From FPU1955 to Wealth thermalization

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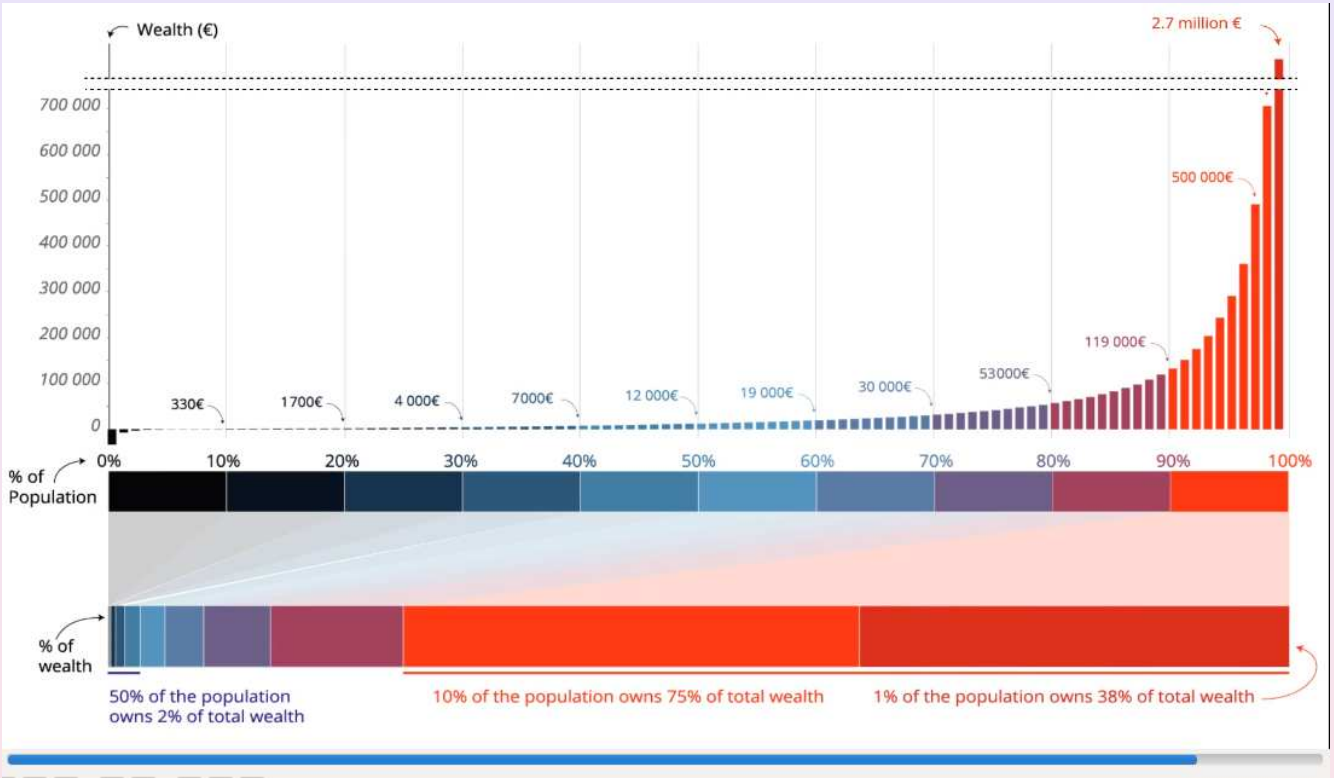


World wealth inequality (Piketty Lab 2021): 50% own 2%, top 10% own 75%
Wealth Thermalization Hypothesis (WTH) and social networks based on https://arxiv.org/pdf/2506.17720; [2506.06534]

Support: ANR OCTAVES, NANOX MTDINA projects

Statistical laws from dynamical equations

* Boltzmann themalization, H-theorem (1872)
* Fermi, Pasta, Ulam, Tsingou + MANIAC I (1955) $\rightarrow$ "The results show very little if any, tendency toward equipartition of energy among the degrees of freedom."

* Proximity to completely integrable models: Zabusky (1965) KdV solitons; Gardner, Greene, Kruskal, Miura (1967); Toda lattice (1967) ...
* Below or above chaos border: Chirikov, Izrailev (1966); ... Livi, Pettini, Ruffo, Vulpiani (1987); ... Benettin, Christodoulidi, Pono (2013) ....

CONCLUSION: FPU model is non-generic

Amicus Plato, sed magis amica veritas (post Socratem)

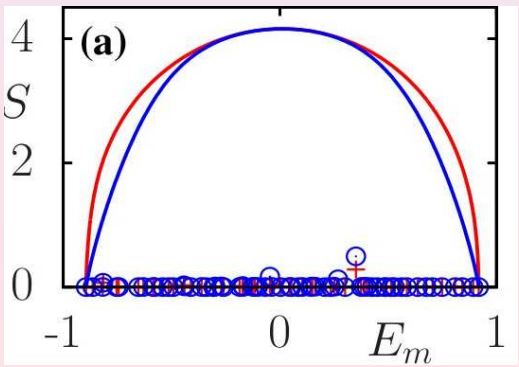
we love FPU model but should go forward

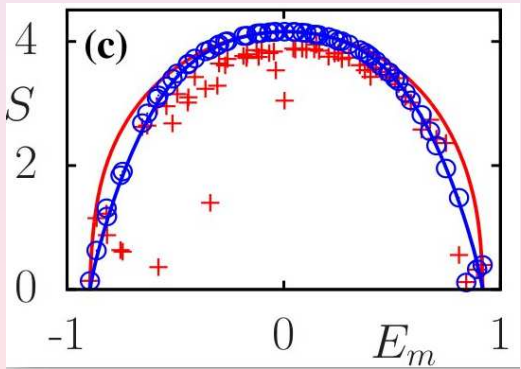
Nonlinear perturbation of Random Matrix Theory

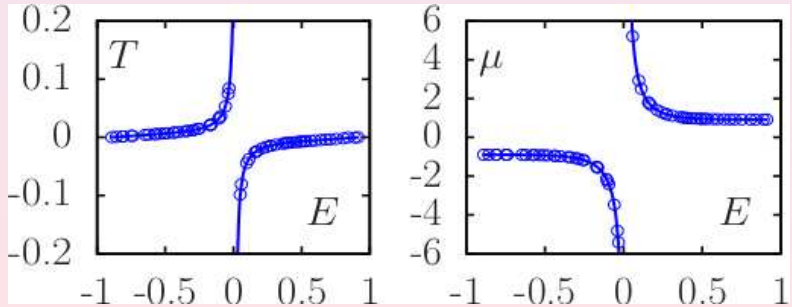
RMT of Wigner (1955) for universal features of complex nuclei, atoms, molecules + nonlinearity $\rightarrow$ generic model of dynamical thermalization:

$$i\hbar\frac{\partial\psi_n(t)}{\partial t} = \hat{H}_0\psi_n(t) + \beta|\psi_n(t)|^2\psi_n(t)$$

here $\hat{H}_0$ is random matrix of size N; $\hbar = 1$; $-1 \leq E_m \leq 1$ energy spectrum (or oscillator frequencies) of eigenstates φ_m; β is nonlinear perturbation.
Integrals: total energy $E = \sum_m E_m|C_m|^2 + \beta\sum_n |\psi_n|^4/2$, norm $\sum_n |\psi_n|^2 = 1$;
quantum von Neumann entropy $S = S_q = -\sum_m \rho_m \ln \rho_m$; $\rho_m = |C_m|^2$
classical Boltzmann entropy $S_B = \sum_m \ln \rho_m + const$; $dS/dE = 1/T$
Rayleigh-Jeans thermal distribution with temperature $T(E)$ and chemical potential $\mu(E)$: $\rho_m = T/(E_m - \mu)$; (classical limit of Bose-Einstein distribution)
e.g. Landau, Lifshitz; V.E.Zakharov et al. (Springer 1992)

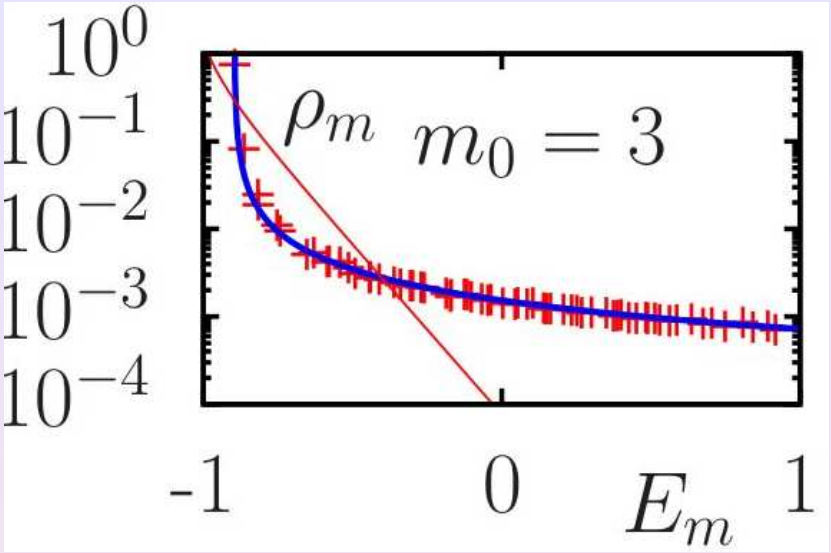


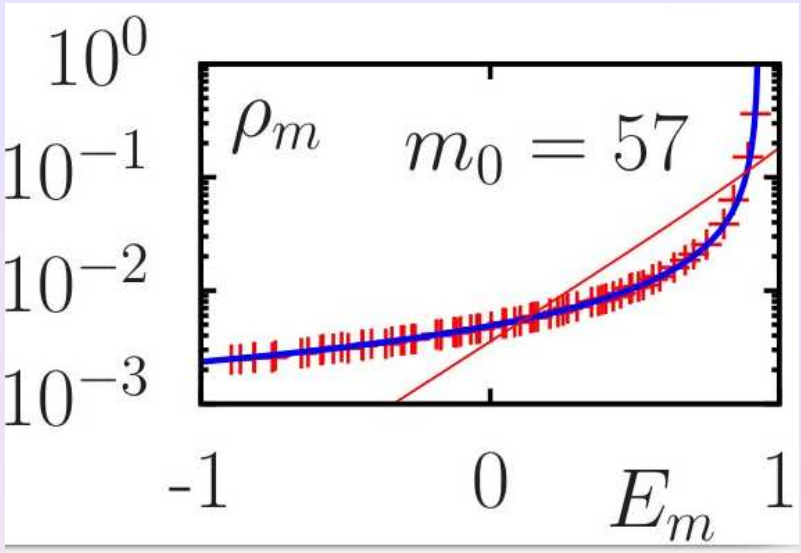




$N = 64$; $\beta = 0.02, 1$; $t \approx 2^{17}(red); 2^{24}(blue)$; RJ in blue; BE in red
Frahm, DS $\rightarrow$ PRL 150 years after Boltzmann (1872): $\beta \lessgtr \beta_c \Rightarrow$ KAM

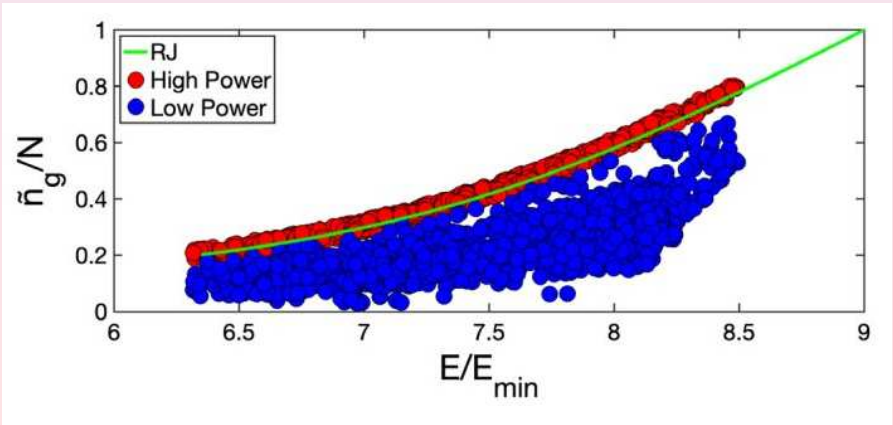
Rayleigh-Jeans thermal distribution in RMT





$N = 64; \beta = 1; 2^{23} \leq t \leq 2^{24}$; red crosses (data); RJ as blue curve; BE as red line; $T > 0$ at left, $T < 0$ at right

Optical fiber observation of RJ at $T < 0$:
Picozzi et al. PRL 130, 063801 (2023)



Dynamical Rayleigh-Jeans condensation

HUGE probability at lowest mode at $T \sim E_2 - E_1$

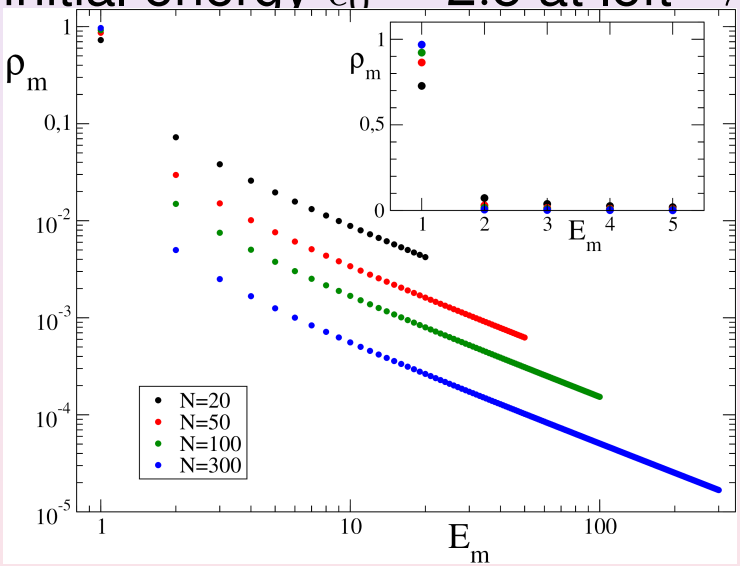
H.Fröhlich Phys. Lett. A (1968) - from BE to RJ condensate +pump/absorb (?)

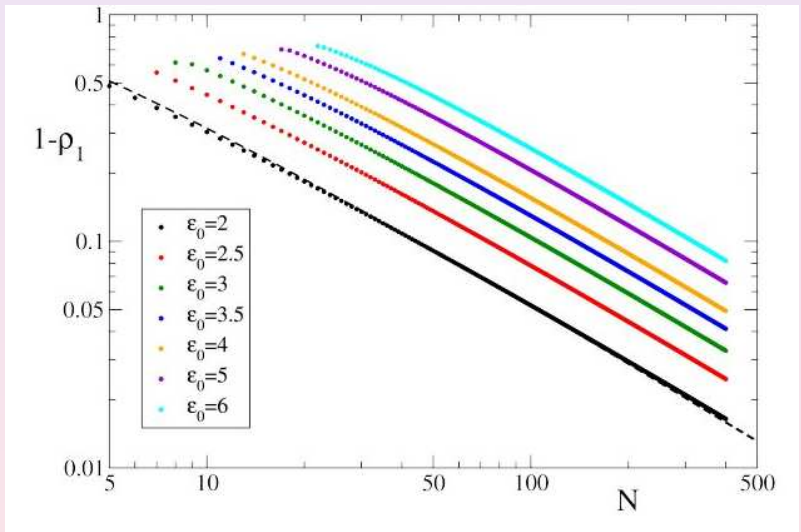
Picozzi et al. PRL (2005), Picozzi et al. PRA (2011)

optic fiber experiments $\rightarrow$ self-cleaning: Wabnitz et al PRL (2016), Picozzi et al PRL (2020); Babin et al PRL (2022); Christodoulides et al Nat Phys (2022)

RJ standard (RJS) model: N equidistant levels $E_m = m\Delta$, $1 \leq m \leq N$; $\Delta = 1$

initial energy $\epsilon_0 = 2.5$ at left $\rightarrow$ results from 2 integrals of energy and norm



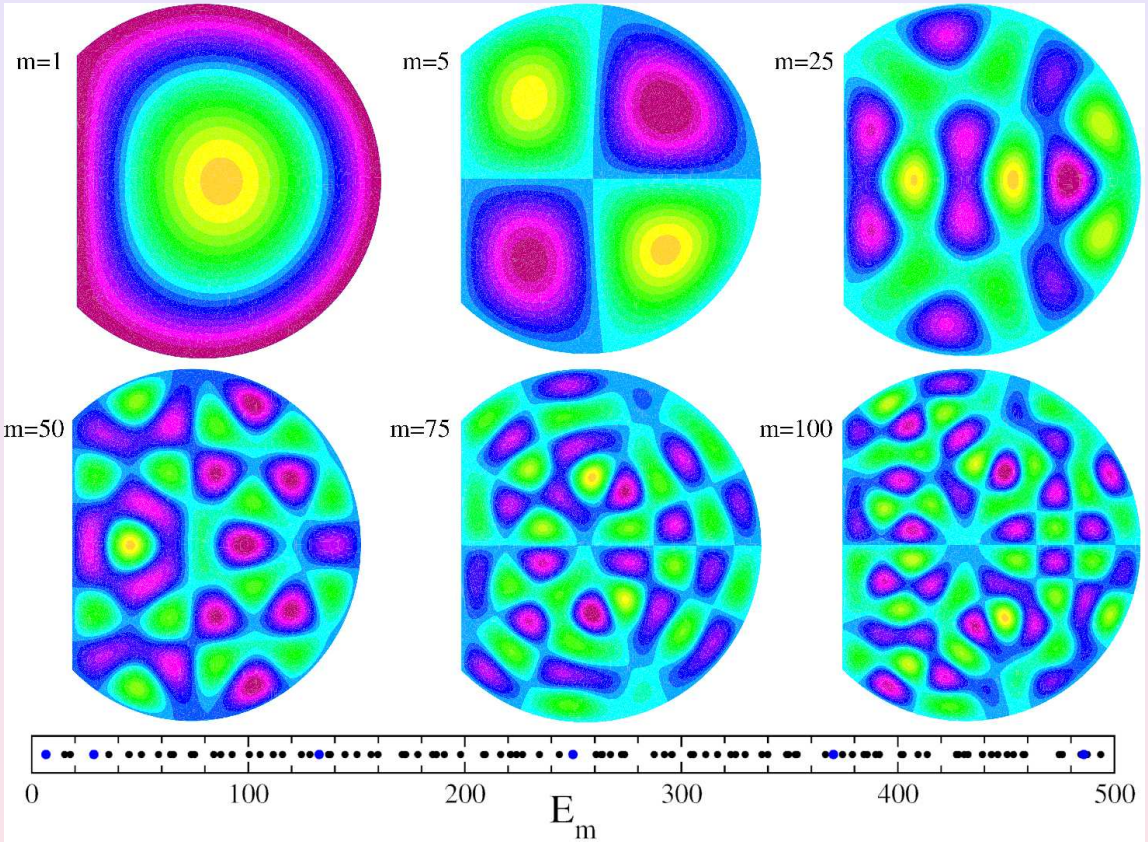


Ermann, Chepelianskii, DS arXiv (2025) (condensate results from 2 integrals)

thermalization in optical multimode fibers: attributted to Kolmogorov-Zakharov turbulence; no links to chaos, chaos border and KAM; specific case of frequency degeneracy when KAM is not applied

Dynamical RJ thermalization in quantum chaos fibers

D-shape fiber for RMT modeling; z-axis plays the role of time

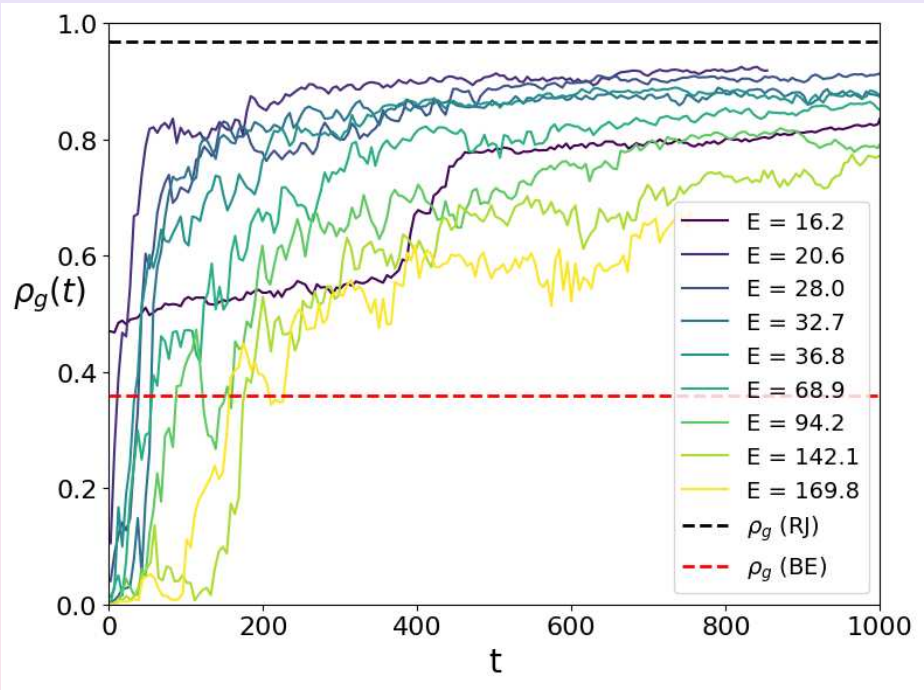


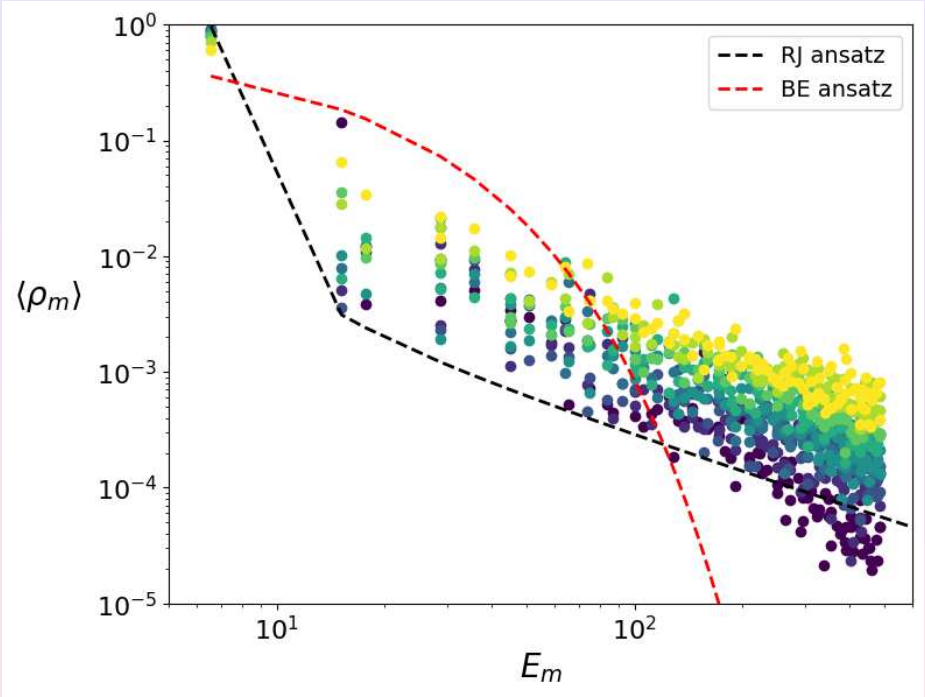
$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi(\vec{r},t) + \beta|\psi(\vec{r},t)|^2\psi(\vec{r},t)$ (in fiber $t \to z$)

here $\hbar = m = 1$, disk radius $R = 1$, RMT level spacing statistics; constant density of states

Fibers: Dynamical RJ condensation

$\beta = 10$; always condensation since $N = \infty$;
dashed curves: RJ (black), BE (red)

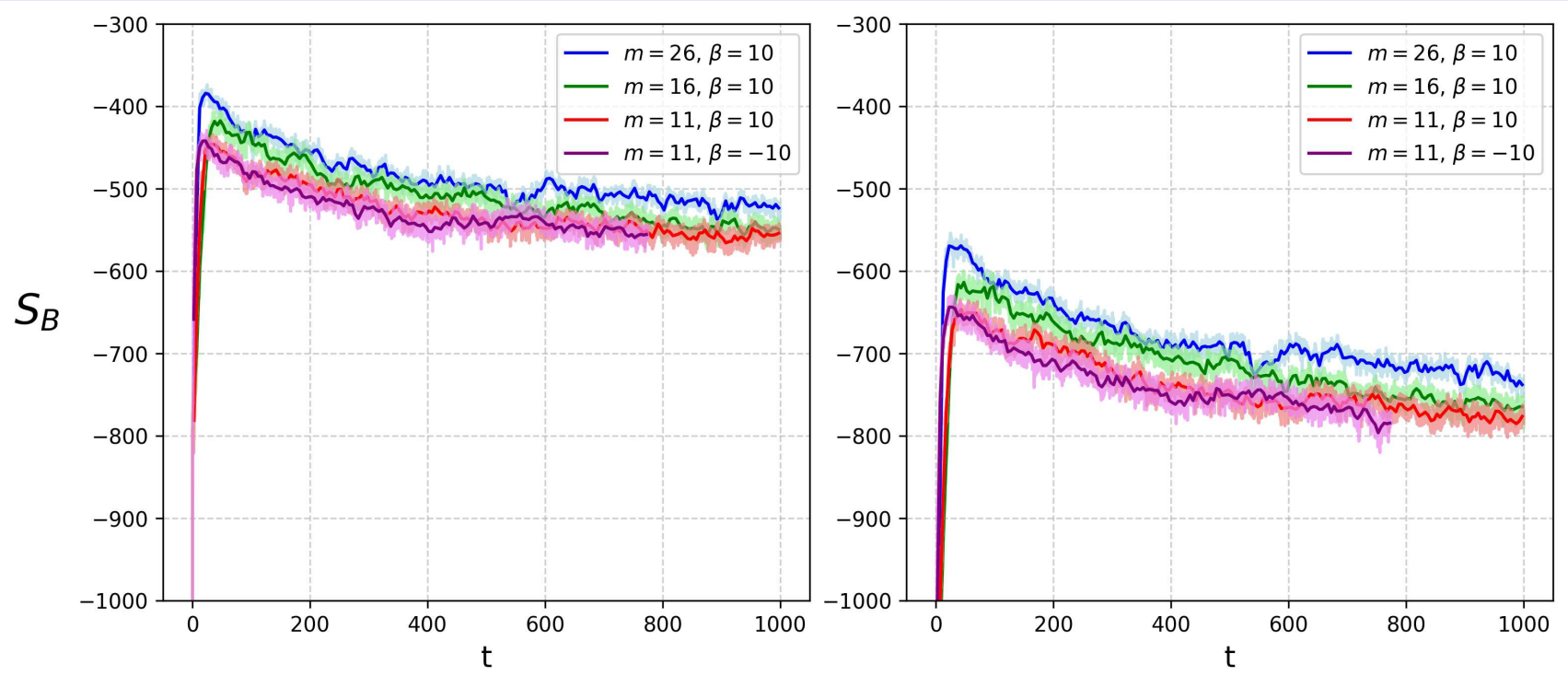




Many interesting features: Talanov et al. wave collapse at positive energies and $\beta \approx -10$; vortexes in superfluid light liquid with transition at $\beta \approx 44$ ...

Boltzmann H-theorem ?

H-theorem: $S_B(t)$ is monotonously growing function or constant



problem: $S_B = \sum_m \ln \rho_m(t)$, $1 \leq m \leq N = \infty$; $(dS/dE = 1/T$ for RJ)
cut-off at $m_{max} = 75; 150$ (left; right)

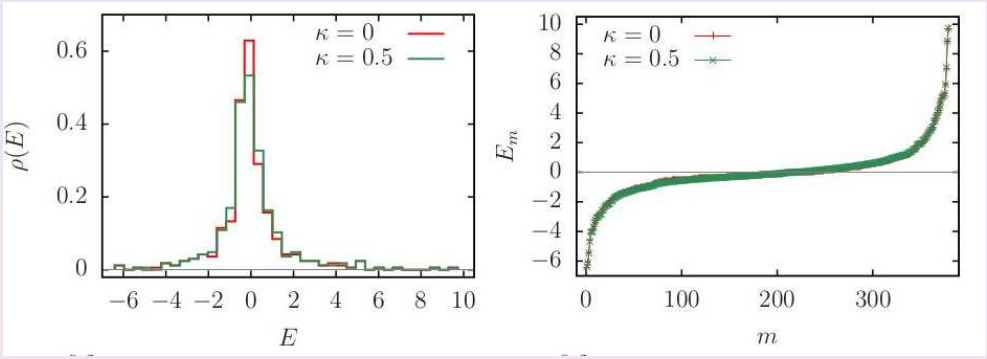
OPEN PROBLEM

Dynamical RJ thermalization in social networks

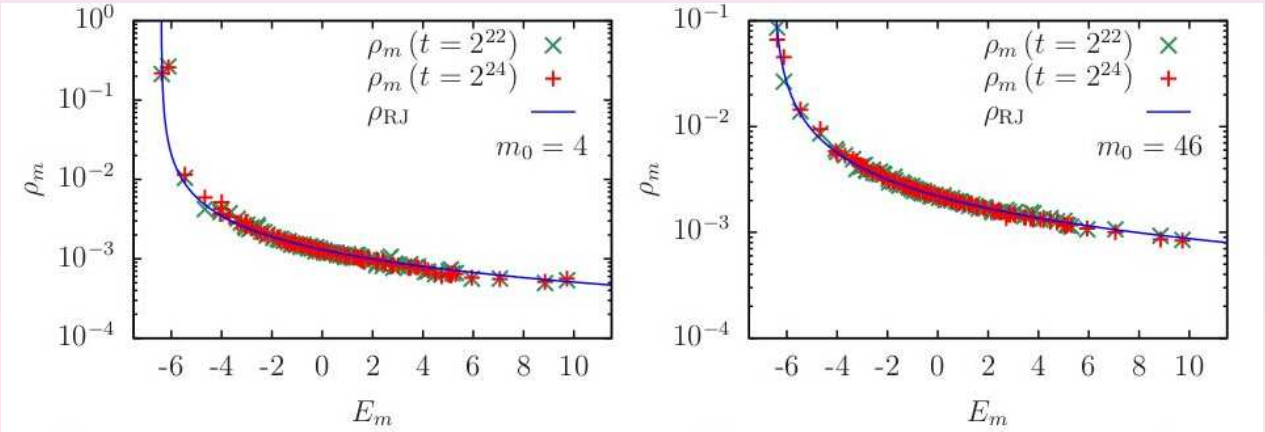
Undirected networks of scientists $N = 379$, network of politicians from Facebook $N = 5908$ from Mark Newman data page; Erdös number $\approx 4 - 5$

Hamiltonian $H = A + \kappa H_0$ with A adjacency matrix and H_0 RMT, $\kappa = 0.5$ for smoothing; nonlinearity β as for RMT model above

density of states, energy spectrum $N = 379$



RJ distribution $\beta = 10$, $N = 379$, initial states $m_0 = 4; 46$



$\rightarrow$ Dynamical RJ thermalization in human society (Frahm, DS 2025)

Wealth Thermalization Hypothesis (WTH)

World wealth inequality (Piketty et al (2021):
bottom 50% own 2%, top 10% own 75%, top 1% own 38%

Various models:
Angle (1986) ... Redner et al (1998) ... Bouchaud-Mezard (2000) ...
Chakrabarti et al (2013)...

Yakovenko et al - Boltzmann-Maxwell distribution (2000-2009),
Boghosian et al - wealth instead of income,
2 intergals of wealth (=energy), norm; kinetic model (2017)

DESCRIPTION: Lorenz curve (1905), Gini coefficient (1914):
cumulated fraction of wealth w vs cumulated fraction of households h

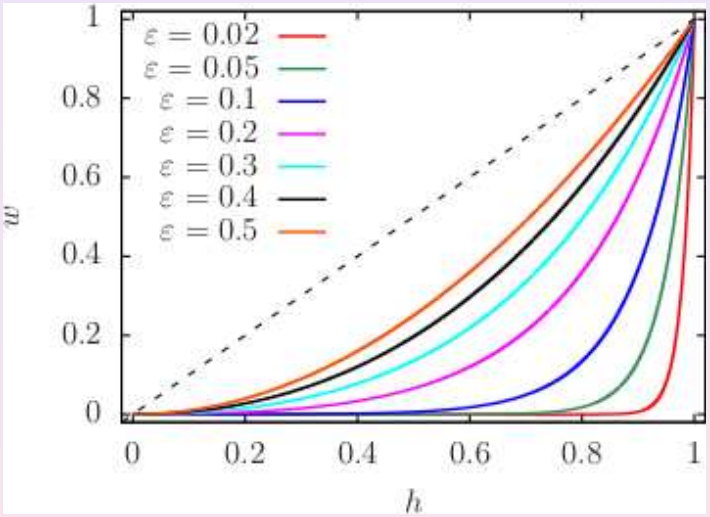
WTH $\rightarrow$ RJ thermal distribution:

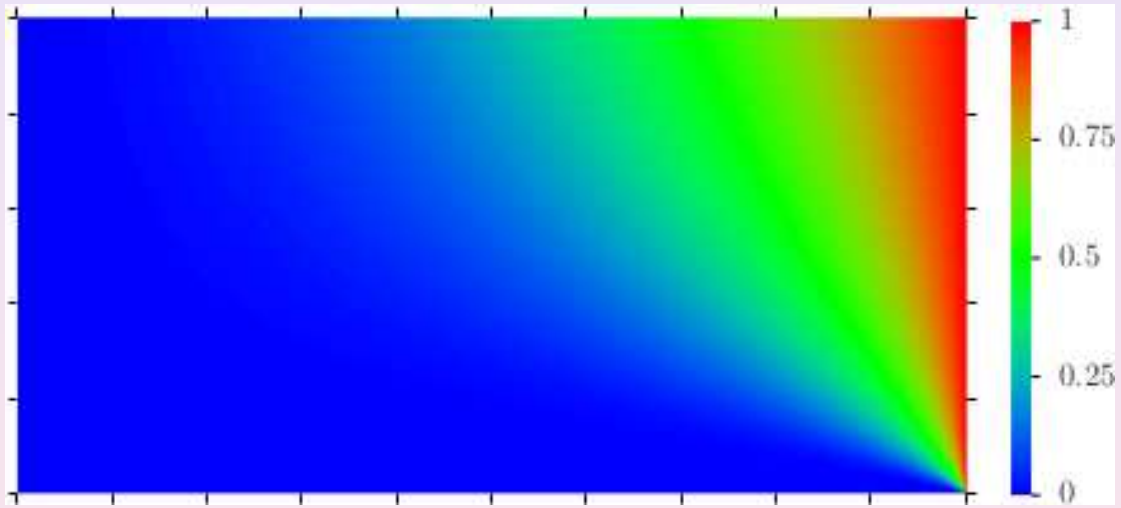
$$\rho_m = T/(E_m - \mu); \quad [E = \sum_m E_m \rho_m \, , \; \sum_m \rho_m = 1]$$

wealth is energy, 2 integrals of total energy E and norm (particles, agents),
$T = T(E), \mu = \mu(E)$

WTH and Lorez curves

RJS model of equidistant spectrum; $\varepsilon = E/B$, $B = N$ is energy/wealth band width; $E_m = m/N (N = 10000)$; w is cummulated wealth ($\sum_m E_m \rho_m / E = w$), h is cumulated norm ($\sum_m \rho_m = h$)

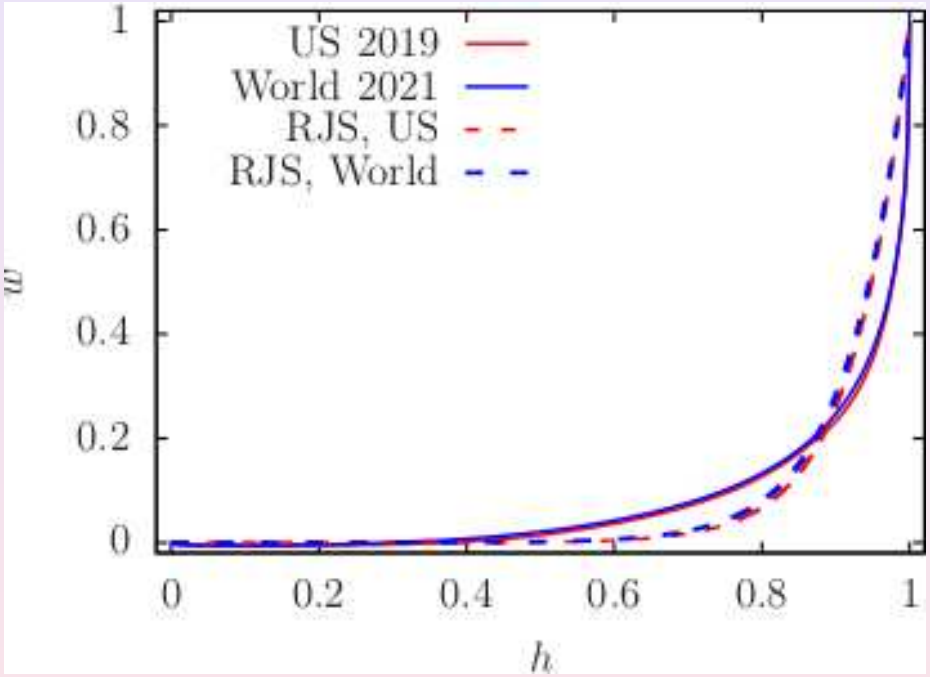


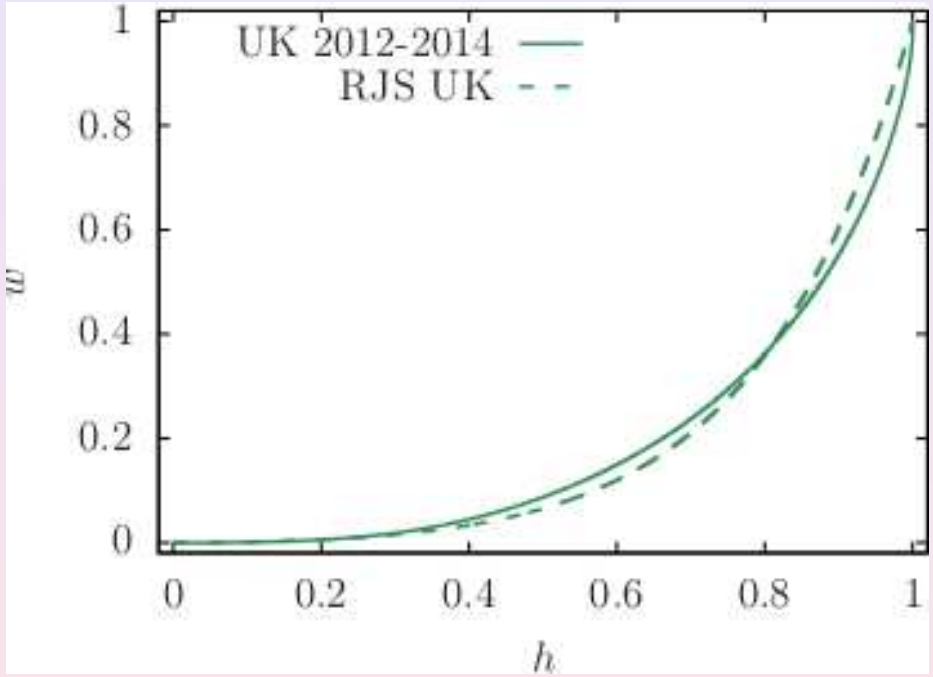


Left: Lorenz curves of RJS at various ε;
Right: w by color in plane $0 \leq x = h \leq 1$, $0 \leq y = \varepsilon \leq 0.5$ ($T > 0$).
Gini coefficient G is doubled area between diagonal and Lorenz curve ($0 \leq G \leq 1$)

WTH theory and Lorez curves of real data

RJS model of equidistant spectrum; $\varepsilon = E/B$, $B = N$ is energy/wealth band width; $E_m = m/N (N = 10000)$; w is cummulated wealth ($\sum_m E_m \rho_m / E = w$), h is cumulated norm ($\sum_m \rho_m = h$)

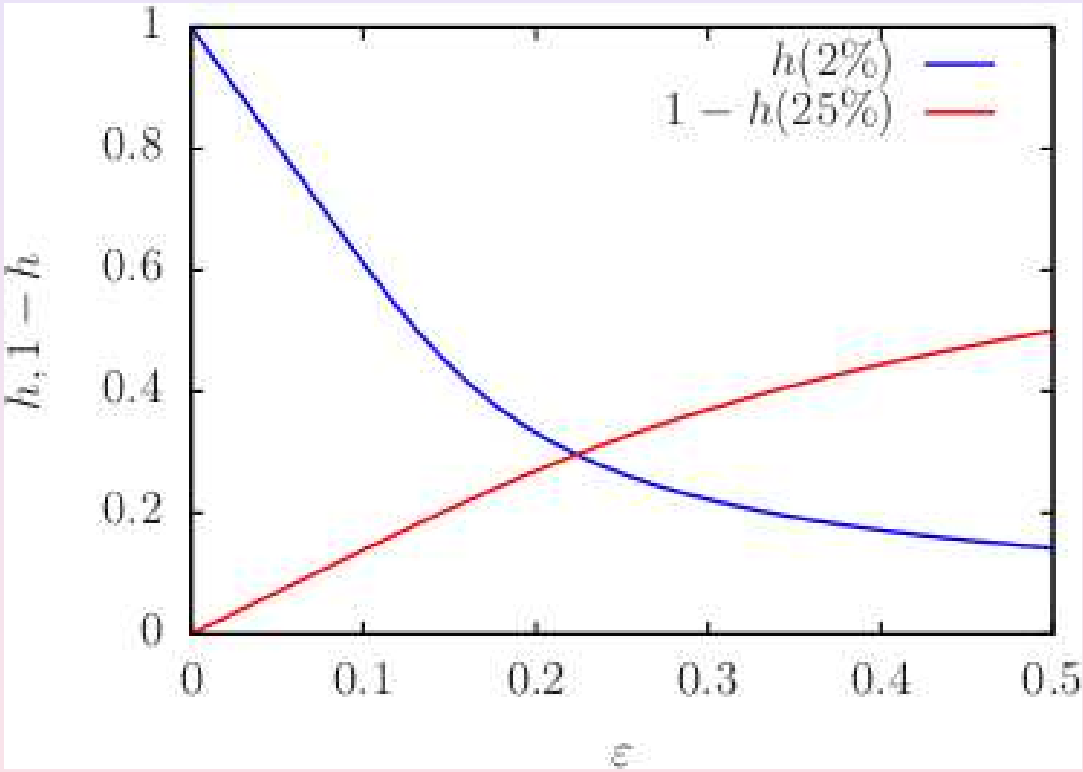




Left: US2019 and World2021(Piketty) data (overlaped curves) and RJS theory (dashed); same Gini $G = 0.852$ ($\varepsilon = 0.0782$)
Right: UK2012-2014 data and RJS theory, same Gini $G = 0.626$ ($\varepsilon = 0.1996$)

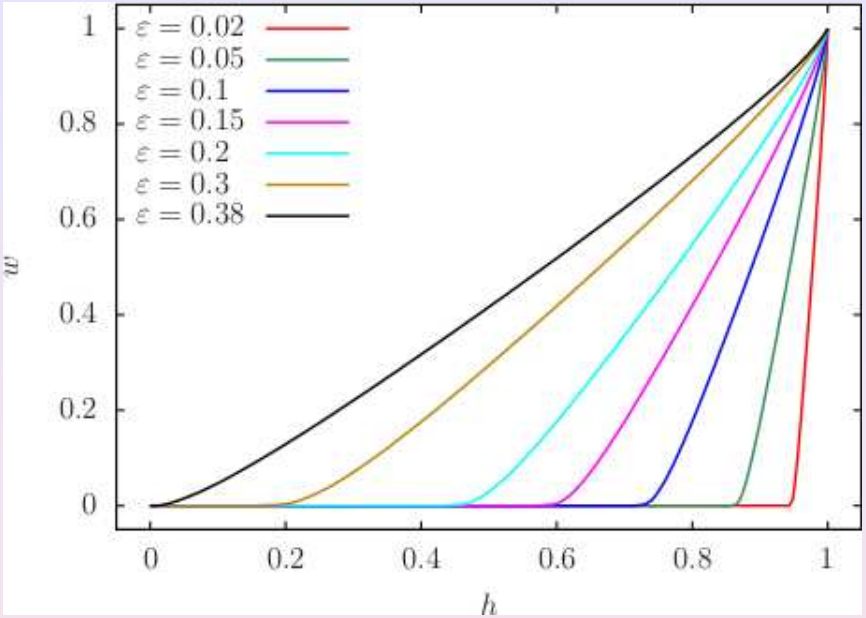
WTH theory: poor and oligarchic phases

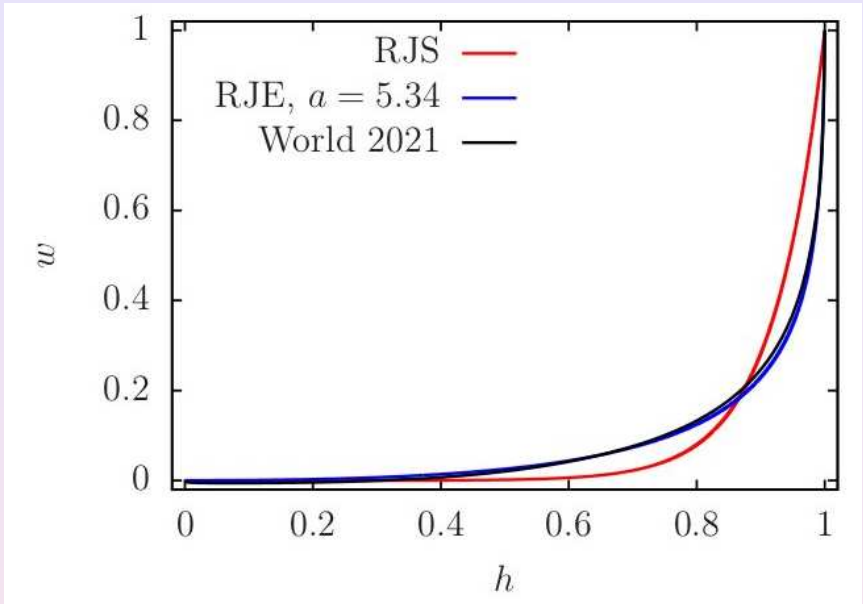
RJS model of equidistant spectrum; $\varepsilon = E/B$, $B = N$ is energy/wealth band width; $E_m = m/N (N = 10000)$; w is cummulated wealth ($\sum_m E_m \rho_m / E = w$), h is cumulated norm ($\sum_m \rho_m = h$)



Dependence of fraction of poor households $f_p = h(2\%)$ (owning 2% of wealth) and fraction of rich oligarchic households $f_r = 1 - h(25\%)$ (owning 75% of wealth) on the rescaled energy $\varepsilon = E/B$ for the RJS model.

WTH theory: social network and density decrease at high E_m

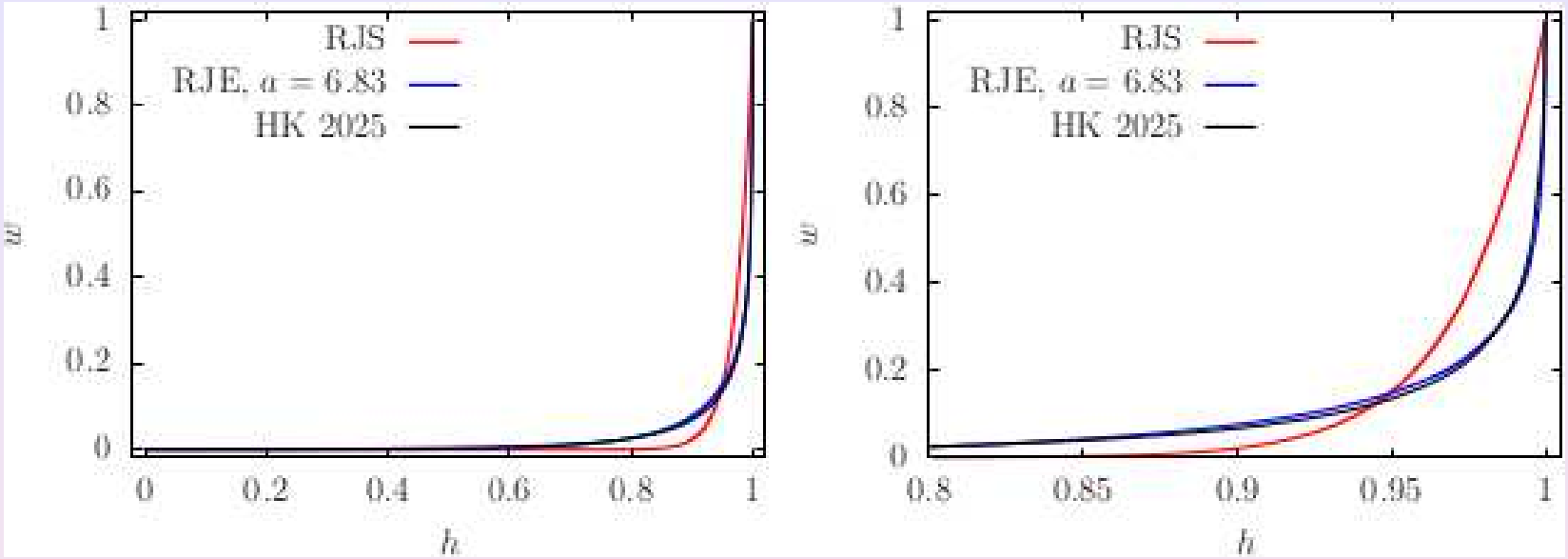




LEFT: netscience network $N = 379$ model; no social stratification

RIGHT: $\nu(E_m) = dm/dE_m = N/(1 + aE_m)$ density decrease at high E_m (RJE); Comparison of Lorenz curve of World 2021 (black) with RJS model (red; $N = 10000$) and RJE model at $a = 5.34$ (blue curve; $N = 10000$). The rescaled energy values $\varepsilon = 0.00855$ (RJE) and $\varepsilon = 0.0791$ (RJS) are obtained by matching the Gini coefficient $G = 0.8420$; a is from a fit of reconstructed spectrum ($E_m = [\exp(am/N) - 1]/a$).

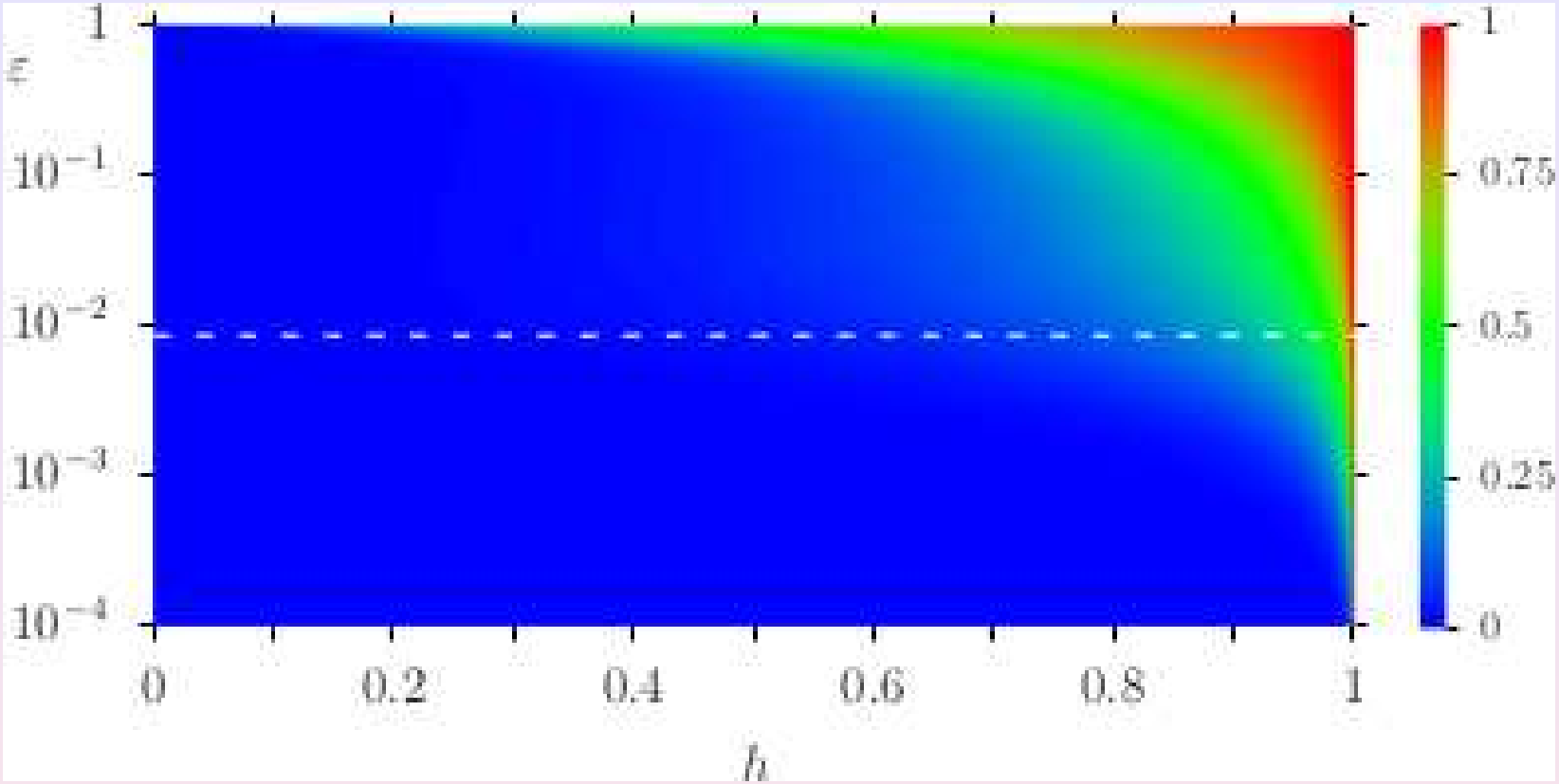
WTH theory: Stock Exchange of Hong Kong



Comparison of Lorenz curve of Hong Kong Stock Exchange at 19 June 2025 (black) abd curves for RJS model (red; $N = 10000$) and RJE model with $a = 6.83$ (blue; $N = 10000$). The rescaled energy values $= 0.000838$ (RJE) and $\varepsilon = 0.0264$ (RJS) are obtained by matching the Gini coefficient $G = 0.947$. The value of a is obtained by a fit from the reconstructed spectrum. The left (right) panel shows the full range $h \in [0, 1]$ (zoomed range $h \in [0.8, 1]$).

Also good description for SE of New York, London

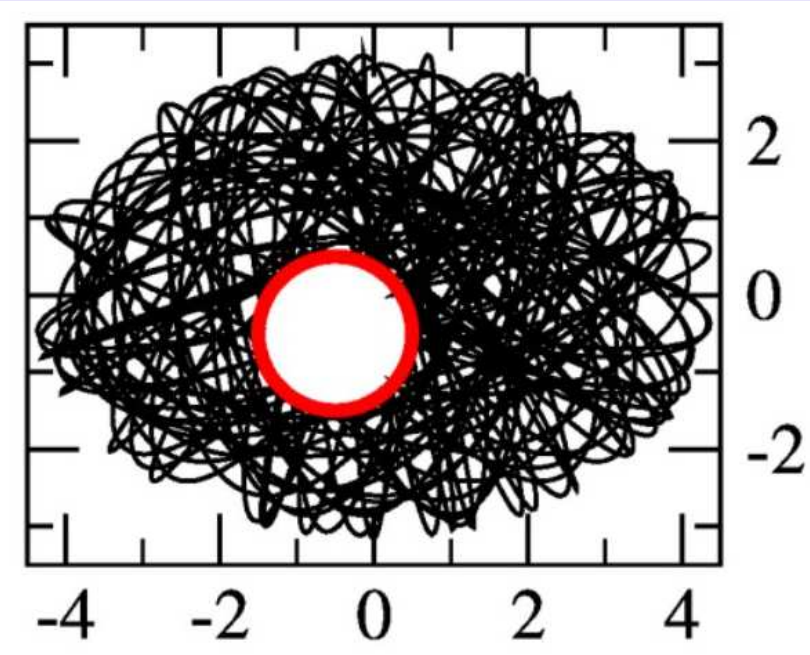
WTH theory for the Whole World 2021

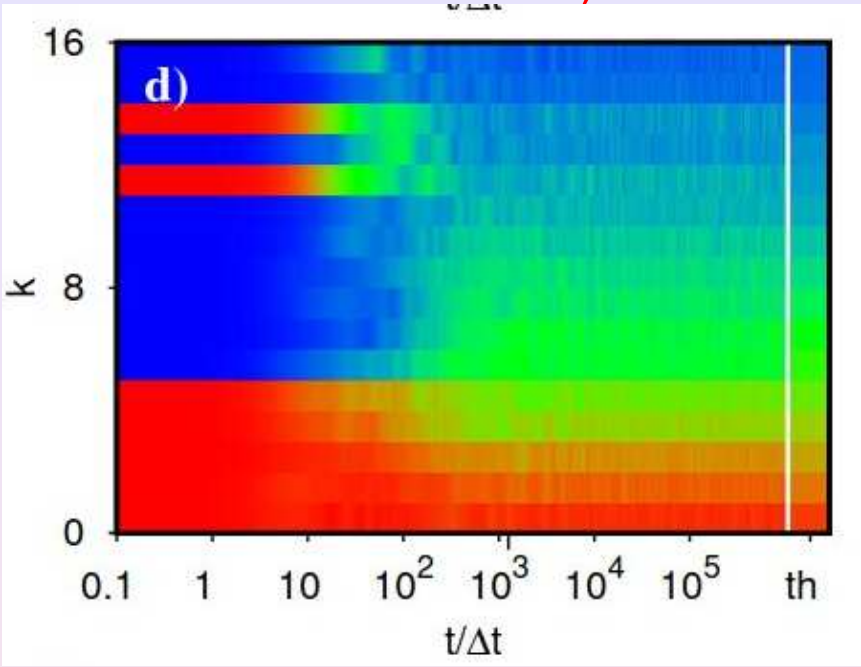


Color plot of cumulated wealth w from RJE model ($N = 10000, a = 5.34, \varepsilon = 0.00855$, Gini as for Piketty World Lorenz curve 2021). The x-axis corresponds to the fraction of households $h \in [0, 1]$ and the y-axis to the rescaled energy $\varepsilon = E/B \in [10^{-4}, 1[$ in logarithmic representation. The white dashed line corresponds to Piketty Wolrd 2021 data.

Quantum Dynamical Thermalization (QDT)

Sinai billiard $\rightarrow$ Sinai oscillator (Ketterle BEC PRL 1995 3d)





Time evolution of orbital filling factors for 7 fermionic atoms in Sinai oscillator, 5 in ground state, 2 excited; effective interaction $A = 3.5$; Fermi-Dirac theoretical distribution on the right of white line
(Frahm, Ermann, DS Cond. Mat. (2019))

Åberg criterion for QDT: intercation coupling matrix elements U_c are higher then energy level spacing Δ_c between coupled many-body states so that $A = U_c/\Delta_c > 1$ S. Åberg PRL **64**, 3119 (1990);
Jacquod, DS PRL (1997); Mirlin et al PRB (2016);
Single state QDT in 16 qubit quantum computer Benenti et al EPJD (2001)