; 20% at $\lambda = 1$ ($N \approx 200000$, $\alpha = 1$). [Frahm, Georgeot, DS arxiv:1105.1062 (2011)]

Invariant subspaces size distribution



$F(x)$ integrated number of invariant subspaces with size larger that d/d_0; $x = d/d_0$, d_0 is average size of subspaces (top: Cambridge, Oxford 2002-2006; middle: all others; bottom: Wikipedia CheiRank). Curve: $F(x) = 1/(1+2x)^{3/2}$.

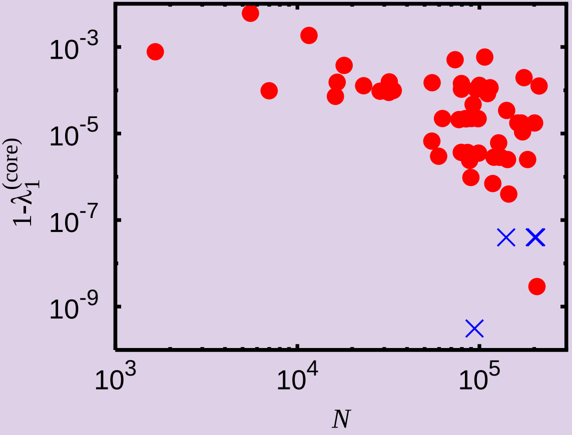
PageRank at $\alpha \to 1$



Top: Cambridge, Oxford 2002-2006; bottom: all others ($\alpha = 1 - 10^{-8}$).

$P = \frac{1-\alpha}{1-\alpha S}\,\frac{1}{N}\,e \;\; ; \quad P = \sum_{\lambda_j=1} c_j\,\psi_j + \sum_{\lambda_j \neq 1} \frac{1-\alpha}{(1-\alpha)+\alpha(1-\lambda_j)}\,c_j\,\psi_j$

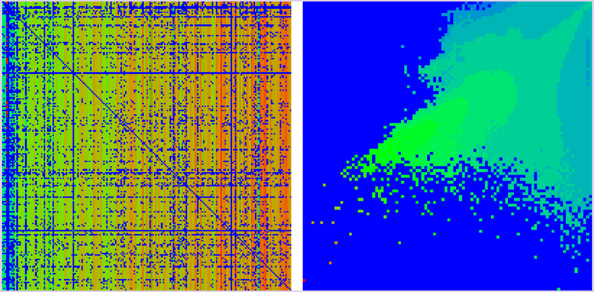
Gap of core space at $\alpha = 1$



Gap vs N for universities Glasgow, Cambridge, Oxford, Edinburgh, UCL, Manchester, Leeds, Bristol and Birkbeck (2002-2006) and Bath,Hull,Keele,Kent,Nottingham, Aberdeen, Sussex, Birmingham, East Anglia, Cardiff, York (2006). Red dots are for gap $> 10^{-9}$ and blue crosses (moved up by 10^9) are for Cambridge 2002, 2003 and 2005 and Leeds 2006 with gap $< 10^{-16}$; point at $2.91 \cdot 10^{-9}$ is Cambridge 2004.

Google matrix of Twitter

entier Twitter network 2009 => 41 million users



K.Frahm, DS arXiv:1207.3414[cs.SI]

Brain network model

E.Izhikevich, G.Edelman PNAS 105, 3593 (2008)

PNAS

Large-scale model of mammalian thalamocortical systems

Eugene M. Izhikevich and Gerald M. Edelman*

The Neurosciences Institute, 10640 John Jay Hopkins Drive, San Diego, CA 92121

Contributed by Gerald M. Edelman, December 27, 2007 (sent for review December 21, 2007)

The understanding of the structural and dynamic complexity of mammalian brains is greatly facilitated by computer simulations. We present here a detailed large-scale thalamocortical model based on experimental measures in several mammalian species. The model spans three anatomical scales. (*i*) It is based on global (white-matter) thalamocortical anatomy obtained by means of diffusion tensor imaging (DTI) of a human brain. (*ii*) It includes multiple thalamic nuclei and six-layered cortical microcircuitry based on *in vitro* labeling and three-dimensional reconstruction of single neurons of cat visual cortex. (*iii*) It has 22 basic types of neurons with appropriate laminar distribution of their branching dendritic trees. The model simulates one million multicompartmental spiking neurons calibrated to reproduce known types of responses recorded *in vitro* in rats. It has almost half a billion synapses with appropriate receptor kinetics, short-term plasticity, and long-term dendritic spike-timing-dependent synaptic plasticity (dendritic STDP). The model exhibits behavioral regimes of normal brain activity that were not explicitly built-in but emerged spontaneously as the result of interactions among anatomical and dynamic processes. We describe spontaneous activity, sensitivity to changes in individual neurons, emergence of waves and rhythms, and functional connectivity on different scales.

brain models | cerebral cortex | diffusion tensor imaging | oscillations |

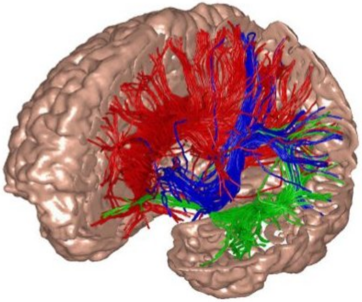


Fig. 1. The model's global thalamocortical geometry and white matter anatomy was obtained by means of diffusion tensor imaging (DTI) of a normal human brain. In the illustration, left frontal, parietal, and a part of temporal

Google matrix analysis: $N = 10^4$ nodes, $N_\ell = 1960108$ links
O.Zhirov, DS arXiv:1002.4583[cond-mat] (2010)

Towards Google matrix of brain

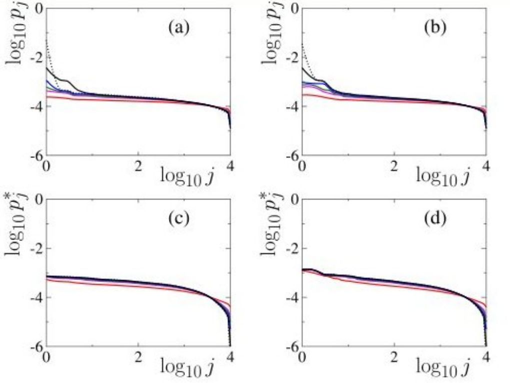


Fig. 2. (Color online.) PageRank p_j for the Google matrix of brain model at $\alpha = 0.6$, 0.85, 0.9, 0.95 and 0.99 shown by red, magenta green, blue and black solid curves (full curves from bottom to top at $\log_{10} j = 0.3$); j marks the index of nodes ordered according to the decreasing order of PageRank. The dotted black curve corresponds to $\alpha = 0.999$ and demonstrates strong dependence of the PageRank on α in the vicinity of $\alpha = 1$. Panels (a) and (b) correspond to unweighted and weighted links. For panels (a) and (b) the values of PAR are $\xi = 8223$ and 8314, 6295 and 6040, 5570 and 5046, 3283 and 3367, 28.4, 90.0, 1.09 and 1.19 for $\alpha = 0.6$, 0.85, 0.9, 0.95, 0.99, 0.999 respectively. Panels (c) and (d) show the dependence of the influence-PageRank $p^*(j)$ on j for the same values of α as for top panels respectively for unweighted and weighted links (for $\alpha > 0.6$ there is a strong overlap of curves).

Towards Google matrix of brain

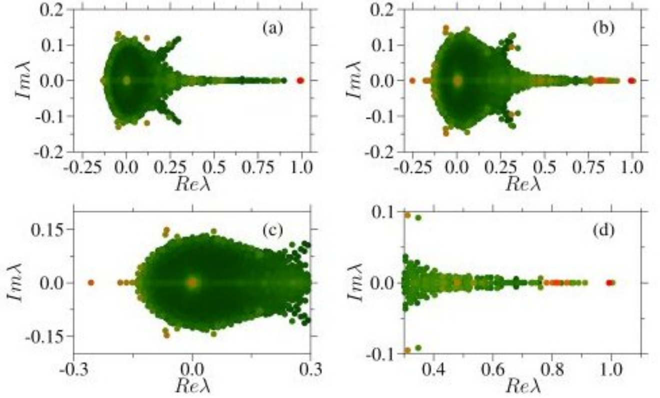


Fig. 3. (Color online.) Spectrum of eigenvalues of the Google matrix **G** of brain at $\alpha = 0.99$ in the complex plain λ for (a) unweighted and (b) weighted links in the neuronal network. Panels (c) and (d) show zooms of data of panel (c). The color shows the degree of localization of eigenvectors of **G** being proportional to the value of PAR ξ and changing from one (red/light gray) to maximal value (dark green/black).

Google Matrix Applications

practically to everything ....

more data at
http://www.quantware.ups-tlse.fr/QWLIB/2drankwikipedia/ .../tradecheirank/

References:

R1. S.Brin and L.Page, *The anatomy of a large-scale hypertextual Web search engine*, Comp. Networks ISDN Systems **30**, 107 (1998)

R2. A.A. Markov, *Rasprostranenie zakona bol'shih chisel na velichiny, zavisyaschie drug ot druga*, Izvestiya Fiziko-matematicheskogo obschestva pri Kazanskom universitete, 2-ya seriya, **15** (1906) 135 (in Russian) [English trans.: *Extension of the limit theorems of probability theory to a sum of variables connected in a chain* reprinted in Appendix B of: R.A. Howard *Dynamic Probabilistic Systems*, volume 1: *Markov models*, Dover Publ. (2007)].

R3. D.Austin, *How Google Finds Your Needle in the Web's Haystack*. AMS Feature Columns, http://www.ams.org/samplings/feature-column/fcarc-pagerank (2008)

R4. Wikipedia articles *PageRank*, *CheiRank*, *Google matrix* (2008-2011)

R5. D.Fogaras, *Where to start browsing the web?*, Lect. Notes Computer Sci. **2877**, 65 (2003)

R6. V.Hrisitidis, H.Hwang and Y.Papakonstantinou, *Authority-based keyword search in databases*, ACM Trans. Database Syst. **33**, 1 (2008)

R7. A.D.Chepelianksii, *Towards physical laws for software architecture* arXiv:1003.5455[cs.SE] (2010)

R8. A.O.Zhirov, O.V.Zhirov and D.L.Shepelyansky, *Two-dimensional ranking of Wikipedia articles*, Eur. Phys. J. B **77**, 523 (2010)

R9. S.M. Ulam, *A Collection of mathematical problems*, Vol. 8 of Interscience tracs in pure and applied mathematics, Interscience, New York, p. 73 (1960).

References (continued):

R10. K.M.Frahm and D.L.Shepelyansky, *Ulam method for the Chirikov standard map*, Eur. Phys. J. B **76**, 57 (2010)

R11. L.Ermann and D.L.Shepelyansky, *Ulam method and fractal Weyl law for Perron-Frobenius operators*, Eur. Phys. J. B **75**, 299 (2010)

R12. L.Ermann, A.D.Chepelianskii and D.L.Shepelyansky, *Fractal Weyl law for Linux Kernel Architecture*, Eur. Phys. J. B **79**, 115 (2011)

R13. L.Ermann and D.L.Shepelyansky, *Google matrix of the world trade network*, arxiv:1103.5027 (2011)

R14. L.Ermann, A.D.Chepelianskii and D.L.Shepelyansky, *Towards two-dimensional search engines*, arxiv:1106.6215[cs.IR] (2011)

R15. K.M.Frahm, B.Georgeot and D.L.Shepelyansky, *Universal emergence of PageRank*, arxiv:1105.1062[cs.IR] (2011)

Books, reviews:

B1. A. M. Langville and C. D. Meyer, *Google's PageRank and beyond: the science of search engine rankings*, Princeton University Press, Princeton (2006)

B2. M. Brin and G. Stuck, *Introduction to dynamical systems*, Cambridge Univ. Press, Cambridge, UK (2002).

B3. E. Ott, *Chaos in dynamical systems*, Cambridge Univ. Press, Cambridge (1993).