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CORRELATION PROPERTIES OF DYNAMICAL CHAOS IN HAMILTONIAN SYSTEMS

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The structure of chaos border in phase space and its impact on the correlation and other statistical properties of the chaotic motion are considered. We conjecture that such a structure is described by a chaotic renormalization group. The effects of an external noise and of dissipation are discussed.

1. Introduction

In this paper we discuss some statistical properties of dynamical chaos in the Hamiltonian systems with divided phase space. By dynamical chaos we term the random, in Alekseev's sense, motion of a completely deterministic (= dynamical) system. According to the Alekseev–Brudno theorem (see ref. 1) this randomness is equivalent to the exponential divergence of close trajectories. However, this does not imply an exponential correlation decay, nor even the decay at all. The corresponding "anomalies" in the motion statistical properties, particularly, occur when the "chaos border" in the phase space is present which separates a chaotic and regular components of the motion. Such anomalies had been observed, apparently first, in ref. 2 and subsequently were studied in other papers [3, 4]. Our experience suggests that the most efficient approach to this problem would be investigating the statistics of the Poincaré recurrences as has been implicitly done in ref. 2. As to the correlation decay it is usually of a complicated oscillatory nature, and it may be even random as a simple example due to Nagashima [5] demonstrates.

Following ref. 3, we are going to consider, as a typical example, the stochastic layer of a nonlinear resonance where the motion can be approximately described by the mapping [6]

$$\bar{y} = y + \sin\theta; \quad \bar{\theta} = \theta - \lambda \ln|\bar{y}| + \nu. \tag{1}$$

Here the phase θ specifies the system position along the layer while the action y does so across it; the parameter λ controls the layer width $|y| \lesssim \lambda$, and ν affects the structure of the layer edge (see below). A local (about $y = z$) motion of the system (1) is described by the standard map

$$\bar{P} = P + K\sin\theta, \quad \bar{\theta} = \theta + \bar{P}, \tag{2}$$

where new action $P = \lambda(z-y)/z - \lambda\ln|z|$, and new parameter $K = -\lambda/z$. Unlike eq. (1) the map (2) has no global chaos border but its parameter has the critical value $K = K_{cr} \approx 1$ which separates the finite ($K < K_{cr}$) and infinite ($K > K_{cr}$) motions.

An invariant curve of map (2) with the rotation number r (the motion frequency $2\pi r$) becomes critical at certain $K = K_r$ when the curve is destroyed. The structure near the critical curve of irrational r is characterized by some scaling (renormalization group) which depends on the continued fraction representation of $r = 1/(m_1 + 1/(m_2 + \cdots \equiv [m_1, m_2, \ldots]$ with integers $m_i \geq 1$. The convergents $r_n = p_n/q_n = [m_1, \ldots, m_n] \to r$ correspond to the strongest resonances near the critical curve which determine the renormalization group. The latter is a sort of "dynamical system" in a functional space of mappings. The logarithm of spatial scale (the serial number n of convergent r_n) plays the role of "time", the asymptotics $n \to \infty$ corresponding to indefinitely small scales that is to the local structure. Due to discreteness of the sequence r_n the renormalization group is also discrete that is, to say, it represents some "renormmap" (of the structure on two different scales).

The simplest scaling relates to the "golden mean" $r = r^{(1)} = [1, 1, \ldots]$. This case has been studied thoroughly since the classical paper by Greene [7] (for survey of other papers see ref. 8). In this case the renormmap has a saddle fixed point that is a universal map exists which turns out to be close to map (2). That sort of scaling is also called the scale invariance (of the motion local structure), and it is directly related to the homogeneity of the $r^{(1)}$ continued fraction. In a simple theory [9] all the scaling factors are related to the asymptotic value of the denominator ratio for two successive convergents $q_{n+1}/q_n \to s_0$ $(= 1 + r^{(1)}$ for $r^{(1)})$, and they differ from the exact values [8] by less than one per cent.

For a generic (almost any) critical curve we conjecture the scaling to be chaotic since the continued fraction elements m_i of the corresponding r form a random sequence [10]. In this case the scale invariance holds at average only, over many iterations of the renormmap. For dissipative systems a similar possibility was considered recently in ref. 11.

The structure of a chaos border ($|y| \approx \lambda$ for the map (1)) depends on the marginal rotation number $r = r_0(\lambda)$ which, in turn, is determined by the fractal diagram of the function $K_{cr}(r)$ for map (2). This diagram, studied in ref. 12, looks like an infinite, scale invariant hierarchy of local maxima, inside each maximum on one scale being *two* principal maxima of close K_{cr} values on the next scale (see fig. 2 in ref. 12)*. Such a doubling results in formation of the set of "most stable" $r = r_s$ which is an uncountable Cantor set of zero measure. The continued fraction elements for $r_s = [m_i]$ take on two values only: $m_i = 1, 2$, and the sequence m_i is random for almost all r_s. We conjecture that for almost all λ the marginal values r_0 (λ) belong to the set r_s, and, hence, the scaling at the chaos border is also chaotic. In this case the corresponding renormmap would have a sort of "chaotic saddle". The asymptotic structure at the chaos border appears also to be universal for any canonical two-dimensional mapping.

2. Chaos border and statistical "anomalies"

The scale invariant structure of chaos border drastically changes statistical properties of the chaotic motion. In fig. 1a the integral probability $P(\tau)$ for the Poincaré recurrences after time interval τ is presented for two mappings with divided phase space. The circles show data from ref. 3 for the map (1) with $\lambda = 3$ (a single trajectory by 10^8 iterations). Crossings of the line $y = 0$, or transitions from one half of the layer $(y > 0)$ to the other $(y < 0)$ and vice versa, were recorded. The solid curves present the results of the unique computation due to Karney [4] for the map

$$\bar{y} = y + 2(x^2 - K), \quad \bar{x} = x + \bar{y} \bmod L, \tag{3}$$

with $K = 0.1$ (1600 trajectories by 2×10^9 iterations each!). An overall dependence $P(\tau)$ appears to be similar for both mappings even though the global structure of their motions is quite different: a bounded stochastic layer for map (1), and a bounded stable region for map (3). At average, the

*This visual picture is similar to and confirms the principal "two resonance" approximation assumed in ref. 13 to develop a particular renormalization technique.

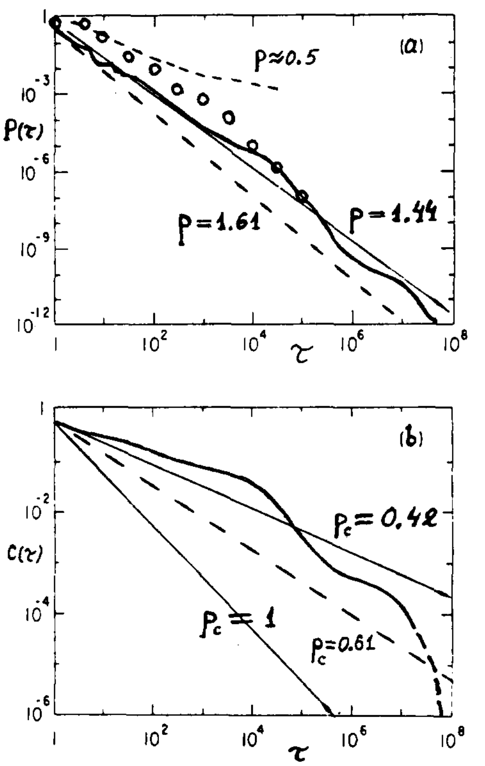

Fig. 1. Statistical properties of motion in presence of the chaos border: (a) the Poincaré recurrences; (b) the correlation decay. Two solid curves represent Karney's numerical data [4] for map (3); the circles do so for map (1) (after ref. 3). The straight lines are power law dependences with the exponent p given. The dashed curve shows the effect of external noise (9).

dependence

$$P(\tau) \propto \tau^{-p} \tag{4}$$

is a power law one with the empirical exponent $p = 1.44$ (the solid straight line in fig. 1a). The dashed line with $p = 1.61$ imposes an upper limit for p. These data are to compare with the mean value $\langle p \rangle = 1.45$ over various λ as given in ref. 3. Apparently irregular oscillations are also characteristic for the dependence $P(\tau)$. These oscillations, as well as the p value, do not depend on initial conditions of the motion, even though the latter is highly unstable. Thus, they relate not to some fluctuations but, rather, to the structure of the layer edge that depends, in turn, on the marginal r_0.

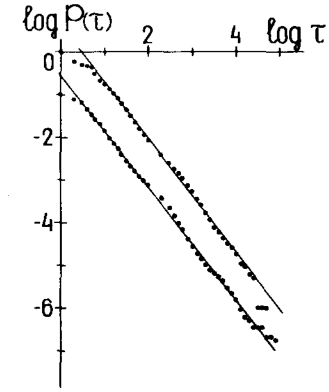

Fig. 2. Statistic of the Poincaré recurrences in map (1) for the "golden" marginal rotation number $r_0 \approx 0.617 = [1,1,1,1,1,1,3,\ldots] \approx r^{(1)}$; 10^7 iterations. The upper points are for $\lambda = 3.22$ and $\nu = 0; p = 1.338$ (the straight line). The lower points, shifted for convenience by $\Delta \log P = -1$, are for $\lambda = 2.65$ and $\nu = -0.185 \times 2\pi$; $p = 1.321$ (the straight line). The logarithms here and below are decimal.

To get some insight into the problem we undertook the special computation in which the parameters λ and ν were chosen in such a way to provide $r_0 \approx r^{(1)} = [1,1,\ldots]$ and, thus, to "kill" at least the initial oscillation in $P(\tau)$. Note that oscillation amplitude as well as the period are the bigger the larger initial differences $(m_i - 1)$. The results of this computation are presented in fig. 2. The oscillation has been considerably reduced, indeed, that qualitatively confirms its relation to the structure of the chaos border. Moreover, without oscillation the p value can be measured much more accurately, and it becomes more reliable, too. The least square fit within the interval $1 \leq \log \tau \leq 4$ gives

$$p = 1.338 \pm 0.014 \quad \text{and} \quad p = 1.321 \pm 0.014$$

for the two cases in fig. 2. Both values are compatible with each other and with the value

$$p = 4/3 \tag{5}$$

we assume for further analysis.

A qualitative explanation of the dependence (4), based upon the effect of trajectory "sticking" at the chaos border, has been given in ref. 3 (see also ref. 9). Here we attempt to relate the dependence

(4) to numerical data [6] on the diffusion rate in the standard map (2) near the critical $K = K_{cr} + \epsilon$ ($\epsilon \ll 1$) as well as to the theory of the latter process developed in ref. 14. Assuming, like in ref. 3, the recurrence time $\tau \sim \tau_n$ to be of the order of the transit time from one scale to the next bigger one, we estimate τ_n via a rough analogy between the transition among integer resonances in the standard map and that among scales near the chaos border. According to ref. 14, in the former case the transit time

$$\tau_{ST} \sim \epsilon_{ST}^{-3} \tag{6}$$

that satisfactorily agrees with numerical data. We assume, further, that near the chaos border the parameter $\epsilon_n \approx \Delta y_n = y_n - y_0$ where y_0 indicates the border. This is suggested by the fact that the average derivative $\langle dK/dz \rangle_\theta$ is a constant at the border. Since the time scale $T_n \sim x_n^{-1/2}$ (see ref. 9) we have $\tau_n \sim T_n \epsilon_n^{-3} \sim x_n^{-7/2}$. On the other hand, the measure scale $\mu_n \sim x_n$, and the motion ergodicity implies (see ref. 3): $\tau P(\tau) \sim \langle \tau \rangle \mu$; $\langle \tau \rangle \sim 1$ (this holds for $p > 1$ only, otherwise $\langle \tau \rangle = \infty$). Whence $p = 9/7 \approx 1.29$ that is fairly close to the empirical value (5) thus justifying the above assumptions. The crucial assumption is that the transit time between the scales is determined by the local parameter ϵ, and it grows indefinitely as $\epsilon \to 0$.

For additional examination of this assumption we studied numerically the Poincaré recurrences to the line $P = 0$ in the standard map at $K = K_{cr} = 0.971635$. The least square fit of the data in fig. 3 within the interval $2 \le \log \tau \le 5$ reveals a power law dependence with the exponent $p = 0.975 \pm 0.013$. The actual value of the rotation number reached in 10^8 iterations was $r_0 = 0.3797$ ($1 - r^{(1)} = 0.3819\ldots$). Note that the data for $\tau \gtrsim 10^5$ are unreliable due to a poor statistic while the initial behavior of $P(\tau)$ is affected by the phase oscillations on the integer resonance $P = 0$. Since the exponent p is very close to unity the correlation decay is nearly absent (see eq. (7) below). This indicates a very long transit time under $\epsilon \approx 0$ on all the scales. Note the crucial difference from the chaos border where ϵ is assumed to grow inward the chaotic component.

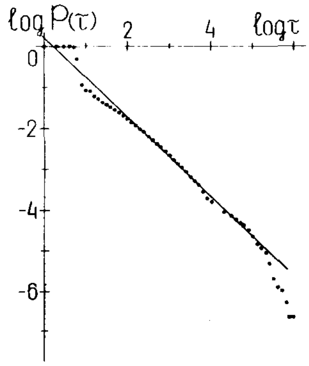

Fig. 3. Same as in fig. 2 for the standard map (2) with $K = K_{cr} = 0.971635$ ($r = 1 - r^{(1)}$); 10^8 iterations; $p = 0.975$ (the straight line).

The external border of a chaotic component is not the only chaos border since near the former an intricate network of internal borders exists surrounding stable periodic trajectories. This question is discussed in refs. 4, 9. If the structure of any chaos border is universal it would not change the motion statistical properties and, particularly, the exponent p. However, the internal borders with their own rotation numbers may cause oscillations in $P(\tau)$ even though $r_0 = r^{(1)}$ at the external border. We observed this indeed in numerical simulations. On the other hand, it was checked that in the absence of oscillations (see figs. 2, 3) the most long recurrences were related to trajectory sticking just near the external border.

Consider the particular correlating function which is constant within a strip along the whole chaos border. Then, the correlation $C(\tau)$ is merely proportional to the measure of a region in which the sticking time is $> \tau$. From ergodicity (see ref. 3):

$$C(\tau) \sim \mu(\tau) \sim \tau P(\tau), \tag{7}$$

hence the correlation decay is also a power law one with the exponent $p_c = p - 1$. The exact relation between $C(\tau)$ and $P(\tau)$ is given in ref. 4. An example of the correlation for map (3) is shown in fig. 1b taken from ref. 4. The empirical value of exponent $p_c = 0.42$ is certainly less than 0.61 (the dashed line), and, in any event, $p_c < 1$ (the lower line). The latter inequality implies a qualitative change in the statistical properties. Particularly, the diffusion equation, related to the correlation under consideration, proves to be completely inapplicable since the process is essentially non-Markovian (see also [4]). For example, in case of map (1) the quantity

$$S^2 = \left\langle \left(\sum_{i=1}^{t} y_i \right)^2 \right\rangle \propto t^d, \quad d = 3 - p > 1 \qquad (8)$$

grows faster than time. We checked the relation $d + p = 3$ numerically for various λ, and we have found it to hold within 10 per cent for $t \lesssim 10^4$ while oscillation in $P(\tau)$ was not very important. It is quite likely that such statistical anomalies were observed experimentally in ref. 15 (see [9] for discussion).

3. Noise and dissipation

To study the effect of external noise we added into the right-hand side of the first equation (1) the term $f\xi_n$ where ξ_n are independent random quantities distributed homogeneously within the interval $(-\frac{1}{2}, \frac{1}{2})$, and where f is a constant. An example of noise influence upon the statistic of Poincaré recurrences is shown in fig. 1a for $f = 0.3$ (the dashed curve). From some $\tau = \tau_f$ on, which depends on the noise power f^2, the recurrence probability falls down much slower, approximately as $1/\sqrt{\tau}$ that corresponds to a free diffusion inward the stable region.

The value τ_f can be estimated from the condition that the scale Δy, related to the recurrence time τ_f, is passed over due to the external noise in the same time: $\Delta y \sim f\sqrt{\tau_f} \sim \tau_f P(\tau_f)$. Whence, the recurrence probability for $\tau \gg \tau_f$ is given by

$$P(\tau) \approx 0.7f/\sqrt{\tau}, \qquad (9)$$

where the factor 0.7 has been obtained from the numerical simulation. If the stable region is of a finite size the dependence (9) is going to change, finally, into an exponential one.

In presence of dissipation the chaos border and chaotic attractor are generally incompatible due to the capture of trajectory into one of the stable regions [16]. Therefore, one would expect both the correlation as well as Poincaré recurrences decay exponentially. To see whether this conjecture is true we did undertake numerical simulations for the modified map (1) in which the variable $\bar{y}$ was additionally multiplied by the dissipation parameter γ $(0 < \gamma < 1)$. A chaotic attractor occurs at sufficiently large dissipation whose critical value can be approximately evaluated from the condition for the absence of stable fixed points.

As well as for $\gamma = 1$, the statistics of Poincaré recurrences to the line $y = 0$, or the transitions between two symmetric halves of the attractor, were studied. Typical dependences $P(\tau)$ for $\lambda = 3$ and $\gamma = 0.5$ (curve 1), and $\gamma = 0.2$ (curve 2) are shown in fig. 4. Unlike the Hamiltonian case (fig. 2) the recurrence probability is well described by the simple exponential law (the Poisson distribution)

$$P(\tau) = AW^{\tau-1} = A\,e^{(\tau-1)/\tau_\gamma}, \quad \tau > 1, \qquad (10)$$

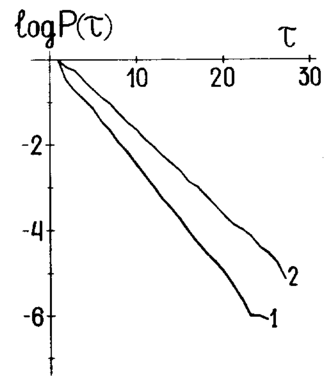

Fig. 4. Same as in fig. 2 with dissipation and $\lambda = 3$; $\nu = 0$: (1) $\gamma = 0.5$; (2) $\gamma = 0.2$.

where $(1 - W)$ means the transition probability onto the other half of the attractor, and τ_γ is the relaxation time. The empirical values of the distribution parameters are as follows: $W = 0.56$; $A = 0.79$ ($\gamma = 0.5$); $W = 0.64$; $A = 1.26$ ($\gamma = 0.2$). For $\delta = 1 - \gamma \ll 1$ these parameters can be evaluated from the diffusion equation (see ref. 17) to give

$$P(\tau) = \sqrt{2\delta}\,\exp(-\delta\tau), \quad \tau \gg 1. \tag{11}$$

For example, at $\lambda = 10^3$ and $\delta = 0.049$ the ratio $-\ln W/\delta = 0.93$, and $A^2/2\delta = 0.97$. Transitions across the line $\theta = 0$ (instead of $y = 0$) were also investigated. The distribution (10) holds in this case as well, for instance, at $\lambda = 3$; $\gamma = 0.5$ the value $W = 0.32$. Similar results have been obtained also for some other models with a chaotic attractor.

4. Conclusion

Numerical evidence strongly supports the existence of a universal critical behavior for the Hamiltonian two-dimensional maps near the chaos border. This behavior is qualitatively understood in terms of a particular renormalization group. It is conjectured that generally this group is chaotic which results in the average scaling only. The latter appears to receive some support from an irregular oscillation observed in the Poincaré recurrences. Whether it does affect the average critical exponent, i.e. whether the difference between the special value $p = 4/3$ (5) and the average one $\langle p \rangle = 1.45$ [3] is real, remains an open question.

It is also found that the critical behavior under consideration is not structurally stable with respect to the external noise, and dissipation. In the latter case the "normal", exponential, relaxation is recovered.

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