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Localization of Quasienergy Eigenfunctions in Action Space

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It is shown that the localization length of quasienergy eigenfunctions is determined by the classical diffusion rate: $l=D/2$. The new numerical method of minimal Lyapunov exponent for the calculation of l is proposed and applied to the quantum standard map and Lloyd model.

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A dynamical approach to the problem of the quantum limitation of classical chaos,[1-3] which plays a significant role in the excitation of atoms by a strong monochromatic field,[4] is proposed. This method is based on the observation that the properties of quantum quasienergy eigenfunctions can be determined by the dynamics of a classical Hamiltonian system with many degrees of freedom. We discuss here also the possibility of using such an approach for the problem of one-dimensional Anderson localization in solid-state systems.[5] The analogy between the problems of Anderson localization and quantum limitation of chaos was established by Fishman, Grempel, and Prange.[6]

Let us consider the system with the Hamiltonian $H=H_0(\hat{I})+V(\theta)\delta_T(t)$, where $\hat{I}=-i\,\partial/\partial\theta$, $\delta_T(t)$ is the periodic delta function, θ is the phase variable, $\hbar=1$, and H_0 is dimensionless.[1-3,6] The classical equations of motion are

$$\bar{I}=I-\partial V/\partial\theta,\qquad \bar{\theta}=\theta+T\,\partial H_0(\bar{I})/\partial\bar{I}. \tag{1}$$

Here $\bar{I}$ and $\bar{\theta}$ are the values of the variables I and θ after one period of time T. If the resonances overlap,[7] then the action grows without limit according to the diffusion law: $\langle(\Delta I)^2\rangle = D\tau$, where τ is the number of periods. In the region of strong stochasticity the phases $\theta(\tau)$ are independent and random. So the diffusion rate is equal to $D_{ql}=\int_0^{2\pi}(V')^2\,d\theta/2\pi$. The same expression for D_{ql} can be obtained in the quasilinear approximation.[8,9] The quasiclassical condition has the form $D \gg 1$, $T \ll 1$.[2,3]

As an example of such a system we consider the quantum standard map described by the Hamiltonian[1-3,6,10]

$$\hat{H}=\hat{I}^2/2+k\cos\theta\,\delta_T(t), \tag{2}$$

where k is a parameter characterizing the magnitude of the perturbation. The classical dynamics is described by the well-known standard map:

$$\bar{p}=p+K\sin\theta,\quad \bar{\theta}=\theta+\bar{p}, \tag{3}$$

where $p=TI$ and $K=kT$ is the classical parameter of stochasticity.[7-9] The diffusion rate for action I is equal to $D=D_0(K)/T^2$, where $D_0(K)$ is the diffusion rate in the standard map. Numerical experiments[1-3,6,10] with the quantum standard map have shown that in the course of time, $\langle I^2\rangle$ stops growing. This means that the external field effectively excites only a finite number of unperturbed levels ($\Delta n=\Delta I\sim l$). It is natural to interpret this effect as resulting from the localization of quasienergy eigenfunctions.[3,6] The following theoretical estimate has been obtained in Refs. 2 – 4:

$$l=\alpha D, \tag{4}$$

where α is an unknown numerical constant. This relation is valid when the field excites a large number of levels ($D \gg 1$). This was confirmed indirectly by numerical experiments with the quantum standard map[3] and a highly excited hydrogen atom in a monochromatic field[4] by measurement of the stationary dis-

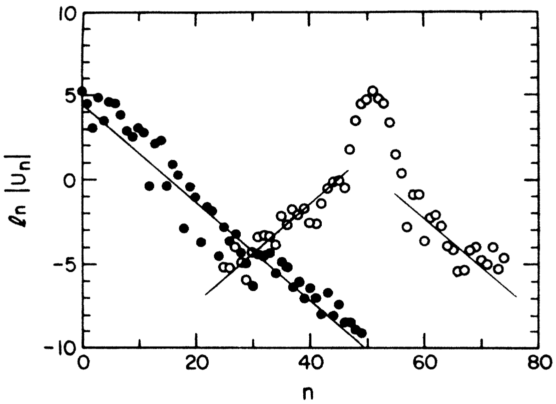

FIG. 1. Localization of the quasienergy eigenfunctions in the quantum standard map ($k=2.8$, $T=4.867$). The open and filled circles represent numerical data from Ref. 6. The straight lines correspond to the value of l obtained by the method of minimal Lyapunov exponent.

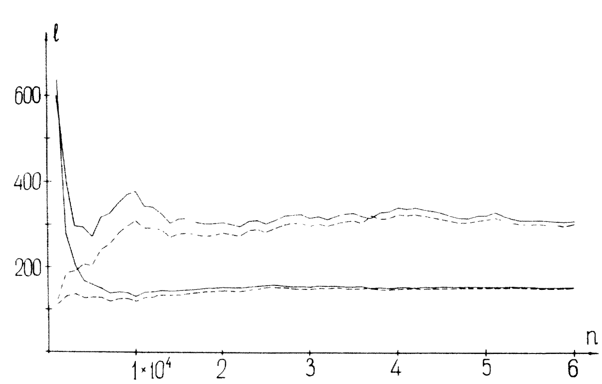

FIG 2. An example of a calculation of the localization length for the quantum standard map ($k=40$, $k=10$). The solid lines correspond to positive Lyapunov exponents and the dashed lines to negative. Two minimal exponents are shown. The fast decay of the Bessel function allows $k/2$ to be used in place of N.

tribution $\bar{f}_n$ on the unperturbed levels.

To calculate l directly from an eigenfunction, let us consider the equation for the eigenfunction with quasienergy ω[6]:

$$u_n^- = \exp\{i[\omega - TH_0(n)]\}u_n^+,$$
$$u^+(\theta) = \exp[-iV(\theta)]u^-(\theta). \quad (5)$$

Here $u^\pm$ are the values of the function u before and after a kick $\delta(t)$ and $u_n^\pm$ are the Fourier coefficients of $u^\pm(\theta)$. It is convenient to introduce $\bar{u} = e^{\pm iV/2} \times u^\pm/g$, where g is some arbitrary real function of θ. Then $u^+ = ge^{-iV/2}\bar{u}$, $u^- = ge^{iV/2}\bar{u}$, and from (3) we obtain

$$\textstyle\sum_r \bar{u}_{n+r} W_r \sin(\chi_n + \phi_r) = 0. \quad (6)$$

Here

$$W(\theta) = g\exp(-iV/2) = \textstyle\sum_r W_r \exp[i(r\theta + \phi_r)],$$

$\chi_n = [\omega - TH_0(n)]/2$, and we consider the case $W(\theta) = W(-\theta)$ only. In Ref. 6 the function $g = 1/\cos\frac{1}{2}V$ was implicitly taken. Such a choice leads to a nonphysical singularity which does not allow for an analysis of the wide class of potentials with $V(\theta) \geqslant \pi$. However, the choice of g is arbitrary and does not influence the localization in the original system (5). So, for example, in the quantum standard map it is convenient to take $g=1$. The formula (6) gives the relation between one-dimensional Anderson localization and localization of quasienergy eigenfunctions in an external field. The Hamiltonian of the corresponding solid-state problem has the form

$$\hat{H}_{ss} = \cos\tfrac{1}{2}\hat{V}\tan(\tfrac{1}{2}\omega - \tfrac{1}{2}T\hat{H}_0)\cos\tfrac{1}{2}\hat{V} - \tfrac{1}{2}\sin\hat{V}.$$

If in (6) only W_r with $|r| \leqslant N$ differs from zero, then the formula (6) determines the dynamics of some Hamiltonian system $[W(\theta) = W(-\theta)]$ with N degrees of freedom in which the serial level number n plays the role of discrete time. It is well known that in the case $N=1$ the localization length is determined by the single positive Lyapunov exponent which gives the rate of exponential decay of eigenfunctions.[5,6,11] It appears that the calculations of l for $N>1$ have not been carried out. For $N>1$, there are N pairs of Lyapunov exponents $\gamma_i^+ = -\gamma_i^- \geqslant 0$.[8] The asymptotic decay rate of the quasienergy eigenfunctions $u_n \propto \exp(-\gamma_0|n|)$ is then determined by the minimal positive Lyapunov exponent $\gamma_0 = 1/l$ (see Fig. 1). The condition for exponential localization is $\gamma_0 \neq 0$. A numerical method for calculating all of the Lyapunov exponents is described in Ref. 8. An example of the calculation of l by this method is shown in Fig. 2.

To determine the value of α in (4), let us consider the Lloyd model.[12] It is obtained from (6) when $W_0\exp(i\phi_0) = 1 - iE$, $W_{\pm 1}\exp(i\phi_{\pm 1}) = ik$, $W_r = 0$ for $|r| > 1$, and χ_n are randomly distributed on the interval $[0, \pi]$.[6] Then the diffusion rate in (1) is $D = D_{ql} = 2(4k^2 - E^2)^{1/2}$ (for $D \gg 1$). The comparison of D with the exact value

$$l = \left[\operatorname{inv}\cosh\left(\frac{1}{4k}\{[(2k+E)^2+1]^{1/2} + [(2k-E)^2+1]^{1/2}\}\right)\right]^{-1}$$

(see Ishii[13] and Refs. 5 and 6) in the region $l \gg 1$ gives $\alpha = \frac{1}{2}$.

In the quantum standard map we have $W_r = J_r(k/2), \phi_r = -\frac{1}{2}\pi r$. In this model the χ_n are not random and both

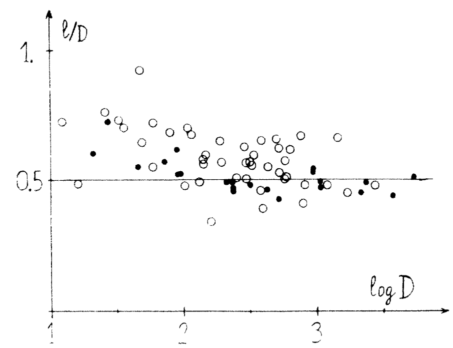

FIG. 3. The ratio $\alpha = l/D$ for different values of the diffusion rate D in the quantum standard map (open circles) and in the Lloyd model with many neighbors (filled circles). Here and in Fig. 5 the logarithm is decimal.

D and l depend on the classical parameter of stochasticity K. A comparison between numerical data and the theory (4) gives satisfactory agreement for the value $\alpha = \frac{1}{2}$ (see Fig. 3). The parameters k and K in Fig. 3 vary within the intervals $5 \leqslant k \leqslant 75$ and $1.5 \leqslant K \leqslant 29$ and $T/4\pi$ is a typical irrational number, $T \leqslant 1$. The scatter of points in Fig. 3 is mainly due to the fact that some of experimental points are not far in the quasiclassical region ($T \sim 1$). An example of the dependence $l(K)$ is shown in Fig. 4. It is clearly seen that according to expression (4) the localization length reproduces the oscillations of the classical diffusion rate.

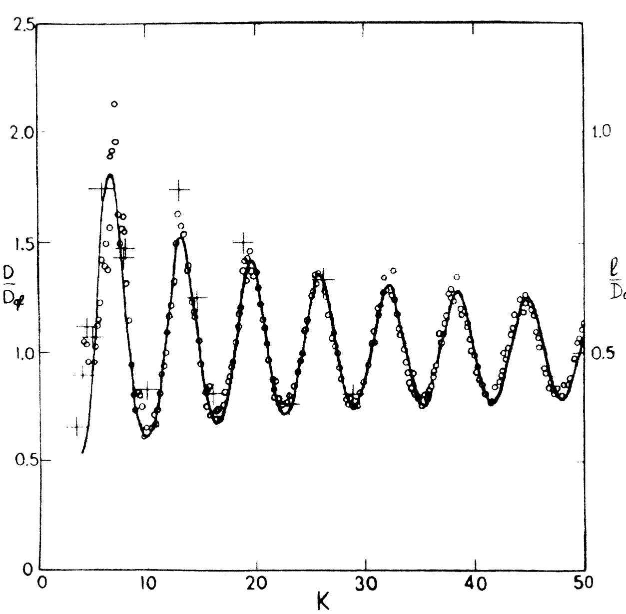

FIG. 4. The dependence $l(K)$ in the quantum standard map (crosses; $k = 30$). The curve and circles show the theory and numerical data for the diffusion rate $D(K)$ from Ref. 9, $D_{ql} = k^2/2$.

The obtained average value $\langle\alpha\rangle = 0.57$, with root mean square deviation $\Delta = 0.11$, significantly differs from the value obtained in Ref. 3, $\langle\alpha\rangle = 1.04$, $\Delta = 0.20$. The cause of this discrepancy is apparently related to the fact that in Ref. 3 l was determined from the stationary (time averaged) distribution $\bar{f}_n \propto \exp(-2|n|/l_s)$ (here we have introduced the index s). If initially only the $n = 0$ level were excited, then this distribution would be given by $\bar{f}_n = \sum_m |\phi_m(0)|^2 \times |\phi_m(n)|^2$, where $\phi_m(n)$ is the eigenfunction with quasienergy ω_m. In Ref. 3 on the assumption that $|\phi_m(n)|^2 \propto e^{-2|n-m|/l}$ and the fluctuations of $|\phi_m(n)|^2$ are negligibly small it was shown that $l_s = l$. However, the influence of strong fluctuations of $|\phi_m(n)|^2$ may be significant, and may lead to $l_s \neq l$. So, for example, in Anderson localization the fluctuations cause the difference between the rate of exponential decay of the density-density correlation function, which is analogous to $\bar{f}_n$, and the decay rate of the square of the eigenfunction.[5] A comparison of the numerical data[3] for l_s with the results presented in Fig. 5 of this paper shows that $l_s \approx 2l$. The cause of difference between l_s and l is apparently connected with the strong fluctuations of $|\phi_m(n)|^2$. A detailed discussion of the fluctuation properties and the localization in the region $K \leqslant 1$ will be given elsewhere.

Apparently, the analytic expression (4) for l and the

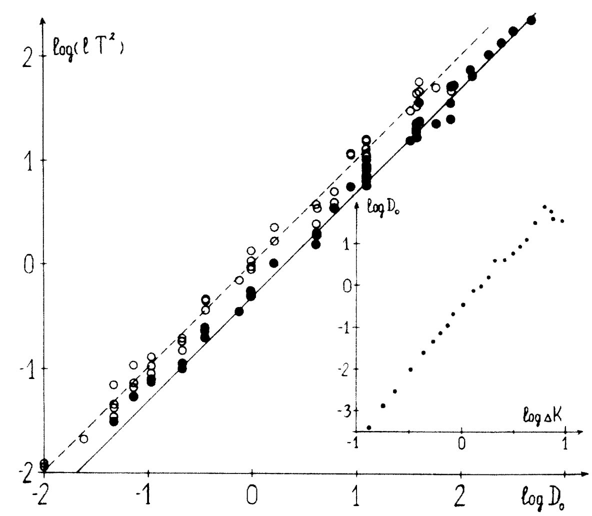

FIG. 5. The dependence of the localization length on the diffusion rate D_0 of the classical standard map. The open circles represent numerical data from Ref. 3 for values of l_s obtained from stationary distributions. The dashed line corresponds to the average value $\langle\alpha_s\rangle = 1.04$. The filled circles show the localization lengths obtained from the quasienergy eigenfunctions by the method of minimal Lyapunov exponent. The straight line shows the theoretical localization $l = D/2$. In the inset the numerical data from Ref. 3 are shown, giving the dependence of D_0 on $\Delta K = K - K_{cr}$, $K_{cr} = 0.971635$.

numerical method of minimal Lyapunov exponent may be used in one-dimensional solid-state problems. As an example, let us consider localization in the Lloyd model with many neighbors: $W_r e^{i\phi_r} = ik$, $W_0 e^{i\phi_0} = 1 - iE$, $W_r = 0$ for $|r| > N$, and the χ_n are random. Then the potential is given by

$$V(\theta) = 2\arctan\left(E - 2k\sum_{r=1}^{N}\cos r\theta\right).$$

For this model, $l = D_{ql}/2 \sim 2kN^2$ (for $E=0$) and the theory gives satisfactory agreement with the numerical data in Fig. 2 which were obtained for parameters in the intervals $0.1 \leqslant k \leqslant 50$, $4 \leqslant N \leqslant 20$. The average value of α obtained from the numerical data was $\langle\alpha\rangle = 0.52$ with $\Delta = 0.07$.

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