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SOME STATISTICAL PROPERTIES OF SIMPLE CLASSICALLY STOCHASTIC QUANTUM SYSTEMS

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Numerical studies are made of simple one- and two-dimensional quantum models which are stochastic in the classical limit. It is shown that the correlation properties of the quantum and corresponding classical motions are only similar for very short time intervals t_s, and that the evolution of the quantum system, unlike the classical one, is stable. The diffusive excitation of the quantum system under a periodic perturbation is limited to a specific time interval $t^* \gg t_s$, during which the diffusion rate is similar to the corresponding classical diffusion rate. For the two-dimensional model, a continuous component in the correlation spectrum survives for an indefinite period $t_w \gg t^*$. It is shown that when the perturbation is quasiperiodic the interval t^* increases sharply.

1. Introduction

There is currently a growing interest in the dynamics of nonlinear quantum systems which are stochastic in the classical limit $\hbar = 0$ (see, for example, refs. 1–9. A study of such systems is important for understanding the statistical properties of quantum systems in the stochasticity domain (for example, the behavior of molecules and atoms in the field of a strong electromagnetic wave [10, 11]), as well as the peculiarities of intramolecular dynamics [17]. A theoretical study of such problems, even in the quasiclassical regime, faces significant difficulties [2–7]. These are due both to the local instability of the classical trajectories, which leads to an exponentially fast spreading of the classical packet, and to an increase of the quantum corrections with time. To study the properties of classically stochastic quantum systems (SQS), numerical experiments have been preformed on a simple model, the quantum rotator with a periodic perturbation [1]. The main result is that the motion of an SQS is similar, under certain conditions, to the stochastic motion of the classical system. For example, a diffusive growth of the rotator energy with time has been observed. Yet, the rate of diffusion decreases substantially with time.

In the present work a number of numerical experiments with simple SQS models is described. The studies presented in section 2 indicate that the correlations in the quantum rotator, unlike the classical one (when the measure of the islands of stability is sufficiently small), do not decay exponentially with time, confirming the theoretical result of ref. 12. Note here that classical correlations may decay in nonexponential way in the systems with large stable component [18]. In section 3, studies of a quantum system with two degrees of freedom are presented. A regime has been found in which the leading degree of freedom, which cannot be excited above a certain level (the quantum limitation of diffusion [1]), affects the second degree of freedom in such a way that the diffusive excitation of the latter lasts much longer than that of the leading degree of freedom (in fact, appears to be unlimited). In section 4 the excitation of the quantum rotator by a quasiperiodic external perturbation (two or three non-commensurable frequencies) is analyzed. The numerical experiments indicate that this case differs qualitatively from the case of a periodic perturbation in that there is no quantum limitation of diffusion, and the growth in energy seems to be unlimited in time.

2. Correlations in the quantum rotator

Let us consider a rotator in an external field with the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2J}\frac{\partial^2}{\partial\theta^2} + \tilde{k}\cos\theta \cdot \delta_{\tilde{T}}(\tau)\,, \tag{2.1}$$

where $\tilde{k}$ is a parameter characterizing the magnitude of the perturbation, $\delta_{\tilde{T}}(\tau) = \Sigma_{n=-\infty}^{\infty}\delta(\tau - n\tilde{T})$ is the periodic delta-function (or periodic kick), J is the moment of inertia of the rotator, and θ is the angular variable. It is assumed below that $J = 1$.

The corresponding classical problem is described by the Hamiltonian

$$H = \frac{p^2}{2} + \tilde{k}\cos\theta \cdot \delta_{\tilde{T}}(\tau)\,. \tag{2.2}$$

The presence of the periodic delta-function makes it convenient to describe the motion of the classical rotator by the mapping

$$\begin{aligned} \bar{p} &= p + \tilde{k}\sin\theta\,, \\ \bar{\theta} &= \theta + \tilde{T}\bar{p}\,, \end{aligned} \tag{2.3}$$

where $\bar{p}$ and $\bar{\theta}$ are the values of the variables immediately following a kick.

The mapping (2.3) has been investigated in detail in ref. 9, where the value $\tilde{k}\tilde{T} \approx 1$ is shown to be the border of stability. At $\tilde{k}\tilde{T} < 1$ the motion is stable and the variation of the quantity p is limited ($|\Delta p| \lesssim \sqrt{\tilde{k}/\tilde{T}}$). If $\tilde{k}\tilde{T} \gg 1$, the motion is stochastic. In this case, for almost any initial conditions (excluding those in the small islands of stability), nearby trajectories diverge exponentially; $d = d_0 \exp(ht)$ where $d = \sqrt{(\tilde{T}\Delta p)^2 + (\Delta\theta)^2}$, and $h \approx \ln(\tilde{k}\tilde{T}/2)$ (at $\tilde{k}\tilde{T} > 4$) is the KS-entropy [9, 13]. Such a local instability of the motion causes the phase θ to become a random variable and the rotator energy E to grow according to the diffusion law,

$$E(t) = \frac{\langle p^2(t)\rangle}{2} \approx \frac{\tilde{k}^2}{4}t + E(t=0)\,. \tag{2.4}$$

In this case the momentum-distribution function has the Gaussian form

$$f(p,t) = \frac{1}{\sqrt{\pi\tilde{k}^2 t}}\exp\left(-\frac{p^2}{\tilde{k}^2 t}\right). \tag{2.5}$$

Here and below, t is the dimensionless time, measured by the number of kicks. The brackets $\langle\ \rangle$ imply the average over a large number of trajectories corresponding to different initial conditions.

It is also convenient to describe the motion of the quantum system (2.1) with a mapping for the wave function ψ [1],

$$\psi(\theta, t+1) = e^{-ik\cos\theta}\frac{1}{\sqrt{2\pi}} \times \sum_{n=-\infty}^{\infty} A_n(t)\exp\left(in\theta - i\frac{Tn^2}{2}\right), \tag{2.6}$$

where

$$A_n(t) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\psi(\theta,t)\,e^{-in\theta}\,d\theta, \quad k = \frac{\tilde{k}}{\hbar}, \quad T = \hbar\tilde{T}.$$

It follows from (2.6) that one kick couples, with exponential accuracy, $\approx 2k$ levels of the unperturbed system. This feature has been used in a previous numerical study [1] of the model (2.1). The mapping (2.6) contains two independent parameters k and T. We set $\hbar = 1$, and then, to get the quasiclassical limit, let $k \to \infty$ and $T \to 0$ in such a way that $kT = \text{const}$. Numerical experiments [1] have shown that when $kT > 1$ and $k \gg 1$, the energy of the quantum rotator system (2.1) grows diffusively at approximately the classical rate for times less than a certain period t^*. But for $t > t^*$, the rate of diffusion drops off and at $t \gg t^*$, the increase in energy practically stops [1, 7]. The time t^* increases as the parameter k increases.

Additional numerical experiments [4] have shown that the t^*-dependence on k may be approximated by the power law $t^* = Ck^\alpha$ (see fig. 1). In this case, t^* is taken as the time t at which the energy of the quantum system differs from the classical value by more than 25%. The root-mean-square values of the parameters C and α are approximately: $\langle \log C \rangle = -0.44$ and $\langle \alpha \rangle = 1.5$. The theoretical dependence

$$t^* = Ck^2 \tag{2.7}$$

obtained in refs. 4 and 7 turns out to be within the spread of the experimental data with $\langle \log C \rangle = -1.19$ (see fig. 1). Hence, when $k \gg 1$ the quantum rotator energy grows, as it does in the classical case, in a diffusive manner during a long period of time.

The theoretical results of ref. 12 predict that correlations in the quantum system after a time

$$t_s \approx \frac{\ln k}{\ln kT} \tag{2.8}$$

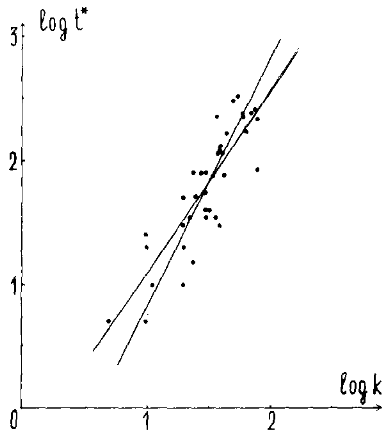

Fig. 1. The dependence of the time t^* of quantum limitation of diffusion on the parameter k. The points stand for experimental values, the two straight lines correspond to a linear interpolation (slope $\alpha = 1.5$) and to the theoretical formula (2.7) (slope $\alpha = 2$); logs are base 10.

do not decrease more rapidly than the inverse of the square root of time ($\tau^{-1/2}$). In the classical case these correlations decay exponentially with t when $kT \gg 1$ (when the measure of the islands of stability is quite small). Thus, for $kT \gg 1$, the quantum and classical correlations should be completely different when $t \gtrsim t_s$ (although, since the absolute magnitude of the correlations proves to be small ($\propto \mathcal{O}(k^{-1})$), they only affect the rotator energy for times $t \gtrsim t^* \gg t_s$).

To verify these predictions [12], a numerical study of the quantum model (2.1) was made to calculate the correlations

$$R_t(\tau) = \langle 0|\cos \hat{\theta}_t \cos \hat{\theta}_{t+\tau} + \cos \hat{\theta}_{t+\tau} \cos \hat{\theta}_t|0\rangle\,, \tag{2.9}$$

where $\cos \hat{\theta}_t = U_t^+ \cos \theta U_t$ is the Heisenberg operator at the moment of time t, U_t is the operator of evolution of the Hamiltonian (2.1) and $\langle 0|\ldots|0\rangle$ stands for the expectation value with respect initial state. In principle, one can consider other correlations, for example, the correlations of $\sin \hat{\theta}$. These correlations behave in qualitatively the same manner as those of $\cos \hat{\theta}$, except with respect to one specific feature which is analyzed below (see section 3).

The numerical algorithm for computing the correlations was to define the wave functions $|\psi_t\rangle = U_t|0\rangle$, $|\varphi_{t+\tau}\rangle = U_\tau \cos \theta |\psi_t\rangle$ and $|\psi_{t+\tau}\rangle = U_{t+\tau}|0\rangle$ by means of eq. (2.6) (using the method described in refs. 1 and 4, and then to calculate the average over θ: $2\,\mathrm{Re}(\langle \psi_{t+\tau}|\cos\theta|\varphi_{t+\tau}\rangle) = R_t(\tau)$.

The results of these numerical experiments are listed in table I. The classical correlations $R_{\rm cl} = (1/\pi)\int_0^{2\pi} \cos\theta_0 \cos\theta_\tau \,\mathrm{d}\theta$ are compared to the quantum correlations $R_{\rm q}(= R_t(\tau)$ in (2.9)) at $t = 0$ and $0 \leqslant \tau \leqslant 7$. The initial classical state is: $p = 0$, $0 \leqslant \theta \leqslant 2\pi$, and the corresponding quantum one is: $\psi(\theta, 0) = (2\pi)^{-1/2}$. It is seen from these data that at $kT = 5$ and $kT = 5 + 2\pi$, when the measure of the islands of stability is negligibly small [9], the classical correlations for $\tau \leqslant 7$ decay quickly with time. The quantum correlations in this case are only close to the classical correlations when $\tau \lesssim t_s \approx 3$. For $\tau \gtrsim t_s$ the quantum correlations are a few times

Table I

τ	R_{cl}	R_q/R_{cl} $k=5$	R_q/R_{cl} $k=20$	R_q/R_{cl} $k=40$	R_q/R_{cl} $k=100$
0	1	1	1	1	1
$kT=2$					
1	0.5767	0.9880	0.9993	0.9998	1.0000
2	0.4986	0.9651	0.9976	0.9994	0.9998
3	0.9614	0.9753	0.9982	0.9996	0.9999
4	0.6794	1.1785	1.0745	1.0294	1.0053
5	0.5688	1.5397	1.0742	0.9552	0.9754
6	0.6504	0.9371	0.9294	1.1233	1.0152
7	0.7648	0.8375	1.0365	0.9678	0.9946
$kT=5$					
1	−0.1310	0.8313	0.9908	0.9977	1.0000
2	0.01229	14.9880	7.4768	2.7307	0.9723
3	0.3384	2.0254	0.8543	1.0774	0.9069
4	0.08002	5.4849	2.7656	1.4921	1.3884
5	0.09999	1.0701	2.3252	1.5372	0.1946
6	0.09167	2.5472	1.9941	2.4490	0.3505
7	0.00965	82.404	−3.8520	12.615	11.703
$kT=5+2\pi$					
1	−0.03770	−0.6053	0.8963	0.9756	0.9960
2	0.08725	−4.0183	0.04544	1.0256	0.9012
3	0.1389	1.7423	1.4104	1.1857	0.2191
4	0.01641	−5.6684	0.4188	4.4308	0.2732
5	0.01945	−9.3060	−9.4807	4.9851	2.0925
6	0.02184	0.6062	2.1323	8.0998	−3.0714
7	0.00752	33.524	8.7181	19.6676	−1.1330

larger than the classical. This is consistent with the theoretical value of t_s (2.8) which is also just a few kick periods in length. As mentioned above, the time t^* at which these correlations begin to effect the energy is usually much larger than t_s. For example, when $k=40$, $kT=5$, the quantum rotator energy differs from its classical value by less than 25% for times $\tau \approx (t^*=120)$ long compared to $t_s \approx 3$. Also, in the case $k=40$, $kT=2$, where the measures of the stable and stochastic components are approximately the same, the classical correlations do not decay with time and the difference between R_{cl} and R_q remains less than 20% for times $\tau \approx 100$ ($\gg t_s \approx 3$). Thus, those characteristics, which do not decrease exponentially with time, for example, the rotator energy or the correlations at $kT=2$, are close to their classical values for times $t^* \gg t_s$. Note also that in the stability region, $kT=0.5$ ($k=20$) the difference between the quantum and classical correlations is at the 0.1% level for $\tau \approx 20$ (at $k=5$, $\tau \approx 20$ it is about 10%). The typical behavior of the quantum correlations is shown in fig. 2. It is seen that there are some residual correlations that do not decrease with time. These correlations do decrease as k increases, but an explicit form of the dependence on k has not been found because of the sharp increase in the required memory and computing time with increasing k.

The magnitude of the residual correlations may be evaluated as follows. Let $\tau \gg t^*$. Then the wave function $|\phi\rangle = e^{\pm i\theta} U_\tau^+ e^{\pm i\theta} U_\tau |0\rangle$ contains approximately $\sqrt{k^2 t^*}$ harmonics (at $\tau \gg t^*$ the increase in energy practically stops). Since $\langle \phi | \phi \rangle = 1$, the average amplitude Q of a typical harmonic is roughly determined from the condition $Q^2 \sqrt{k^2 t^*} \sim 1$. Then, from the relation $R(\tau) \sim \langle 0 | \phi \rangle \sim Q$ and (2.7) we get the estimate

$$|R_t(\tau)| \sim (k^2 t^*)^{-1/4} \sim k^{-1}, \quad t+\tau \gg t^* . \tag{2.10}$$

At $t+\tau \ll t^*$ the number of harmonics in $|\phi\rangle$ will be of the order of $\sqrt{k^2 \tau}$ and hence, on this time scale, the correlations decay with growing τ,

$$|R_t(\tau) \sim (k^2 \tau)^{-1/4}, \quad t_s \lesssim t+\tau \ll t^* . \tag{2.11}$$

This decay is very slow and since the parameter $(t^*)^{1/4}$ is not very large, the non-decreasing-with-time residual correlations are observed in the numerical experiment practically immediately (see fig. 2).

It is worth noting that, according to the estimates (2.10), (2.11) and the results of ref. 12, quantum correlations also do not decay exponentially in systems where the measure of the islands of stability is strictly equal to zero (e.g., the system (2.1) with the perturbation potential

$$V(\theta) = \begin{cases} -\dfrac{\theta^2}{2}, & 0 \leqslant \theta \leqslant \dfrac{\pi}{2}, \\ \dfrac{(\theta-\pi)^2}{2} - \dfrac{\pi^2}{4}, & \dfrac{\pi}{2} \leqslant \theta \leqslant \pi, \end{cases}$$

$$V(\theta) = V(-\theta), \quad V(\theta) = V(\theta + 2\pi), \quad kT > 4 .$$

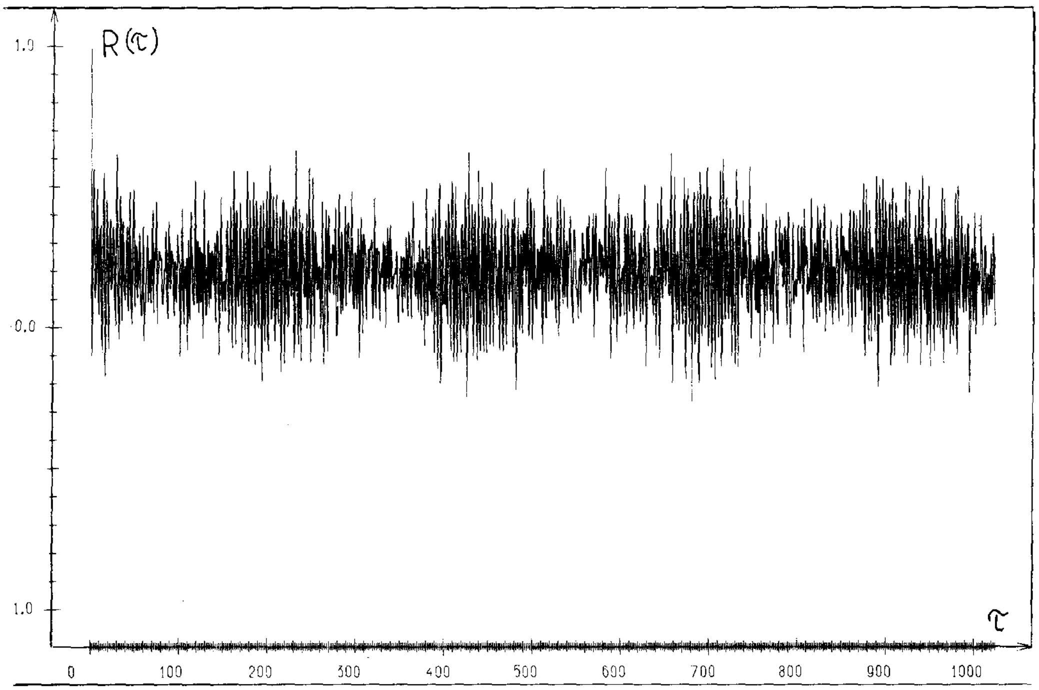

Fig. 2. The dependence of quantum correlations R (see (2.9)) on τ for the system (2.1) at $k = 5$, $kT = 5$, $t = 100$, $\tau = 1024$.

This result is supported by numerical experiments which show that in such a quantum system, unlike its classical counterpart, there is no exponential decay of correlations.

An interesting feature of the quantum correlations of the operator $\cos\hat{\theta}$ is that $R_t(\tau) > 0$ for almost any τ. As a result, the frequency spectrum of the correlations is sharply peaked at $\omega = 0$. The properties of this frequency spectrum are discussed in detail in section 3. It should be mentioned that in the quantum model, not only is the exponential decay of correlations absent but the KS-entropy h is zero as well [12] (in the classical system, $h \approx \ln(kT/2) > 0$ at $kT > 4$ [9]). In quantum case h is defined as KS-entropy for classical map obtained by C-number projection of Heisenberg equations on a quantum states basis [2]. Another definition used in [16] also gives $h = 0$ for systems with a discrete spectrum of motion. By virtue of this, the quantum system does not exhibit the local instability of motion which occurs in the classical model (2.2) when $kT > 1$. The presence of the local instability ($h > 0$) causes the dynamics of a classical system to be non-reversible. It is, of course, true that the equations of motion of the system with the Hamiltonian (2.2) are reversible (the system's Hamiltonian is symmetric with respect to time-reversal at the moments of time $T/2 + lT$ (where l is an integer), and therefore the substitution $p \to -p$ at any of the moments $T/2 + lT$ will exactly reverse the configuration space trajectory). However, an arbitrarily small perturbation ϵ will result in a significant change in the trajectory after a time period $t_\epsilon \approx |\ln \epsilon|/h$.

In connection with this, in the numerical experiments where the rount-off errors are at the level $\epsilon \approx 10^{-12}$ (BESM-6), the classical system is, in practice, irreversible (see fig. 3). At the same time, the dynamics of the quantum system prove to be completely reversible (the accuracy of return is at the level computer accuracy $\sim \epsilon$). Moreover, reversibility occurs even when a small random

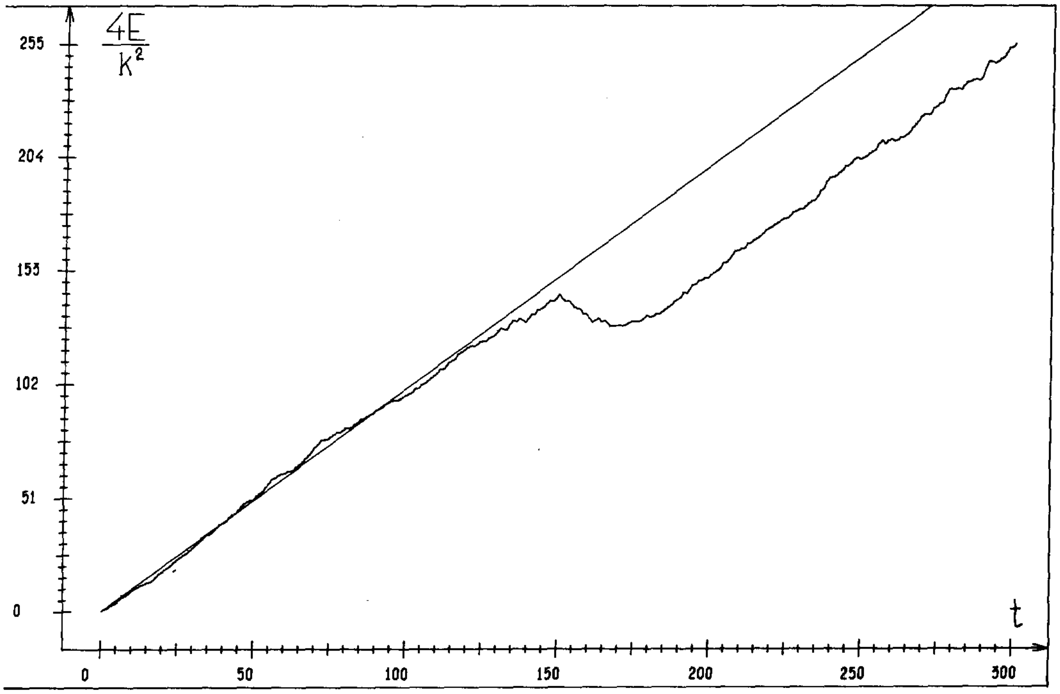

Fig. 3. The time dependence of the energy of a classical rotator (2.2) when the motion is reversed at the moment $t = 150$; the evolution of the system is reversed at the moment $t = 150$; the evolution of the system is irreversible ($kT = 5$).

change $\Delta\varphi = 0.1$ is added to the phases of the Fourier-components A_n of the wave function ψ at the moment of reversal (see fig. 4).

The total number of energy levels used in these numerical experiments was $N = 2049$ $(-1024, 1024)$. The initial conditions were varied: in one case, only the zero level was excited ($n_0 = 0$, with a uniform distribution on over θ), in the second, many levels were excited corresponding to a Gaussian distribution $|A_n|^2 = (\pi(\Delta n)^2)^{-1/2} \exp(-n^2/\Delta n^2)$ with width $4 \lesssim \Delta n \lesssim 20$. Just as in refs. 1 and 4, no significant dependence of the motion of initial conditions was observed.

3. Two-dimensional model

The numerical experiments of the preceding section have shown that the statistical properties of a quantum system are much weaker than those of the corresponding classical system. Of further interest is the effect of such a system on a second degree of freedom to which it is weakly coupled.

When the coupling is weak, the influence of the second degree of freedom on the first may be neglected and the excitation of the second degree of freedom is determined by the statistical properties of the motion of the first degree of freedom.

As an example, consider a system with the Hamiltonian

$$\hat{H} = \frac{\hat{p}_1^2}{2} + \omega\hat{p}_2 + (k\cos\theta_1 + \epsilon\cos\theta_1\cos\theta_2)\delta_T(\tau)\,, \tag{3.1}$$

where

$$\hat{p}_1 = -\mathrm{i}\frac{\partial}{\partial\theta_1}, \qquad \hat{p}_2 = -\mathrm{i}\frac{\partial}{\partial\theta_2}, \qquad \hbar = 1\,.$$

By solving the Schroedinger equation with the

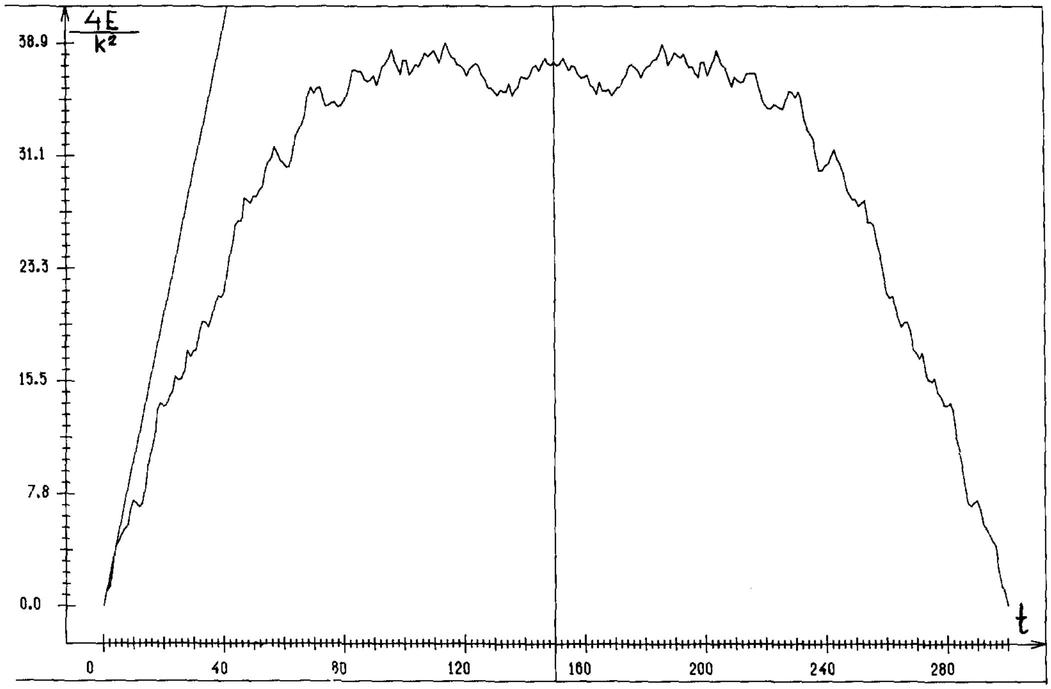

Fig. 4. The time dependence of the quantum rotator energy (2.1) when the motion is reversed and a random phase $0 < \Delta\varphi < 0.1$ is added to each amplitude A_n at the moment of time $t = 150$; the evolution of the quantum system is entirely reversible ($k = 20$, $kT = 5$); the straight line corresponds to the classical diffusion (2.4), the vertical line corresponds to the moment of time reversal.

Hamiltonian (3.1), one gets a mapping for the wave function over one period T:

$$\psi(\theta_1, \theta_2, t+1) = \exp(-\mathrm{i}(k\cos\theta_1 + \epsilon\cos\theta_1\cos\theta_2)) \times \frac{1}{2\pi} \sum_{n_1, n_2 = -\infty}^{\infty} A_{n_1 n_2}(t)\, \mathrm{e}^{\mathrm{i}(n_1\theta_1 + n_2\theta_2)}\, \mathrm{e}^{-\mathrm{i}(Tn_1^2/2 + \omega T n_2)} . \qquad (3.2)$$

Let us consider the case where $\epsilon \ll 1$ and only the ground level of the second degree of freedom is initially excited ($A_{n_1 n_2} = A(n_1)\delta_{n_2,0}$). Then the number of excited levels is determined by the statistical properties of the system (2.1). Indeed, from the equations for the Heisenberg operators we have

$$\hat{p}_2(t) = \hat{p}_2(0) + \epsilon \sum_{t_1=1}^{t-1} \cos\hat{\theta}_1(t_1) \sin(\hat{\theta}_2 + \omega T t_1) . \qquad (3.3)$$

From (3.3) and (2.9) one can obtain the number of excited levels n_2,

$$\langle n_2^2 \rangle = \frac{\epsilon^2}{2} \left[\sum_{t_1=0}^{t} \left(\sum_{\tau=0}^{t-t_1} R_{t_1}(\tau)\cos\omega T\tau - \frac{1}{2} R_{t_1}(0) \right) \right] . \qquad (3.4)$$

Due to the exponential decay of correlations in the classical case in the sums in (3.4), the main contribution was given by the terms with $\tau = 0$. Therefore the diffusive excitation occurs over both the first (2.4) and the second degrees of freedom. For the classical system then, the number of excited levels in the second degree of freedom is approximately

$$\langle n_2^2 \rangle \approx \frac{\epsilon^2}{4} t . \qquad (3.5)$$

In the quantum system, the presence of residual correlations (see fig. 2) leads to a sharp restriction of energy growth in the first degree of freedom at $t > t^*$ (the influence of the second degree of freedom may be neglected, since $\epsilon \ll 1$). The influence of the residual correlations on the excitation of the second degree of freedom requires further investigation. If the frequency spectrum $\tilde{R}(\nu)$ of the correlations $R(\tau)$ is purely discrete (this occurs when the quasienergy spectrum [14] of the system (2.1) contains only discrete levels), then for those values of the parameter ωT which coincide with the discrete frequencies of $R(\tau)$, $\langle n_2^2 \rangle$ will increase quadratically $\langle n_2^2 \rangle \sim t^2$. If the spectrum $\tilde{R}(v)$ contains a continuous component (this may occur only if the spectrum of quasienergies turns out to be continuous), $\langle n_2^2 \rangle$ grows diffusively with time ($\langle n_2^2 \rangle = D\epsilon^2 t/4$), with the diffusion factor $D \sim \tilde{R}(\omega T)$.

The motion of the quantum system (3.1) was studied numerically by means of the formula (3.2). The parameter ϵ was chosen to be equal to 10^{-5} (variations of ϵ within the interval 10^{-3} to 10^{-5} resulted in the quantity $\langle n_2^2 \rangle / \epsilon^2$ remaining unchanged to an accuracy of up to 0.1%). A finite number of levels $-400 \leqslant n_1 \leqslant 400$, $-2 \leqslant n_2 \leqslant 2$, were used in the run, and, because of the smallness of ϵ, $\langle n_2^2 \rangle$ was determined only by the probability W_{n_2} of finding the system at the levels $n_2 = \pm 1$. During the whole period of running, $W_{n_2 = \pm 2} < 10^{-14}$ and, in view of this, it was assumed that the influence of the $n_2 = \pm 2$ levels could be neglected. The excitation of these levels was calculated to verify the validity of this assumption. Another method for determining this validity was to check the conservation of the probability $W = \Sigma_{n_1, n_2 = -\infty}^{\infty} |A_{n_1 n_2}|^2 = 1$. In all cases, the error δW for the total probability did not exceed 10^{-3} and the dynamics of excitation of the first degree of freedom (e.g., $\langle n_1^2 \rangle$) coincided with the case $\epsilon = 0$ to an accuracy of up to 0.1%.

The numerical experiments indicate that the excitation of the second degree of freedom depends substantially on the parameter ωT. There are three different situations:

1) For the second degree of freedom, just as for the first the quantum limitation of the diffusion is observed; at $k = 5$, $T = 1$, this occurs for $\omega T = 1$, 1.5, 1.87, 2.37, 2.42.

2) For some values of ωT, resonant excitation of the second degree of freedom ($\langle n_2^2 \rangle \approx t^2$) is observed. This takes place for $\omega T = 0$, 0.5, 1.27, 1.71 if $k = 5$, $T = 1$.

3) In still other cases ($k = 4$, $T = 1$; $\omega T = 2.4$, 2.5, 2.52) diffusive excitation has been observed. It is worth noting that for $\omega T = 2.5$, $\langle n_2^2 \rangle$ grows practically linearly with time up to $t = 2000$ (see fig. 5), while the diffusion limitation on the first degree of freedom occurs in a few kicks ($t^* = 5$). For $\omega T = 2.4$, 2.52, a linear growth was observed during the total time of computation ($t = 750$) with the average diffusion factors $D_q / D_{cl} \approx 0.7$, 2 respectively, where $D_{cl} = \epsilon^2/4$.

When the initial conditions were changed, substantial variations in the motion were not observed (for example, resonances occured at the same values of the parameter ωT). However, changes in the parameters k and T (even at $kT = \text{const.}$) produced large variations in the dependence on ωT (for example, at $k = 10$, $T = 0.5$, $\omega T = 1.27$, diffusion limitation was observed instead of resonance). The only exception was the value $\omega T = 0$, for which $\langle n_2^2 \rangle$ grew quadratically with time for all investigated values of the parameters k and T in the regions $k < 1$, $k > 1$ at $kT < 1$ and $kT > 1$. In the classical stability region $kT < 1$, $k > 1$, the dependence of $\langle n_2^2 \rangle$ on t seemed to be close to the classical dependence, where resonant excitation also occurs due to the stability of the classical motion. Hence, the resonant growth of $\langle n_2^2 \rangle$ in the quantum system at $kT > 1$, $\omega T = 0$ can be interpreted as an indication of the presence of a stable quantum component. Of course, this problem should be examined in more detail.

It should be noted that if the Hamiltonian (3.1) contained $\sin \theta_1$ instead of $\cos \theta_1$, the excitation of the second degree of freedom would be determined by the correlations of $\sin \theta_1$. But since in this case $\langle n_2^2 \rangle \sim \langle n_1^2 \rangle$ at $\omega = 0$, the quantum limitation of the diffusion should occur in both the first and second degrees of freedom. The existence of resonances for $\omega T \neq 0$ indicates the presence of a

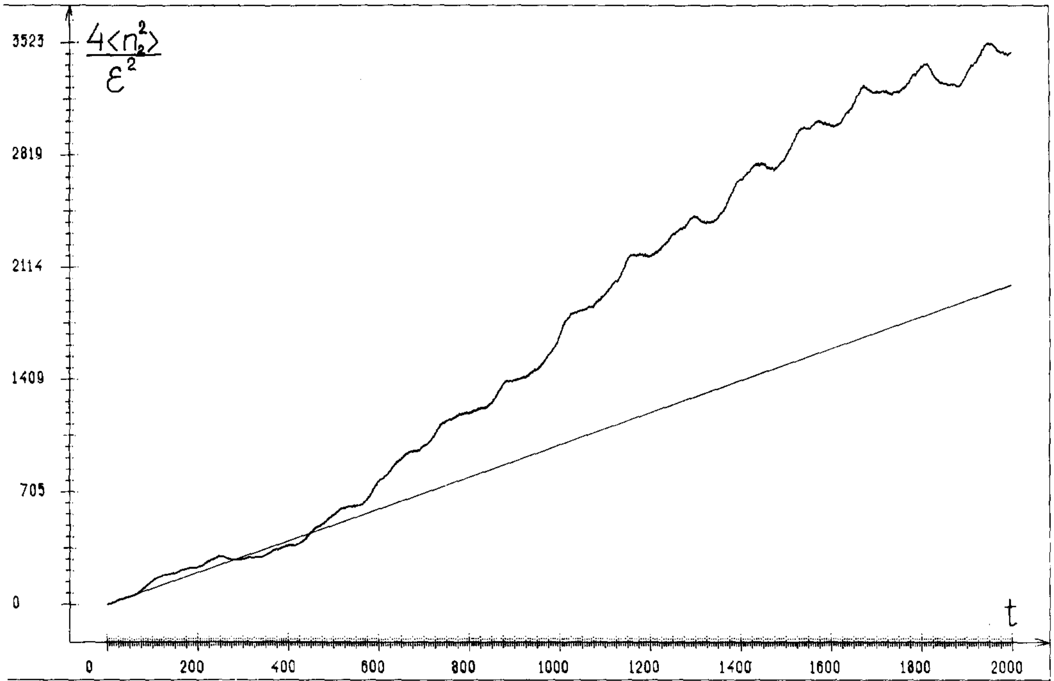

Fig. 5. The dependence of $\langle n_2^2\rangle$ on time for the system (3.1) with $k = 5$, $T = 1$, $\omega T = 2.5$, $\epsilon = 10^{-5}$, $t = 2000$. The straight line corresponds to the classical diffusion (3.5).

discrete component in the spectrum of correlations $R(\tau)$, and, as a consequence (see above), of discrete levels in the spectrum of quasienergies ($\epsilon_{n'} - \epsilon_n = \omega$). On the other hand, a diffusive growth of $\langle n_2^2\rangle$ at some value of ωT indicates that there exist in the spectrum of quasienergies a continuous zone of width $\Delta\epsilon_z \gtrsim 0.02$ ($k = 5$, $T = 1$). However, the finiteness of the computing time t_c admits only the lines with $\Delta\epsilon \gtrsim 1/Tt_c$, and therefore, strictly speaking, one can only affirm that in the zone of quasienergies $\Delta\epsilon_z$, the spectrum is either continuous or consists of closely-spaced discrete lines which are separated by a distance $\Delta\epsilon_L \lesssim 5 \times 10^{-4}$. Summarizing, one can say that, besides the two time scales of the motion of the quantum system (2.1) t_s and t^* (see section 2 and ref. 7) there is another time scale t_w on which some weak statistical properties are still conserved. It is on this time scale that the diffusive excitation of the second degree of freedom in (3.1) occurs. It is significant that t_w greatly exceeds t^* and t_s ($t_w \gg t^* \gg t_s$). In the case where $k = 5$, $T = 1$ we have $t_s \approx 1$, $t^* \approx 5$, $t_w \approx 2000$ (see fig. 5). The question of how to determine the time scale t_w, whether it is finite or infinite, requires further examination. If $t_w = \infty$, the spectrum of quantum correlations and the spectrum of quasienergies will both contain continuous components. It is worth noting that continuity of the spectrum of quasienergies does not necessarily imply continuity of the spectrum of correlations. For example, at $T = 4\pi$ (the case of a quantum resonance [1, 15]) the wave function is given by $\psi(\theta, t) = \exp(-ikt \cos\theta)\psi(\theta)$ and according to (2.9), the spectrum of correlations consists of only one discrete line. On the other hand, the spectrum of quasienergies for this case is continuous [15].

4. A model with non-commensurable frequencies

In addition to the case considered in the foregoing section, a situation can occur in which the motion over one degree of freedom, during a certain period of time, may be considered periodic and known. When this is the case, the motion over the other degree of freedom is determined by a field of known external forces. For the system (3.1), such a situation arises when $p_2 \gg \epsilon$. In this case, one can assume, as a first approximation, that the coordinate θ_2 varies periodically with time: $\theta_2(\tau) = \theta_2(0) + \omega_0\tau (\omega_0 = \omega)$. The dynamics of the first degree of freedom are then described by the Hamiltonian (2.1) with the time-dependent parameter k: $k(\tau) = k + \epsilon \cos \omega_0 \tau$. Experiments show that the dynamics of the quantum system with $k(\tau)$ varying periodically in time differ substantially from the cases considered in refs. 1, 4 and 7, where $k = \text{const.}$ and $k \approx t^{\alpha} (\alpha < \frac{1}{2})$. Therefore, a study of this system is important for understanding SQS properties. When the time-dependent parameter $k(\tau)$ is introduced into the classical system (2.2), the motion becomes stochastic when $(k + \epsilon)T \gtrsim 1$ (as it does in the case of constant k). The phase θ then varies randomly and the rotator energy grows diffusively with time

$$E(t) = \frac{k_{\mathrm{ef}}^2}{4} t + E(0), \tag{4.1}$$

where $k_{\mathrm{ef}}^2 \approx k^2 + \epsilon^2/2$. Thus, the dynamics of classical systems with constant and varying k have no principal distinctions. On the other hand, the dynamics of the quantum rotator are quite different for these two cases.

One difference is that, with k constant, there is quantum limitation of the diffusion (see ref. 1 and section 2). This results in the fact that at $t \gg t^*$ (see (2.7)) there is practically no growth of the rotator energy. The numerical experiments carried out with the model (2.1) with $k(\tau) = k + \epsilon \cos \omega_0 \tau$ have shown that, in the case where the frequencies ω_0 and $\Omega = 2\pi/T$ are non-commensurable and $\epsilon \gtrsim 1$, the quantum rotator energy grows diffusively with time, and the diffusion factor is close to the classical one. At $k = 0$, $\epsilon = 7$, $\epsilon T = 7$, $\omega_0 T = 2$, a diffusive growth of the energy continues throughout the running period $t = 1000$ (see figs. 6 and 7), and the distribution over energy levels is nearly Gaussian (2.5) with $k = k_{\mathrm{ef}}$. In the case where $k = 7$, $T = 1$, $\epsilon = 0$, the time of diffusive growth is only $t^* \approx 10$. If the frequencies are commensurable ($\omega_0 T = 2\pi p/q$, where p and q are integer non-commensurable numbers), quantum limitation of diffusion is observed, and the length of time t^* during which the diffusion slows down increases with increasing q; $t^* \approx 60$ at $\omega_0 T = \pi$; $t^* \approx 400$, $\omega_0 T = 2\pi/3$; $t^* \approx 450$, $\omega_0 T = 2\pi/5$; $t^* > 1000$, $\omega_0 T = 8\pi/13$ for $k = 0$, $\epsilon = 7$, $T = 1$. The case $\omega_0 T = 0.1$ is interesting. Because the time necessary for a phase shift is $t_{\mathrm{p}} \approx 2\pi/\omega_0 T \approx 60 > t^* \approx 10$, the diffusion drops off when t reaches t^* and the growth of energy practically ceases. However, after a time $\approx t_{\mathrm{p}}$ the phase variation becomes significant and the energy grows again. Thus, a step-like diffusive growth of energy with time occurs.

A diffusive energy growth was also observed in the essentially quantum region at $k = 0$, $\epsilon = 3.5$, $\epsilon T = 7$, $\omega_0 T = 2$ (see fig. 8). For small values, $\epsilon \lesssim 4.5$, the quantum limitation of diffusion was observed, and as ϵ was varied from 1 to 4.65, the time t^* increased from $t^* \approx 1$ to $t^* > 2500$ (see fig. 9). Within the interval $1 \leqslant \epsilon \leqslant 4.65$, the dependence of t^* on ϵ is nearly exponential. However, experimental limitations prevent measurements of this dependence when $\epsilon > 4.65$. Apparently, t^* will continue to increase exponentially with increasing ϵ (some estimates for t^* (k_{ef}) are given at the end of this section). However, the fact that t^* increases sharply (by three orders of magnitude) with increasing ϵ from 1 to 3.5, suggests that there is some $\epsilon_{\mathrm{cr}} \approx 3.5$ above which a practically unlimited excitation of the quantum rotator takes place. This value of ϵ_{cr} is only $s_{\mathrm{q}} = \epsilon_{\mathrm{cr}}/\epsilon_1 \approx 3.5$ times higher than the quantity $\epsilon_1 \approx 1$ which corresponds to the quantum border of stability [10]. For example with the parameter values $k = 0$, $\epsilon = 1$, $T = 5.6$, and $\omega_0 T = 2$, the ratio of the quantum and classical diffusion coefficients at the time $t = 200$ is approximately equal to $D_{\mathrm{q}}/D_{\mathrm{cl}} \approx 2.4 \times 10^{-4} \ll 1$.

At $k > \epsilon \gtrsim 2$, the rotator is also diffusively excited (with $D_{\mathrm{q}} \approx D_{\mathrm{cl}}$, and a widening Gaussian

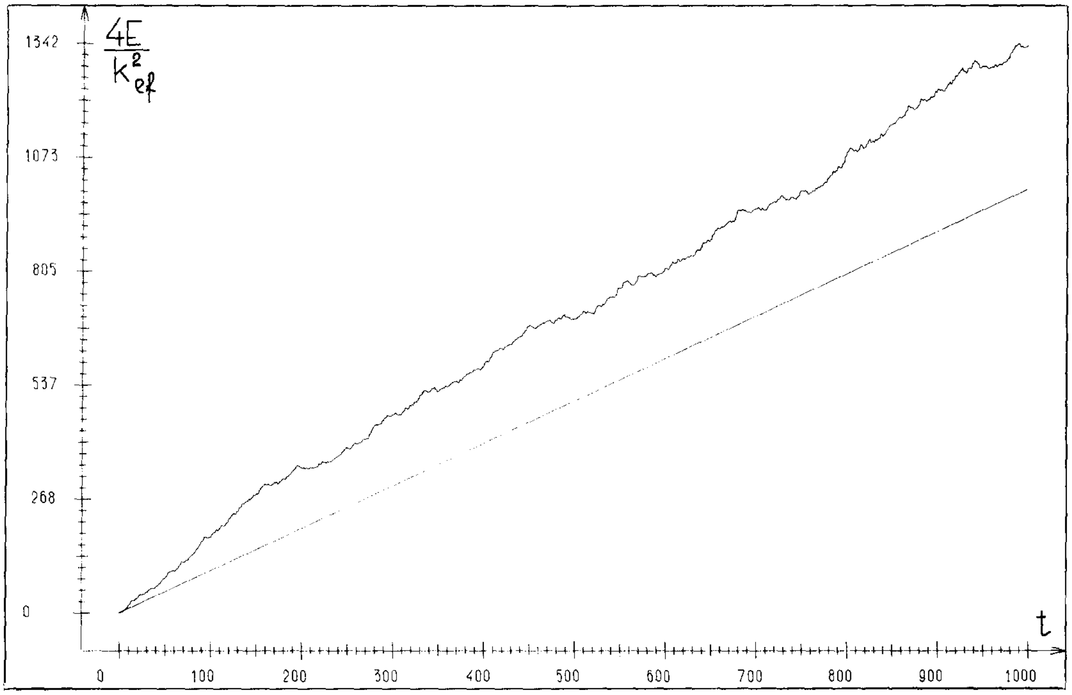

Fig. 6. The time dependence of the rotator energy for the system (2.1) with $k(\tau) = k + \epsilon \cos \omega_0 \tau$ at $k = 0$, $\epsilon = 7$, $\epsilon T = 7$, $\omega_0 T = 2$. The straight line corresponds to the classical diffusion (4.1).

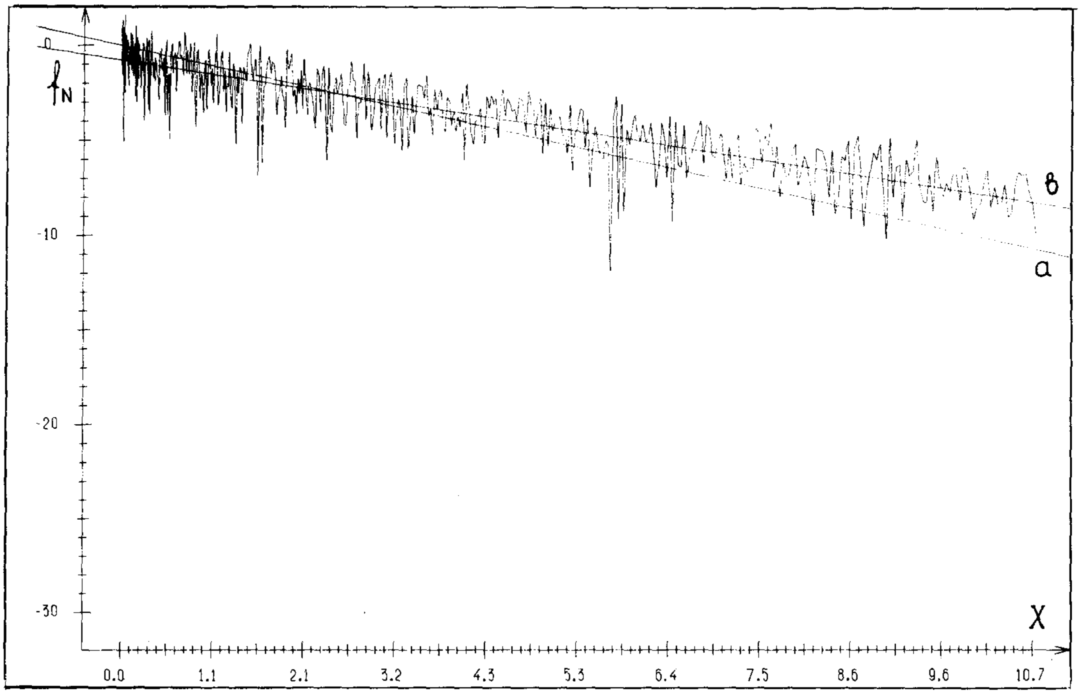

Fig. 7. The distribution function over the unperturbed levels of the system (2.1) with $k(\tau) = k + \epsilon \cos \omega_0 \tau$ and the parameter value of fig. 6. The coordinates are normalized, $f_N = |A_n|^2 \sqrt{\pi k_{ef}^2 t}$, $x = n^2/k_{ef}^2 t$, and the moment of time is $t = 1000$. The straight line "a" gives the theoretical classical distribution (2.5) with $k = k_{ef}(f_N = e^{-x})$, and the straight line "b" denotes the linear interpolation. The broken line is the experimental result.

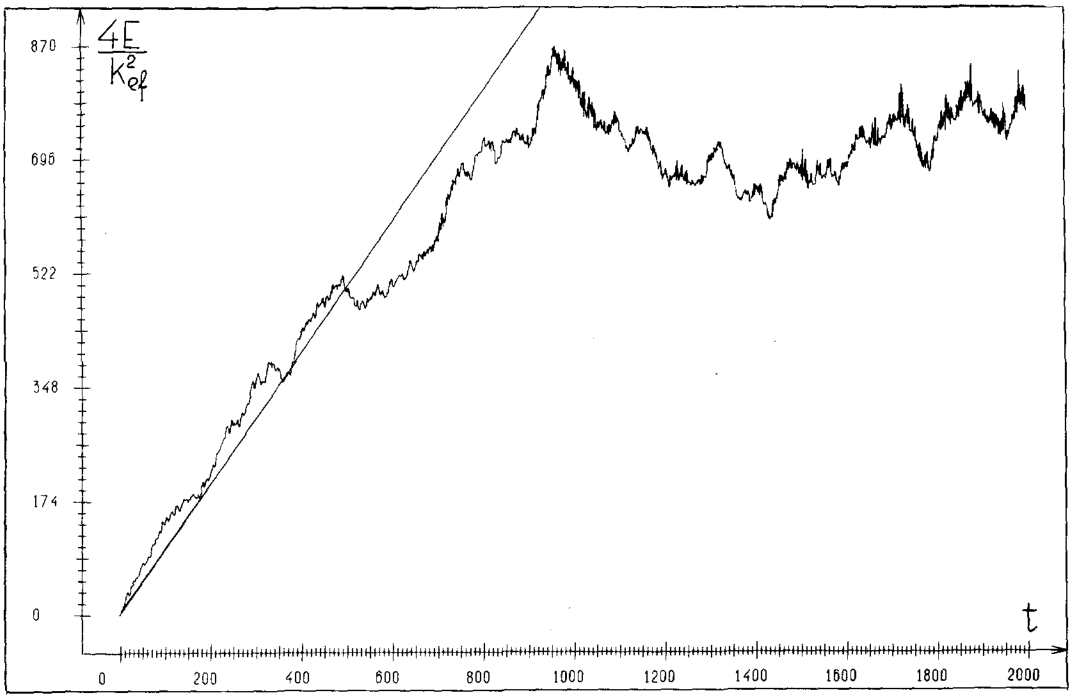

Fig. 8. The same as fig. 6 but with $k = 0$, $\epsilon = 3.5$, $\epsilon T = 7$, $\omega_0 T = 2$.

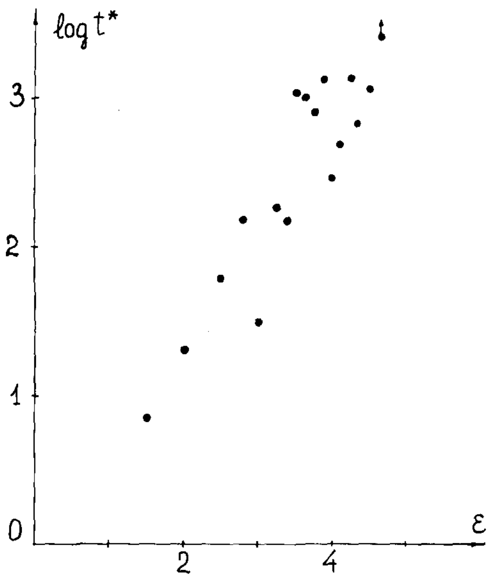

Fig. 9. The dependence of the time t^* on the parameter ϵ for the model (2.1) with $k(\tau) = k + \epsilon \cos \omega_0 \tau$ at $k = 0$, $\epsilon T = 7$, $\omega_0 T = 2$; log is base 10.

energy level distribution which is similar to the corresponding classical distribution). Note that for $k = 10$, $T = 0.5$, $\epsilon = 0$, the time at which quantum limitation of diffusion occurs is $t^* \approx 25$, while at $\epsilon = 2.5$ ($\omega_0 T = 1$) the energy growth continues during the entire period of computation, $t = 1000$. Fourier analysis shows that $\tilde{R}(\nu)$, in contrast to the case where $\epsilon = 0$, does not contain clearly observed peaks. This implies a continuous spectrum of the motion for the quantum system (where the accuracy of experimental resolution is $\Delta\omega \approx 10^{-3}$).

At $\epsilon \lesssim 2$ and $k \gg 1$, the rate of diffusive excitation of the quantum rotator coincides, for a period of time, with the classical rate. It then decreases to a certain limiting value $\bar{D}_q$ and does not decrease further during the entire period of running ($t = 300$). The dependence of the ratio $\bar{D}_q/D_{cl}$ on ϵ is illustrated in fig. 10. For comparison, this figure also presents the dependence of $\bar{D}_q/D_{cl}$ on ϵ when $k(t) = k + \epsilon\xi(t)$, where $\xi(t)$ is randomly

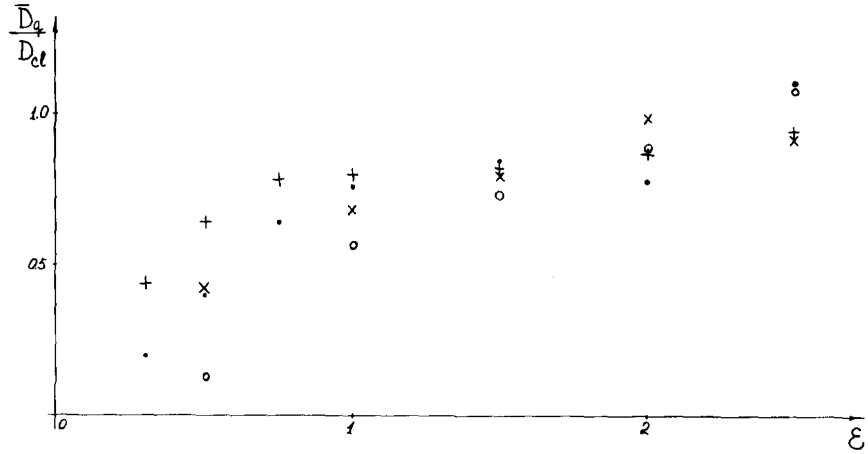

Fig. 10. The dependence of the limiting value of the diffusion factor $\bar{D}_q$ on ϵ in the model with $k(\tau) = k + \epsilon \cos \omega_0 \tau$, $t = 300$, $\omega_0 T = 1$; the circles "○" give the dependence when $k = 10$, and × when $k = 20$. Also shown is this dependence in the model with the random perturbation: $k(t) = k + \epsilon\xi(t)$, $kT = 5$; here the dots "·" give the $k = 10$ dependence and plusses "+" the $k = 20$ dependence.

varied with time in the interval $[-1, 1]$. Just as in the case where $k(t)$ varies periodically, when $\epsilon \gtrsim 2$, D_q approximately equals D_{cl}. And when $\epsilon < 2$, there is a limiting coefficient $\bar{D}_q < D_{cl}$ which decreases with ϵ. At the same values of ϵ, k, $T(\epsilon < 2)$, the diffusion rate $\bar{D}_q$ for the rotator model with randomly varying $k(t)$ turns out to be larger than the value $\bar{D}_q$ in the model with periodically varying $k(t)$ (see fig. 10). However, the qualitative form of the ϵ-dependence seems to be the same.

To summarize, there is a quantum border of stability at $\epsilon \approx 1$. At $\epsilon \ll 1$ the diffusive excitation of the rotator is sharply reduced, and at $\epsilon \gtrsim 2$, both for the periodic (with non-commensurable frequencies) and the random variation of $k(t)$, the rotator energy grows diffusively during the entire period of computation ($t = 10^3$) with the diffusion rate $D_q \approx D_{cl}$. One should mention that the diffusive excitation observed at $\epsilon > 1$ occurs also in the essentially quantum region $T > 1$ (for example, for $k = 10$, $\epsilon = 2.5$, $T = 4.6$, $\omega_0 T = 1$). In the region of classical stability, $(k + \epsilon)T \ll 1$, the energy variation is limited just for the classical system.

To conclude this section, some theoretical estimates are derived. Consider the case in which the two frequencies are commensurable, i.e. $\omega_0 T = 2\pi p/q$. Here the perturbation is periodic with period q. The time $\tau^* = t^*/q$ is determined by the distance between the discrete levels of the quasienergy $\tau^* = t^*/q \approx 1/\Delta$ (a similar method for estimating t^* was used in [7]). The quantity Δ is determined by the number N_ψ of effectively excited levels of the quasienergy, where N_ψ is approximately equal to the number of excited unperturbed levels $N_\psi \approx k_{ef}\sqrt{t^*}$. From these relations we obtain an estimate for t^*,

$$t^* \sim k_{ef}^2 q^2 . \tag{4.2}$$

Now let $\omega_0 T = 2\pi p/q + \delta$, where δ is a small deviation. During the time $t \lesssim t_p \sim \delta^{-1}$ the system moves in approximately the same manner as when $\omega_0 T = 2\pi p/q$. If $t_p \sim \delta^{-1} \gtrsim t^* \sim k_{ef}^2 q^2$, quantum limitation of diffusion occurs, although when $2t_p > t > t_p$, a diffusive growth may begin again, as it does when $k = 0$, $\epsilon = 7$, $\epsilon T = 7$, $\omega_0 T = 0.1$. Hence, the quantum limitation of diffusion will be observed (at least on some time scale) if

$$\delta(q) \lesssim \frac{1}{q^2 k_{ef}^2} . \tag{4.3}$$

The total measure of all deviations grows with increasing q,

$$\sum_{q=1}^{q_{cr}} \sum_{p=1}^{q} \delta(q) \sim \frac{\ln q_{cr}}{k_{ef}^2} \sim 1$$

and becomes equal to 1 at $q_{cr} \approx e^{\gamma k_{ef}^2}$. From this, one can obtain an estimate for t^* in the case of two non-commensurable frequencies,

$$t^* \sim k_{ef}^2 e^{2\gamma k_{ef}^2}, \tag{4.4}$$

where γ is a certain numerical constant. The estimates (4.2) and (4.4) coincide qualitatively with the available experimental data (see fig. 9) but a more exact comparison fails because of a sharp increase in t^* with increasing k_{ef} and q. In the case where there are three non-commensurable frequencies (e.g., $k(\tau) = k + \epsilon \cos \omega_0\tau \cos \omega_1\tau$) where $\omega_0 T = 2\pi p_0/q_0 + \delta_0$, and $\omega_1 T = 2\pi p_1/q_1 + \delta_1$, the time t^* is approximately $(q_0 q_1 k_{ef})^2$, for $\delta_0 = 0$, $\delta_1 = 0$. In order for diffusive limitation to occur it is required that $t^* < \min(\delta_0^{-1}, \delta_1^{-1})$. It seems to follow that the measure of such frequencies is small at large k_{ef} ($\Sigma_{q_0, p_0, q_1, p_1} \delta_0\delta_1 \approx k_{ef}^{-4} \ll 1$). Therefore, in this case of three or more non-commensurable frequencies, an unlimited diffusive excitation of the quantum rotator should occur for almost any ω_0, ω_1. At $k = 0$, $\epsilon = 3.5$, $\epsilon T = 7$, $\omega_0 T = 2$, $\omega_1 T = 2^{1/4}$, the experimental value of the time t^* is greater than the run time $t = 2000$.

Similar estimates may be made for nonlinear systems with two (or more) degrees of freedom that are similarly influenced by periodic perturbations. As an example let us consider the model (3.1) with the substitution $\omega\hat{p}_2 \to \omega\hat{p}_2^2/2$. Since this model has two degrees of freedom, the number of excited quasienergy levels, $N_\psi \approx \Delta n_1 \Delta n_2 \approx k_{ef}\epsilon t^*$ is larger than t^* if $k_{ef}\epsilon \gtrsim 1$ and therefore $t^* = \infty$. Thus, it seems possible that in nonlinear quantum systems with N degrees of freedom ($N \geqslant 2$) and periodic or quasiperiodic perturbations, the quantum limitation of diffusion is absent.

5. Concluding remarks

The studies described here show that the statistical properties of an SQS are much weaker than those of the corresponding classical stochastic system. For example, the quantum systems are exempt from the exponential decay of correlations (section 2 and ref. 12) which occurs in the classical systems when the measure of the islands of stability is negligibly small. The statistical properties of the classical and quantum systems correspond only for short periods of time $t_s \propto \ln(1/\hbar)$ (2.8). At $t \gtrsim t_s$ the correlations in the quantum and classical systems become completely different (see table I and fig. 2). Likewise the KS-entropies of classical and quantum systems are quite different: in an SQS the KS-entropy is $h = 0$ [12, 16], while in the corresponding classical system, $h > 0$. Note also that in an SQS, $h = 0$ not only when the spectrum of the motion of the quantum system is discrete (this case was considered in [16]), but also when this spectrum is continuous. For example, in the model (2.1) at $T = 4\pi p/q$ (the quantum resonance [1]), the quasienergy spectrum is continuous [15], but h is still equal to zero [12]. Note here that the influence of quantum resonances on the motion of the system is not significant for nonresonance values of T (except small regions near resonance values [15]). Indeed, according to numerical experiments the behavior of the system with unperturbed (nonresonance) spectrum $E_n = \frac{1}{2}n\sqrt{5}$ is qualitatively the same as in the case of the main rotator model (2.1) with nonresonance value of T. The consequence of zero KS-entropy is the stable reversibility of the quantum evolution (see section 2, fig. 4), which is absent in classical stochastic systems (fig. 3).

At the same time, the weaker statistical properties, for example, diffusion, are conserved in an SQS for much longer times $t^* \propto 1/\hbar$ ($t^* \gg t_s$). For the system (2.1) with one degree of freedom and a periodic external force, the time t^* grows with an increase in the quasiclassical parameter according to (2.7). The numerical experiments show (section 3) that a continuous component in the spectrum of correlations and a diffusive excitation of the other degree of freedom at definite frequencies persist for long time $t_w \gg t^* \gg t_s$. At $k = 5$, t_w exceeds t^* by nearly three orders of magnitude (section 3). The question of what determines this third time scale, and whether it is finite or infinite, remains open.

Numerical experiments with a one-dimensional model and a two-frequencies external force (section 4) have shown that in such a system the diffusive time scale t^* increases very sharply (probably, exponentially) as the quasiclassical parameter k_{ef} increases (see fig. 9 and (4.4)). Apparently, if one exceeds the quantum border of stability ϵ_1 by $s_q \approx 3.5$, the time scale t^* increases by three orders of magnitude. If there are three non-commensurable frequencies (or more) the diffusive time scale appears to be infinite (see the estimates of section 4). In this case, unlimited diffusive excitation of the quantum rotator occurs and the quantum correlations decay according to a power law (2.11). Hence, the quantum system possesses the property of mixing.

Because an external force with non-commensurable frequencies may always be approximately represented by additional degrees of freedom, the ultimited diffusive growth of the energy at an almost classical diffusion rate is possible in quantum systems with two or more degrees of freedom and an external periodic driving force. This diffusive excitation takes place if the classical stochasticity criterion is satisfied and the quantum border of stability for the perturbation is exceeded [10].

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