Hydrogen Atom in Monochromatic Field: Chaos and Dynamical Photonic Localization

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Abstract—We discuss the quantum localization phenomenon that strongly limits any quantum process of diffusive ionization that may be started in systems subjected to a periodic perturbation. In the case of a highly excited hydrogen atom in a monochromatic field, this phenomenon is theoretically analyzed by reducing the dynamics to appropriate mappings. We show that, if the field strength is less than a "delocalization border," the distribution over unperturbed levels is exponential in the number of absorbed photons and we determine the corresponding localization length. Using the mapping description, we show that the excitation process occurring in a two-dimensional atom goes essentially along the same lines as in the one-dimensional model. We support these predictions by results of numerical simulation, and we discuss the possibility of their experimental verification.

I. INTRODUCTION

MUCH theoretical and experimental work has been recently devoted to the investigation of the behavior of highly excited atoms in microwave fields. This line of physical research prompted by the influential experiments of Bayfield and Koch in 1974 [40] has by now revealed a remarkable variety of themes of absolute interest.

The present day technique allows for preparation of highly excited states (up to principal quantum number $n \sim 300$ [1]) with a long lifetime.¹ Since these states are endowed large matrix elements, they are very sensitive to external fields. Therefore, the microwave ionization of Rydberg atoms may find application in the field of microwave radiation detection.

From a general physical viewpoint, this problem lies at the intersection of several lines of contemporary research, so that methods and ideas originally developed in quite different areas find here a common ground of application.

The first and perhaps the most important of these themes is chaos. As a matter of fact, even the simplest theoretical model—a classical, one-dimensional Kepler atom in a monochromatic electric field—shows that the onset of chaotic motion gives an essential contribution to the classical excitation process. On the other hand, a Rydberg atom is an essentially quantum object. The study of its microwave excitation provides, therefore, an ideal testing

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ground for the existence of quantum "chaotic" phenomena, and brings the long debated and sometimes abstract question about the existence and nature of quantum chaos very close to physical application. In particular, the deep phenomenon of quantum suppression of chaotic diffusion, that previous theoretical studies on quantum chaos indicate a typical occurrence, may find here its first experimental verification. This phenomenon is due to quantum interference effects that prevent any diffusive-like excitation process from going on indefinitely, and can be considered as a dynamical version of the Anderson localization well known to solid-state physicists. From a conventional quantum theoretical viewpoint, a large number of photons is required in order to ionize Rydberg atoms by microwave fields. The results obtained in this field do, therefore, provide a better understanding of multiphoton processes and collisionless ionization of atoms and molecules in laser fields. For example, a somewhat unexpected result is that the most efficient ionizing process is not a single-photon but a multiphoton one [12].

The theory developed for this problem [2], [3], which lies on the elusive borderline between classical and quantum mechanics, borrows ideas from both these disciplines, as well as from localization theory. A ground work for this theory was provided by the results of previous investigations of a rather abstract model now known as the "kicked rotator," which, in spite of its unphysical character, conveys the essentials of the present day understanding of the effect of quantization on classical chaotic motion [4], [11]. We have here one more illustration of how useful the investigation of properly constructed abstract models may prove as an approach to more physical and complicated problems.

It is important to remark that the main impulse to the development of the present theoretical frame was not given by new laboratory experiments but by numerical simulation on super computers. Obvious limitations of computer performances enforced the use of simplified models. The first of these was a one-dimensional model, which yields an approximate description of the excitation of atoms initially prepared in very extended states. However simplified, this 1-D model turned out to reproduce the *essential qualitative features* of the real problem, as was made apparent by subsequent numerical simulations of more complete models. According to the picture emerging from the 1-D simulations, the quantum excitation process is determined by a sort of competition be-

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¹The highest level observed in space up to now is near 732 [1].

tween the diffusion that would be predicted on classical grounds and the "localizing" effect of quantum interference. The latter tends to arrest the diffusive broadening of the wave packet at a maximum spread ("localization length") on the order of the classical local diffusion coefficient. Since the latter sharply increases with the level number, it may happen that diffusion does never actually come to an end, with a consequent enhancement of the ionization rate.

The quantitative conditions for this diffusive ionization are determined by the classical chaotic border—below which diffusion cannot even start—and by the *delocalization border* that defines the critical value above which the increase of the classical diffusion coefficient triggers an unending escape of the wave packet, in spite of the paralyzing effect of quantum interference. Within the "window" of field strength defined by these two critical values, the classical diffusion is "frozen" by quantum effects and a quasi-stationary distribution over the unperturbed levels is reached.

This is, in summary, the theory presented in [2], [3], and [12]. An unsatisfactory aspect of that theory was its incapability of accounting for the form of the localized distribution. In particular numerical experiments, [2] and [3] show that in the tail of the steady-state distribution a chain of equidistant multiphoton peaks appears, which could not be described by that theory. For this reason, it was not possible to estimate ionization rates.

Recently, we developed an alternative theoretical approach [13], [14] which confirms the older results and moreover allows for a gross description of the whole distribution, including its peak structure. This approach was based on the construction of a map describing the evolution in one orbital period. The explicit form of this map establishes a close connection with the rotator problem and leads us to predict that the localized distribution should be exponential in the number of absorbed photons. This allows for a new, simple interpretation of the delocalization phenomenon: delocalization takes place when the number of photons within the localized distribution becomes comparable to the number of photons required for ionization.

The strategy of reducing the dynamics to a suitable mapping proved very fruitful also in the analysis of a 2-D model. In that case, we found that, due to Coulomb degeneracy, the 2-D excitation in energy proceeds in a quite similar way as in the 1-D model [15]. This result suggests the important conclusion that the localization phenomenon, originally predicted for the 1-D model, is not a peculiarity of that simplified model, but is typical in highly excited atoms.

However, this phenomenon has not yet been experimentally observed. The reason is that available experimental data refer to the region $\omega_0 = \omega n_0^3 < 1$ where, according to our theory [2], [3], there is either stable (even classically) behavior, or diffusive excitation (delocalization). In other words, there is no "localization window" in this region.

In this paper, we illustrate the general theory of photonic localization and its relationship to the mapping description of the dynamics. We shall support this theory with a large amount of numerical data, whose agreement with the theoretical predictions yields solid grounds for the predicted phenomena, and makes the call for experimental verification more urgent. It is worth mentioning in connection with the possibility of laboratory experiments that, owing to the general nature of the localization phenomenon and in view of some numerical simulations of ours in which a quantum defect was introduced, localization should be observable also in Rydberg alkali atoms.

In this paper, we will use atomic units. To facilitate conversion to physical units, we recall that for $n_0 = 100$ the frequency $\nu = \omega/2\pi = 10$ GHz corresponds to $\omega_0 = \omega n_0^3 = 1.51998$ and $\epsilon_0 = \epsilon n_0^4 = 0.1$ corresponds to $\epsilon = 5.14485$ V/cm. According to this choice of units, $\hbar = 1$ throughout the following sections.

II. QUANTUM LOCALIZATION OF DIFFUSIVE EXCITATION

The onset of chaos in a classical system subject to an external time periodic perturbation triggers a process of diffusive excitation. Instead, in quantum systems, this process is usually inhibited by a phenomenon known as quantum localization of diffusion. In this section, we shall analyze this phenomenon in some generality.

Let us consider the excitation process which is induced by an external monochromatic perturbation on a quantum system with a density $\rho(E)$ of unperturbed levels (*E* is the energy). Under appropriate conditions, the initial stage of this excitation process will be characterized by a diffusive growth of the energy. This phenomenon can be reasonably expected at least in two cases.

1) When the *unperturbed* classical motion is chaotic (e.g., a typical strongly nonlinear system). In this case, the perturbation matrix can be assimilated to a random matrix. This may lead to a diffusive growth of energy with time, at least on some initial time scale, provided that some additional conditions are fulfilled. First of all, the perturbation strength V should be larger than the average level spacing, i.e., the inequality $\rho V > 1$, which defines the *quantum stability border* [16], should be larger than the average level spacing ($\omega \rho > 1$); otherwise, a picture in which levels move adiabatically in time would apply, leaving no room for any excitation.

A model yielding diffusive excitation according to the above sketched mechanism was introduced and analyzed by Akulin and Dykhne [17] in order to describe collisionless molecular dissociation in a monochromatic (laser) field. These authors obtained the diffusion rate for the case $1 \ll \rho V \ll \rho \omega$; however, they did not investigate the excitation process for large times, which will be our main concern in the following.

2) If the unperturbed classical system is an integrable one, a diffusive growth of energy can still be started when the perturbation is strong enough to produce a stochastic transition in the classical motion. Notice that the condition $\rho V >> 1$ does not by itself enforce any diffusion; it just means that many neighboring levels will be strongly coupled by the perturbation. Even so, however, invariant curves may exist in the classical phase space, and these in the semiclassical case will restrict also the quantum excitation to narrow energy intervals. In order that diffusion may occur, classical invariant curves must be destroyed. Overlapping of classical resonances [18] yields a quantitative criterion for this destruction. The threshold defined by this criterion is in principle absolutely distinct from the quantum stability border discussed above. Strong resonance overlapping leads to purely chaotic classical motion; this supplies the randomness which is needed for diffusion and which in the previous case 1) was provided by the random character of the matrix elements.

We shall therefore assume that a sort of diffusion has been started, for one of the above illustrated reasons. The really important question is then whether this diffusion will continue indefinitely or not. The answer is in general negative: indeed, quantum interference effects will lead diffusion to a halt after some finite time τ_D [13], [19]. This important phenomenon was first detected in the kicked rotator model [4], [5]; a simple heuristic analysis was then developed, which allowed for an estimate of τ_D . The same analysis can in principle be applied to widely different models.

In order to illustrate this estimate, we shall first assume that the diffusion involves all levels around the initially excited one, with a diffusion rate $D = \langle (\Delta n)^2 \rangle / \tau$ where Δn is the number of excited levels at time τ . This assumption is satisfied when $V \ge \omega$. In the opposite case, the diffusive excitation would proceed via one-photon jumps between neatly distinguished resonant zones. The argument that we shall presently explain would in that case require some modifications that will be discussed later.

Let us consider a wave packet initially concentrated on a given unperturbed level and follow its evolution in the discrete time defined by the number of periods of the external perturbation. The character of this evolution will be determined by the nature of the quasi-energy spectrum, which is a subset of $(0, 2\pi)$ [41]. If this spectrum is pure point, so that only a finite number of quasi-energies can be assumed to significantly contribute in the wave packet dynamics, then the evolution will have a recurrent character, with a recurrence time on the order of $\Delta \sigma^{-1}$, i.e., of the inverse of the average spacing of quasi-energies. Therefore, even the excitation determined by a pure point quasi-energy spectrum may appear diffusive on a short time scale $< \Delta \sigma^{-1}$; nevertheless, a continuous QE spectrum is required for an unlimited diffusion.

If we assume the number of QE significantly contributing in the motion up to time τ to be on the same order as the number $\Delta n(\tau)$ of levels excited up to τ , $\Delta n(\tau) \sim (D\tau)^{1/2}$, then we can write $\Delta \sigma \sim (\Delta n)^{-1}$ (notice that this estimate would be wrong in the presence of approximate resonances, because then some quasi-energies would be very close to each other; indeed, in the case $V < \omega$, it will be suitably modified). This yields the conditions that must be satisfied in order that no recurrence may have appeared up to time $\tau: \tau < \Delta n(\tau)$. By using the diffusive estimate for Δn , we see that this condition is violated for $\tau \ge \tau_D$, with $\tau_D \sim D$. For such times, recurrences will become manifest and this will lead to *localization* of diffusion. Instead of spreading indefinitely, the distribution over unperturbed levels will enter a steady-state oscillatory regime; its "width" will henceforth assume a constant value *l*—the *localization length*—that can be estimated by the value of Δn at time τ_D , i.e., by

$$l \sim \Delta n \sim \tau_D \approx \alpha D. \tag{1}$$

The numerical factor $\alpha \sim 1$ has been introduced in order to turn the previous estimates of orders of magnitude into more precise ones; it should be adjusted by numerical experiments.

The above estimate for the localization length is correct if D does not depend on the level number n (homogeneous diffusion) or at least if D does not change appreciably on the scale of l defined by (1). Otherwise, the formula $\Delta n \sim (D\tau)^{1/2}$ would not hold. In the case of homogeneous diffusion, some definite prediction can be made about the form of the steady-state distribution; indeed, the (timeaveraged) population of unperturbed levels should be given by

$$\bar{f}_n \propto \exp\left\{-2\left|n - n_0\right|/l\right\}$$
(2)

where n_0 is the initially excited level.²

The exponential form of the stationary distribution (2) can be explained by a general argument. Suppose we wish to determine the QE eigenfunction associated with a QE eigenvalue λ . If we expand the unknown eigenfunction in unperturbed eigenfunctions with amplitudes u_n , then we must solve an infinite linear system of equations of the form

$$u_n = \sum_{m=-\infty}^{+\infty} V_{nm}(\lambda) u_m.$$
(3)

The external perturbation does effectively excite—in one period—only a finite number M of unperturbed levels: this implies fast decay of the matrix elements V_{nm} for |n - m| > M. Therefore, with a good accuracy, we can replace the system (3) with

$$\begin{vmatrix} u_{n+2M} \\ \vdots \\ u_{n+1} \end{vmatrix} = M_n(\lambda) \begin{vmatrix} u_{n+2M-1} \\ \vdots \\ u_n \end{vmatrix}$$
(4)

where M is a $2M \times 2M$ matrix depending on n. In many cases of interest, each M_n matrix is a canonical one. It is then a known fact [20] that the asymptotic behavior of solutions of (4) is determined by the minimal Lyapunov exponent associated with the infinite sequence of matrices

²A more precise formula will be used in Section V.

 M_n ; if this exponent is positive, then physically acceptable (i.e., nonexponentially diverging) solutions of (4) must decay exponentially for $n \rightarrow \pm \infty$, and the localization length of these solutions (QE eigenfunctions) will be given by the inverse of this Lyapunov exponent. Positivity of the minimal Lyapunov exponent depends in general on how the matrix M_n depends on n. In localization theory, it is proven that, if the sequence M_n is a stationary random sequence of matrices, then the minimal LE is positive; in general, however, the same can be expected even when M_n is a sufficiently irregular sequence, stationary in a statistical sense. For example, this has been shown numerically to happen in the kicked rotator case. Notice that the identification of the inverse localization length with the minimal LE provides an efficient way for numerically determining the localization length itself [19]-[21].

Numerical results obtained in this way and comparison to exactly solvable models indicate that the factor α in (1) should be given the value 1/2 in order to give the correct localization length for *QE eigenfunctions* [11].

The exponential localization of QE eigenfunctions entails exponential decay of the steady-state distribution also. However, according to numerical results, in order that (1) may yield the correct localization length for the distribution, the value $\alpha = 1$ should be used. This is due to strong fluctuations of the actual QE eigenfunctions around the average exponential law [10], [19].

The localization phenomenon just discussed is a dynamical version of the well-known Anderson localization. Indeed on a formal level (4) may be interpreted as the stationary Schroedinger equation for a particle on a lattice, in which *n* numerates the lattice sites. In the Anderson model, a random potential is present, thanks to which the matrices M_n make up a truly random sequence—which, as already hinted, is enough to guarantee positivity of the minimal Lyapunov exponent and hence exponential localization of eigenstates. The formal analogy between localization in periodically perturbed systems and Anderson localization was pointed out in [7]. We wish to stress, however, that in our case no random element is introduced from outside.

Equation (1) provides a quantitative estimate for the localization length, once the diffusion rate D is known. In order to determine D, we proceed as follows. The change Δn produced by a one-photon transition is $\Delta n = \rho \omega$. Then, if W is the one-photon transition rate, we get

$$D = \langle (\Delta n)^2 \rangle / \tau = 2\rho^2 \omega^2 W 2\pi / \omega$$
 (5)

where the factor 2 accounts for transitions in both directions.

According to Fermi's Golden Rule, $W = (\pi/2) \epsilon^2 \rho |\mu(E, E + \omega)|^2$ where μ is the dipole matrix element and E is field strength. Now (1) becomes

$$l = 2\pi^2 \mu^2 \epsilon^2 \rho^3 \omega \tag{6}$$

l is here the local length in number of unperturbed levels. For reasons that will be explained below, it is more convenient to express the localization length in the number of absorbed photons:

$$l_{\phi} = l/\rho\omega = 2\pi^2 \mu^2 \epsilon^2 \rho^2 = 2\pi D_{\phi} \rho \qquad (7)$$

where $D_{\phi} = \pi \mu^2 \epsilon^2 \rho$ is the diffusion rate in the number of photons, i.e., the squared number of absorbed photons per unit time.

We should at this point recall that (7) was derived under the assumptions $V\rho = \mu\epsilon\rho >> 1$ and $V > \omega$. The meaning of the first assumption is now clear: it implies $l_{\phi} >>$ 1 and therefore corresponds to a quasi-classical regime in which a large number of photons is absorbed and a large number of levels is excited. The second assumption, which reads $\mu \epsilon > \omega$, may now be clarified. Indeed, $\mu \epsilon$ is an estimate for the critical detuning from the resonant Rabi frequency. Then $\mu\epsilon > \omega$ means that the level width associated with one-photon transitions is larger than the spacing of the levels involved in such transitions. Since our final result (7) for l_{ϕ} does not depend on the external frequency ω , we may suspect that (7) holds even for $\mu\epsilon$ $< \omega$. Indeed, even though the argument leading to (1) does not work when $\mu\epsilon < \omega$, and the result (6) for the localization length in the number of levels is no more correct in that case, we shall now show how that argument can be modified so as to work also when $\mu \epsilon < \omega$, yielding again the result (7) for the localization length in the number of absorbed photons.

The number N of photons absorbed at time t grows diffusively: $N \sim (D_{\phi}t)^{1/2}$. However, since $\mu \epsilon < \omega$, only levels close to resonant ones will now be excited; therefore, the excitation will not spread over all levels around the initially excited one, as we previously assumed; instead, it will concentrate on a chain of equidistant (in energy) narrow excited zones of width $\Delta \nu \ll \omega$. Thus, the number of levels excited at time t will now be $\Delta n \approx$ $N\rho\Delta\nu$. In order to estimate t_D —i.e., the time after which diffusion stops—we need the spacing $\Delta \sigma$ of QE eigenvalues, because then $t_D \sim 1/\Delta\sigma$. We are now using real time, and not, as before, the number of periods; therefore, these eigenvalues are to be taken mod ω . Exactly resonant energy values would then correspond to the same unperturbed OE eigenvalue. Since all excited levels lie in zones of width Δv around resonant values. OE levels can be assumed to lie within a single band of width $\Delta \nu$ in (0, ω). Therefore, their average spacing will be $\Delta \sigma \sim \Delta \nu / \Delta n$ ~ $1/N\rho$. This leads to $t_D \sim D_{\phi}\rho^2 = \pi \mu^2 \epsilon^2 \rho^3$ and $l_{\phi} \sim N \sim (D_{\phi} t_D)^{1/2} \sim 2\pi^2 \mu^2 \epsilon^2 \rho^2$ which is the same as (7).

In deriving this result, we estimated $\Delta \nu$ by the Rabi frequency $\mu\epsilon$, which characterizes the critical detuning from resonance. It might appear more natural to estimate $\Delta \nu$ by the one-photon transition rate $\Gamma: \Delta \nu \sim \Gamma \sim \mu^2 \epsilon^2 \rho$ [17]. This would be indeed the correct result when considering transitions from one level into a single zone. However, in the presence of many resonance zones, the average rate of transition $\overline{\Gamma}$ must be renormalized as shown in [22] and yields $\overline{\Gamma} = (\Gamma/\rho)^{1/2} = \mu\epsilon$. Anyway, the above expression for l_{ϕ} does not depend on $\Delta \nu$ and is valid also for $\mu\epsilon < \omega$. In this case, the distribution over unperturbed levels will display a chain of equidistant (in energy) peaks. Moreover, if l_{ϕ} does not depend on energy, then exponential localization *in the number of photons* must be expected in the sense that the probability enclosed in the *N*th peak would be approximately $f_N \sim \exp[-2|N - N_0|/l_{\phi}]$.

An important case in which this situation occurs is provided by the H atom problem that we are now going to discuss.

III. CLASSICAL DYNAMICS OF A ONE-DIMENSIONAL Hydrogen Atom in a Monochromatic Field: The Kepler Map

We shall now study the diffusive excitation occurring in a classical hydrogen atom in a monochromatic, linearly polarized, electric field. We shall first discuss a simplified one-dimensional model, in which the electron moves along a straight line in the direction of the field. Besides greatly simplifying the theoretical and numerical work, this model was also found to correctly describe the excitation of true, i.e., three-dimensional, H atoms initially prepared in very extended states along the field direction [3], [23]. This fact was actually of crucial importance since the study of the 1-D model has led to a clear physical picture of the excitation process of the H atom in a monochromatic field. Moreover, since such extended states (with parabolic quantum numbers $n_1 >> n_2$ and magnetic quantum number m = 0) can be produced in laboratory experiments [24] the one-dimensional theory can be subjected to experimental tests.

The one-dimensional Hamiltonian in atomic units is

$$H = p^{2}/2 - 1/x + \epsilon x \cos \omega t \quad (x > 0) \quad (8)$$

where ϵ and ω are the field strength and frequency.

The classical dynamics described by (8) were investigated in several papers [3], [25]–[30]. A theoretical analysis based on the Chirikov criterion [18] shows that for $\omega_0 = \omega n_0^3 > 1$ and for field strength above a critical value $\epsilon_c = 1/(50\omega_0^{1/3})$, the electron enters a chaotic regime of motion, marked by unlimited diffusion leading to ionization. Numerical analysis confirms these estimates and provides empirical evidence that the diffusion is ruled by a Fokker–Planck law [3], [31]. For $\omega_0 < 1$, numerical results show that the critical field for the chaotic transition approaches the threshold value for ionization in a static field when $\omega_0 \rightarrow 0$.

Here we shall describe a new approach to the classical problem, which allows for a very simple theoretical and numerical analysis and which exposes in a transparent way the connection between the H atom problem and the kicked rotator. In subsequent sections, we will show that this approach leads to a simplified quantal description, and also how the same approach can be generalized to a number of dimensions higher than 1.

The idea is to describe the dynamics by a map rather than by the Hamilton equations. This *map* gives the change of the appropriate dynamical variables over one orbital period of the electron, and, as we shall see, it can be given a simple analytical form, thanks to an important peculiarity of the problem. Indeed, it will appear that the external field fully develops its perturbing influence on the free Keplerian motion of the electron mostly when the electron itself is in the vicinity of the perihelion. This is a consequence of the Coulomb singularity and leads to a kick-like influence of the external perturbation.

In order to obtain the map, we rewrite (8) in actionangle variables (n, θ) :

$$H = -1/2n^{2} + \epsilon n^{2}$$

$$\cdot \cos \omega t \left[3/2 - 2 \sum_{s=1}^{\infty} J'_{s}(s) s^{-1} \cos s\theta \right] \quad (9)$$

where J'_s are the derivatives of Bessel functions. For large $s, s^{-1}J'_s(s) \approx 0.411s^{-5/3}$ which even for s = 1 is correct within 20 percent. We now introduce the "eccentric anomaly" ξ , according to

$$t = n^{3}(\xi - \sin \xi)$$
$$x = n^{2}(1 - \cos \xi)$$
$$\theta = \xi - \sin \xi$$

and also define a new time η , in which the unperturbed ξ motion is uniform. Then we get the following set of equations

$$dn/d\eta = -\epsilon n^{2} \sin \xi \cos \omega t$$

$$d\xi/d\eta = n^{-3} + 2\epsilon n(\cos \omega t)(1 - \cos \xi)$$

$$dt/d\eta = 1 - \cos \xi.$$
 (10)

We wish to use these equations for the purpose of evaluating the change Δn in action between two subsequent passages at the aphelion ($\xi = \pi$). Since we want to perform this evaluation at first order in ϵ , we shall neglect ϵ in the second and third equations (10).

By integrating the equations thus obtained, we get

$$\eta = \xi n^3 + \pi n^3$$
$$\omega t = \omega n^3 (\xi - \sin \xi) + \phi$$

The integration was started with $\eta = 0$, $\xi = -\pi$, i.e., at the aphelion, at a time t_0 ; therefore, $\phi = \omega (t_0 + \pi n^3)$, that is, ϕ is the field phase at the perihelion.

We now substitute this result into the first equation (10). The next passage at the aphelion after t_0 will occur for $\xi = \pi$, so that

$$\Delta n = -\epsilon n^5 \int_{-\pi}^{\pi} \cos\left(\chi(\xi - \sin\xi) + \phi\right) \sin\xi \, d\xi$$
$$= 2\pi\epsilon n^5 J'_{\chi}(\chi) \sin\phi \qquad (\chi = \omega n^3) \qquad (11)$$

where

$$J'_{\chi}(\chi) = 1/2\pi \int_{-\pi}^{+\pi} \sin \left(\chi(\xi - \sin \xi)\right) \sin \xi \, d\xi$$

is the derivative of the so-called Anger function [34].

It is convenient to introduce a variable $N = E/\omega = -1/2n^2\omega$. In the quantum case, the change of N would give the number of absorbed photons. The change in N



Fig. 1. Dependence of the kick amplitude A on $\chi = \omega n^3$.

corresponding to the change in n given by (11) is

$$\Delta N = \Delta n / (n^3 \omega) = (2\pi \epsilon n^2 / \omega) J'_{\chi}(\chi) \sin \phi. \quad (12)$$

For integer values of χ , the Anger functions coincide with ordinary Bessel functions, and for $\chi \rightarrow \infty$ they have the same asymptotic behavior. That is, on defining

$$A(\chi) = (\chi^{2/3}/0.411) J'_{\chi}(\chi),$$

we have $A(\chi) \approx 1$ for $\chi \gg 1$. (See the above given asymptotics of $J'_s(s)$.) Instead, for $\chi \to 0$, $J'_{\chi}(\chi) \sim \chi/2$; the behavior of $A(\chi)$ is illustrated in Fig. 1. We can rewrite (12) as

$$\Delta N = kA(\chi) \sin \phi \qquad k = 0.822\pi\epsilon\omega^{-5/3}.$$
 (13)

A stationary-phase analysis of (11) shows that for large χ , the main change in action occurs within a small interval $\Delta \xi \sim \chi^{-1/3} \ll 1$ near the stationary phase point $\xi = 0$. Therefore, as we anticipated above, for large χ the monochromatic perturbation is mainly effective when the electron is very close to the perihelion.

Now we shall regard N, ϕ as a pair of canonically conjugate variables and seek for a canonical map connecting the values of N, ϕ at consecutive passages at the aphelion. The above developed perturbation theory yields (13) for the change in N at first order and $\Delta \phi = 2\pi \omega (-2\omega \overline{N})^{-3/2}$ for the change in ϕ at zero order. Following a standard procedure [35], we can now look for a generating function $G(\overline{N}, \phi)$ such that the map defined by

$$N = \partial G / \partial \phi \qquad \phi = \partial G / \partial \overline{N}$$

coincides at first order and zero order, respectively, with our perturbative result.

This function is

$$G(\overline{N}, \phi) = \overline{N}\phi + 2\pi (-2\omega\overline{N})^{-1/2} + kA(\overline{\chi})\cos\phi$$
(14)

with $\overline{\chi} = \omega (-2\omega \overline{N})^{-3/2}$. It generates the following map

$$\overline{N} = N + kA(\overline{\chi}) \sin \phi$$

$$\overline{\phi} = \phi + 2\pi\omega(-2\omega\overline{N})^{-3/2}$$

$$+ 3k\omega^2(-2\omega\overline{N})^{-5/2} A'(\overline{\chi}) \cos \phi. \quad (15)$$

Notice that the implicit character of (15) cannot be avoided if a canonical (area preserving) map is required. For the same reason, the second equation (15) contains a

first-order correction to the above perturbative result for ΔN , $\Delta \phi$.

Since $A(\chi) \sim 1$ for $\chi \gg 1$, the map (15) is greatly simplified in the region of large χ , i.e., of large ω_{0^-} where it takes the form of the following "Kepler map:"

$$\overline{N} = N + k \sin \phi$$

$$\overline{\phi} = \phi + 2\pi\omega (-2\omega\overline{N})^{-3/2}.$$
(16)

On the other hand as can be seen from Fig. 1, $A(\chi)$ is already close to 1 for $\chi = 1$; therefore, the map (16) provides an acceptable description of the motion for $\omega_0 \ge 1$. Notice that even though the map (16) is canonical, hence area preserving, it is not defined on all bound states (N < 0); indeed, it carries some bound states into the positive energy region, where ϕ is no longer defined. When this happens, the electron escapes to infinity and ionization occurs.

The effect of a small static field ϵ_s superimposed to the monochromatic field would essentially be a change in the Kepler period. Then for small ϵ_s only the second equation (16) should be modified by adding a term

$$-6\pi\omega\epsilon_{s}(-2\omega\overline{N})^{-7/2}$$

We obtained a numerical check of the validity of the mapping (16) as an approximate description of the dynamics, by the following procedure. By numerically solving the exact equations of motion (10), we computed the sequence $\phi_j = \omega t_j$ of the phases at passages of the electron at the perihelion. Next, by computing $N_j = (-2\omega)^{-1}[(\phi_j - \phi_{j-1})/(2\pi\omega)]^{-2/3}$ we found $g(\phi_j) = k^{-1}[N_j - N_{j+1}]$ and we plotted this against ϕ_j for several values of k. The comparison of the result to the theoretical prediction $k^{-1}[N_{j+1} - N_j] = \sin \phi_j$ is given in Fig. 2; even for ω_0 = 1.5, the agreement between numerical and theoretical data is very good.

IV. DIFFUSIVE EXCITATION AND IONIZATION IN THE 1-D CLASSICAL MODEL

The mapping (16) allows for a straightforward estimate of the critical field value required for the transition to chaotic motion. By linearizing the second equation (16), we obtain the map

$$\overline{N} = N + k \sin \phi$$

$$\overline{\phi} = \phi + T\overline{N}$$
(17)

where $T = 6\pi\omega^2 n_0^5$ (an unessential constant has been dropped in the second equation). Equation (17) is the celebrated standard map. As is well known, the onset of stochasticity for this map occurs when the stochasticity parameter K = kT becomes larger than 1.

Then, defining ϵ_c by $K = \epsilon_0 / \epsilon_c$, we obtain the condition for unlimited chaotic excitation in the form $\epsilon_0 > \epsilon_c$, with ϵ_c given by

$$\epsilon_c \approx 1/(49\omega_0^{1/3}). \tag{18}$$

This estimate follows from the simplified map (17) which is a good approximation to the true dynamics when $\omega_0 >$



Fig. 2. Numerically computed function $g(\phi_j)$ (dots) compared to the theoretical curve sin ϕ (full curve) for the case $\epsilon_0 = \epsilon n_0^4 = 0.04$, $\omega_0 = \omega n_0^3 = 1.5$

1. Instead, for $\omega_0 < 1$ we must use the complete map (15). By linearization, we get again a standard map like (17) with kA in place of k (the first-order correction in the second equation (16) is small and can be neglected). This leads to the new estimate

$$\epsilon_c \approx 1/(49\omega_0^{1/3}A(\omega_0)). \tag{18a}$$

For example, for $\omega_0 = 0.6$, $A(\omega_0) \sim 0.45$ so that (18a) yields $\epsilon_c \approx 0.05$ in satisfactory agreement with the numerically obtained value. (See [3, Fig. 5].) For much lower values of ω_0 , the critical field becomes too large and the mapping description of the dynamics becomes inadequate. For such very small frequencies, the critical field approaches the threshold value for a static field: $\epsilon_s \approx 0.13$.

An example of a phase-space picture obtained by iterating (16) is shown in Fig. 3, which exhibits the typical structure of area-preserving maps. The islands around $\phi = \pi$ correspond to main resonances, whose position in N is determined by the condition $2\pi s = 2\pi\omega(-2\omega N)^{-3/2}$. The scattered points in the figure belong to a single chaotic trajectory.

During the chaotic excitation process described by (16), the stochasticity parameter K grows. The phases ϕ take random and independent values and the trajectory in N becomes similar to a random walk, with the diffusion rate

$$D = \langle \Delta N \rangle^2 \rangle / \Delta t = k^2 / 2 \approx 3.33 \epsilon^2 / \omega^{10/3}$$
 (19)

where t is measured in number of iterations.

As we already mentioned, ionization will occur as soon as a kick carries the orbit into the positive energy region. Then, typically, the orbit will proceed to infinite in physical space and never return back to the nucleus. This is due to the fact that far from the nucleus, the external field produces only small oscillations of the trajectory around its average Kepler motion (ellipse or hyperbola). (From this picture, we may conclude that transitions in the continuum and from continuum into bound states are negligible small. This is the reason why our initial quantum computations [2], [23] carried out in the discrete basis give a correct description of the excitation process.)

Since the change in energy due to a single kick is $\leq \omega k$, the energy of the ionized electron will lie in the band 0 $< E < 0.822\pi\epsilon/\omega^{2/3}$. From this result, we can draw the following conclusion: if a beam of electrons passes close to the nucleus in the presence of a monochromatic field, only electrons with energy smaller than $0.822\pi\epsilon\omega^{-2/3}$ can



Fig. 3. Phase space portrait for the map (16) in the variables $E_0 = \omega N n_0^2 = -n_0^2/2 n^2$, ϕ . Parameter values are $\epsilon_0 = 0.03$, $\omega_0 = 3.5$. Six regular and one chaotic trajectory are shown.

be captured. (For a more detailed discussion see Section IX.)

Also, in order that ionization may occur after just one kick (i.e., after one orbital period) it is necessary that $k > N_I = 1/(2\omega n_0^2)$ (in quantum terms, N_I is the number of photons required for ionization) which gives $\epsilon_0 > \omega_0^{2/3}/5$. For example, if $\omega_0 = 10$, then $\epsilon_0 \approx 0.93 \gg \epsilon_s$. Therefore, in a monochromatic field with $\epsilon_0 \approx \epsilon_s$, ionization may be negligible even after several Kepler periods (see also below).

The time t_l required for ionization can be estimated as the time required to reach the region N > -k, from where the electron is thrown into the positive energy region by just one kick.

From (19) we obtain

$$t_I \sim N_I^2/D \approx (1/13)\omega_0^{4/3}/\epsilon_0^2.$$
 (20)

Consider an ensemble of points initially concentrated on a given value of N with uniformly distributed phases. At a later time $t \ll t_l$, this ensemble will be distributed in N according to a Gaussian law

$$f(N, t) = (\pi k^2 t)^{-1/2} \exp\left[-(N - N_0)^2/k^2 t\right].$$
 (21)

In order to correctly interpretate this result, we must recall that "time" t is here the number of mapping iterations, i.e., the number of orbital periods. In real time, the diffusion coefficient depends on the value of the action and in order to find how the distribution in action space evolves in time, an appropriate Fokker-Planck equation should be solved, as in [3] and [31]. For the same reason, in order to evaluate the physical time required for ionization, a conversion of (20) must be made, from the number of iterations to the number of field periods.

In strictly mathematical terms, this average ionization

time diverges even though every individual stochastic orbit ionizes in a finite time. In fact, the average waiting time before the last ionizing kick is $\int_0^{N_0} (-2\omega N)^{-3/2} f(N)$ dN which is infinite, because f(N) does not vanish at the ionization border. From a physical viewpoint, however, it is sufficient to consider the time needed to reach states very close to the ionization border, because these states will be ionized by any small perturbation. Therefore, the divergent integral above should not be started from zero, but rather from some value to be chosen on the grounds of physical considerations. One possible choice may be to compute the ionization time as the time needed for changing N by a factor two. In this way, one gets a ionization time

$$\tau_I \sim \omega_0 t_I \sim \omega_0^{7/3} / \epsilon_0^2 \tag{22}$$

which coincides with a previously given estimate [3].

V. QUANTIZATION OF THE ONE-DIMENSIONAL MODEL; EXPONENTIAL PHOTONIC LOCALIZATION

The Kepler map formulation of the H atom dynamics discussed in the previous sections predicts diffusive excitation above the chaotic threshold. We now turn to the question of what kind of behavior would be predicted by quantum dynamics for the same 1-D model. According to our general discussion in Section II, we should expect also here the important phenomenon of quantum localization which, indeed, was theoretically predicted and numerically confirmed in [2], [3], [12], [23] and [31].

The analysis developed in these papers provided a clear general picture of the quantum mechanism of excitation, but was not able to explain the multiphoton peak structures which appear on high levels according to numerical experiments and could not therefore provide estimates for the quantum ionization rate.

Let us now see how does the 1-D H atom problem fit into the general picture of localization discussed in Section II. In order to use the estimate (7) for the localization length, we need the expression for dipole matrix elements and for the level density ρ . Semiclassical values of the dipole matrix elements are given by the classical Fourier amplitudes of the perturbation [32]. The latter can be read directly from (9), so that the semiclassical formula sought for is

$$\mu = n^2 s^{-1} J'_s(s) \approx 0.411 n^{-3} \omega^{-5/3}$$
(23)

Since we are interested in matrix elements for one-photon transitions, we have put $s = \omega n^3$; moreover, we have used the asymptotic formula already introduced in Section III. The above expression (23) coincides with the one given in [3]. Substituting [23] and $\rho = n^3$ into (7), we obtain the localization length in the number of absorbed photons

$$l_{\phi} = 3.33\epsilon^2 \omega^{-10/3}.$$
 (24)

Our general discussion (Section II) also leads us to predict the shape of the distribution on levels. Since according to (24), the localization is homogeneous, the distribution should be exponential in the number of absorbed photons. However, on high levels the perturbation strength $V \sim \epsilon \omega^{-5/3} n^{-3}$ becomes much smaller than ω ; therefore, the excitation there will mainly proceed via resonant one-photon transitions and a corresponding peak structure will appear.

An independent derivation of this exponential photonic localization can be obtained by direct quantization of the Kepler map (16). The classical variable N is energy divided on ω ; on the other hand, $\phi = \omega t$ varies from $-\infty$ to $+\infty$, so we can represent quantum mechanically N and ϕ by the canonical pair of operators $\hat{N} = -id/d\phi$, $\hat{\phi} =$ $\phi(-\infty < \phi < +\infty)$. Then replacing N, ϕ in (16) by \hat{N} , $\hat{\phi}$, we get the quantum map that describes the discretetime quantum evolution corresponding to the classical Kepler map (16) in the Heisenberg description. In order to get the quantum map in the Schroedinger description, we first notice that (16) can be factorized as the product of two simpler maps. The first is a "kick" which changes N but not ϕ , followed by a "free" evolution that would be produced by the Hamiltonian $H_0 = 2\pi [-2\omega N]^{-1/2}$ in the unit time.

According to the chosen representation for the operators \hat{N} , $\hat{\phi}$, we can now write the quantum map for wave functions $\psi(\varphi)(-\infty < \varphi < +\infty)$ in the following form:

$$\overline{\psi} = e^{-i\hat{H}_0}\hat{P}e^{-ik\cos\phi}\psi \qquad (25)$$

 \hat{p} is the projection operator on the subspace in which \hat{N} is negative, (the subspace spanned by bound states), i.e., on the subspace where e^{-iH_0} is defined. The map (25) can be significantly simplified by properly exploiting the periodicity in ϕ . Due to this periodicity, the map commutes with translations of ϕ by multiples of 2π , that is it commutes with $e^{-2\pi i \hat{N}}$. This entails that the "fractional part" of \hat{N} is unchanged under the action of the map and is therefore an integral of the discrete-time motion defined by the map itself; it may be called the "quasi-momentum" associated with the coordinate ϕ . If the value of the quasi-momentum is known, the map can be simplified as follows. We first rewrite (25) in the \hat{N} representation, i.e., we write the map for the Fourier transform $\hat{\psi}(\nu)$ of $\psi(\phi)$:

$$\overline{\hat{\psi}} = e^{-2\pi i [-2\omega\nu]^{-1/2}} \theta(-\nu) \int_{-\infty}^{+\infty} d\nu' \, \hat{\psi}(\nu') \, \overline{K}(\nu - \nu').$$
(26)

 θ is the unit step function and represents the projection \hat{P} ; \overline{K} is the Fourier transform of $e^{-ik\cos\phi}$, and has the form

$$\overline{K}(\nu) = (2\pi)^{-1} \sum_{m} g_{m} \delta(\nu - m)$$

where g_m are the Fourier amplitudes for $e^{-ik\cos\phi}$ in [0, 2π].

Substituting this into (26) and writing $\nu = N_0 + N_{\phi}$ with $N_0 = -1/2n_0^2\omega$ and N_{ϕ} integer, we get

$$\overline{\hat{\psi}}(N_0 + N_{\phi}) = e^{-2\pi i [-2\omega(N_0 + N_{\phi})]^{-1/2}} \theta(-N_0 - N_{\phi})$$
$$\cdot \sum_m g_m \widehat{\psi}(N_0 + N_{\phi} - m).$$

We see that $\hat{\psi}$ at a given point λ is uniquely determined by $\hat{\psi}$ at points with the same N_0 , which shows again that N_0 is an integral of the motion. Therefore, if N_0 is known, then $\hat{\psi}$ is completely specified by its values at points N_0 + m, and the state is uniquely defined by a new wave function in $(0, 2\pi)$

$$\psi = \sum_{m} \hat{\psi}(N_0 + m) e^{im\theta}$$

The quantum map for such wave functions is

$$\overline{\psi} = e^{-iH_0} \hat{P} e^{ik\cos\theta} \psi \tag{27}$$

where now $\hat{H}_0 = 2\pi [-2\omega(N_0 + \hat{N}_{\phi})]^{-1/2}$, $\hat{N}_{\phi} = -i\partial/\partial\theta$ in $(0, 2\pi)$ with periodic boundary conditions, and \hat{P} is the projection on bound states $(\hat{N}_{\phi} < -N_0)$. Notice that, given the initially excited state n_0, N_0 is just equal to $-1/2n_0^2\omega = -n_0/2\omega_0$.

The quantum Kepler map (27) establishes a close connection between the hydrogen atom problem and the kicked rotator. We can define quasi-energies λ for this map by $\overline{\psi} = e^{i\lambda}\psi$ but, unlike the rotator case, λ will have a nonvanishing imaginary part describing absorption in the continuum. We remark that an equation for quasi-energy eigenfunctions close to $\overline{\psi} = e^{i\lambda}\psi$ [with $\overline{\psi}$ given by (27)] was derived from the Schroedinger equation by Bersons [36].

The existence of quasi-momentum suggests the interesting prediction, that the distribution of the (real parts) of the quasi-energy levels for the H-atom problem should not show any level repulsion, even in the region of strong classical chaos. To our knowledge, this would be the first physical example of absence of level repulsion in the region of classical chaos.

By iterating the quantum Kepler map (27), we obtain the distribution *in the number of absorbed photons*. This is a different kind of distribution than the usual distribution over the unperturbed levels that would be produced by integrating the Schroedinger equation. A link between the two types of distribution can be established by noticing that the absorption of the different number of photons generates different peaks in the distribution over the unperturbed levels. Therefore, for a given number of photons, we shall identify the probability obtained by means of the Kepler map, with the total probability over all unperturbed levels lying within a single one-photon interval around the peak determined by the absorption of the given number of photons.

Since the diffusion rate for the Kepler map is constant, $D = k^2/2$, then the photonic localization length $l_{\phi} = D$ $= k^2/2$ is likewise constant.

Therefore, the picture would be the same as in the kicked rotator model, were it not for the dissipation introduced by the projector \hat{P} . However, for $l_{\phi} \ll N_I$, the corresponding ionization rate is negligible so that homogeneous exponential localization is to be expected also in this case. In particular, a prediction about the form of the steady-state distribution can be borrowed from results obtained for the rotator model [10], according to which

$$\bar{f}_{N} \sim (1/2l_{\phi})(1+2|N-N_{0}|/l_{\phi})$$

$$\cdot \exp(-2|N-N_{0}|/l_{\phi}).$$
(28)

The value of l_{ϕ} just obtained is the same already obtained by a more general argument in Section II.

By multiplying l_{ϕ} by the number of unperturbed levels ωn^3 lying within a one-photon interval, we obtain the previously derived [2], [3] value of localization length ((11) of [3]), with a slight difference in the numerical factor. As explained in [3], this difference is due to the particular choice of a numerical factor in the classical diffusion coefficient which was made in [2]. Anyway, we wish to emphasize that the previous theory was able to justify the form of the steady distribution in a restricted neighborhood of the initially excited level. Instead, the above presented theory yields an approximate description for the overall distribution, including its peak structure.

If the photonic localization length is large enough, the peak structure will determine a sort of plateau in the distribution over unperturbed levels [2], [3]. Even more importantly, if l_{ϕ} is comparable to the number of photons required for ionization: $N_I = 1/2n_0^2\omega$, then strong ionization will occur. The condition $l_{\phi} \approx N_I$ yields a critical value of the field

$$\epsilon_q \approx \omega_0^{7/6} / (6.6n_0)^{1/2}.$$
 (29)

Across the "delocalization border" defined by (29), a qualitative change occurs, from a regime of localized quantum motion with very small ionization, to a diffusive ionization very similar to the classical one. This "delocalization phenomenon" has been predicted and described on somewhat different grounds in [2], [3] with the same estimate (29) for the delocalization border.

The simple form of the quantum Kepler map is very convenient for estimating ionization rates. The simplest case is that of single-photon ionization in a small field, which corresponds to $k \ll 1$.

From (25), we get immediately the loss of probability after one orbital period: $\gamma_{\phi} \approx (k/2)^2$. In order to obtain the ionization rate Γ_{ϕ} in physical time, we must divide this γ_{ϕ} by $2\pi n_{0}^3$, i.e., by the orbital period. Thus, we find

$$\Gamma_{\phi} = \gamma_{\phi} (2\pi n_0^3)^{-1} = 0.265 \epsilon^2 \omega^{-10/3} n_0^{-3} \qquad (30)$$

which is exactly the standard perturbation-theoretic result [3], [12].

In the localized regime, when $N_l > l_{\phi} > k > 1$, the probability that is removed to the continuum after one kick is approximately equal to the total probability on photonic levels with $N_l - k < N_{\phi} < N_l$. Therefore, the ionization rate (in discrete time, i.e., in number of iterations) is

$$\gamma_{\phi} \sim \sum_{N=N_I-k}^{N_I} \bar{f}_N \sim k \bar{f}_{N_I}$$

where f_N is the averaged distribution on photonic levels. In physical time, we obtain CASATI et al.: HYDROGEN ATOM IN MONOCHROMATIC FIELD

$$\Gamma_{\phi} \sim k \bar{f}_N (k \omega)^{3/2} \sim (\omega^{5/4} / \sqrt{n_0}) (\epsilon_q / \epsilon_0)^{3/2}$$

$$\cdot \exp\left[-2(\epsilon_q / \epsilon_0)^2\right]$$
(31)

where (28) has been used.

The tail of the distribution is very sensitive to small changes of parameters such as, e.g., the field strength. In the average, the distribution f_N will conserve the form (28) but its value at any fixed value of N will fluctuate strongly. Since for $\epsilon_0 \ll \epsilon_q$ the ionization rate is determined by an interval of width $\Delta N \sim k$ in the tail of the distribution [see (31)], and since this interval is smaller than the localization length, these fluctuations will affect also the ionization rate, which will have a very irregular fine structure as a function, e.g., of the field strength.

In the case $\epsilon_0 > \epsilon_q$, localization never takes place and the process of excitation is close to the classical one [3], [31]. The characteristic time of the decay of probability is determined by the time t_l needed to reach the continuum [see (20)].

Finally, from the quantum Kepler map, we can extract a prediction on the energy distribution of ionized electrons, when k >> 1. Indeed, using the expansion

$$e^{-ik\cos\hat{\phi}} = \sum_{N} (-i)^{N} J_{N}(k) e^{iN\phi}$$

we see that, after one kick, the probability that the electron has an energy $N\omega - 1/2n_0^2$ will be $P_N = |J_{N-[N_I]}(k)|^2$ where $[N_I]$ is the integer part of $N_I < N$; this is in agreement with results of [36]. The effect of several subsequent kicks will be an averaging of this value of P_N over different N_I 's within an interval $\Delta N_I \sim k$; anyway, for $N - N_I > k$, P_N will decay sharply (this corresponds to the classically forbidden region).

All the above theoretical results were checked by computer simulations of the quantum H atom, performed by means of the numerical technique described in [3] in the parameter range $1 < \omega_0 < 3.5, 0.02 < \epsilon_0 < 0.16$, with initial conditions $n_0 = 30, 45, 66, 100, 200, 400, 500$. In each case, we computed the steady-state distribution f_N and we analyzed its dependence on $N_{\phi} = N_I - 1/(2n^2\omega)$.

Examples of such distributions are given in Fig. 4(a)-(c). The chain of peaks equally spaced at one-photon intervals is here evident, as well as the exponential character of the distribution.

In order to compute the localization length, the whole range of N_{ϕ} was divided into one-photon intervals and in each interval the maximum of the distribution was taken. The numerical value of the localization length was then found as the slope of a straight line fitting the points thus obtained (for $N_{\phi} > 0$).

In order to check the exponential localization, we plotted the maxima of the distribution against the rescaled number of photons $X = 2N_{\phi}/l_{\phi}$ where l_{ϕ} is the numerically obtained value of the localization length. We also subtracted the constant part of $\ln f_N$, so that perfect exponential localization would correspond to $\ln f_N = -X$. Such a plot for 47 different distributions is shown in Fig. 5. Even though there is some scatter of points, the exponential behavior is fairly evident.

In order to check the theoretical prediction (24) for the localization length, we plotted the rescaled, numerically obtained, localization length as a function of field intensity. The agreement between the numerical results and the analytical expression (24) (which is represented by the straight line in Fig. 6) has been checked in an interval of ten orders of magnitude of field intensity, for 43 cases with theoretical $l_{\phi} > 1$, and it appears quite satisfactory.

Another type of check is shown in Fig. 7 where we plotted the ratio R of the experimental value l_{ϕ} to the theoretical one for the same 43 cases above. The scatter of points around the value R = 1 mirrors the strong fluctuations in the steady-state distributions (see Fig. 5). This scatter is larger, the larger the value of the localization length, because of the small interval of change of $\ln f_N$ and of the strong fluctuations in one-interval localization length. The obtained averaged value of R is 1.23 ± 0.08 to be compared to the theoretical value R = 1.

We also compared the steady-state distribution obtained by numerical simulation of the continuous time Schroedinger evolution, with the steady-state distribution obtained by iterating the quantum Kepler map (Fig. 4). This was done by computing the total probability in each one-photon interval $N_{\phi} - 1/2$, $N_{\phi} + 1/2$ according to the first distribution and by matching the "photonic" distribution obtained in this way with the second distribution. There is a much better agreement around the initially excited level $(N_{\phi} = 0)$ than in the tails of the distributions, which agree only in the average. This fact may be connected with the already mentioned sensitivity of the tail to small parameter changes. The steady-state "Schroedinger" distribution was obtained by time averaging over 50 microwave periods (with $450 < \tau \le 500$); instead, the "Kepler" distribution was averaged over 15 iterations, from the 135th to the 150th, which approximately corresponds to the same physical time. Anyway, in the localized regime $(l_{\phi} < N_I)$, the exact averaging interval is inessential, provided it is not too small or too large.

Since ϵ_0 and ω_0 have the same value in the three cases illustrated in Fig. 4, these three cases correspond to the same classical behavior, which is shown in Fig. 4(b). In contrast to the quantum case, the classical distribution spreads over the unperturbed levels, and strong ionization occurs after a time $t_I \ll (2\pi n_0^3 \Gamma_{\phi})^{-1}$ with Γ_{ϕ} as in (31).

Strictly speaking, the theory discussed in this section refers to the region $\omega_0 > 1$. However, above the classical chaotic threshold, the value of $\omega_0 = \omega n^3$ grows during the process of excitation and it eventually becomes larger than 1 on sufficiently high levels. Therefore, we feel justified in using the same expression (24) for the localization length (that does not depend on the level number *n*) even for $\omega_0 = \omega n_0^3 < 1$. As a consequence, even the delocalization condition (29) still holds. On the other hand, for small ω_0 , the quantum delocalization border is lower than the classical chaotic border; then the latter alone deter-



(b) Fig. 4. Probability distribution on unperturbed levels averaged from 450 to 500 microwave periods (full curve) versus the number of photons $N_{\phi} = (1/2 n_0^2 - 1/2n^2)/\omega$ for the same parameters $\epsilon_0 = \epsilon n_0^4 = 0.03$, $\omega_0 = \omega n_0^3 = 3.5$ as in Fig. 3. The straight-line results from a least squares fitting of the maxima of the distribution in each photon interval. The crosses (+) show the total probability in the interval $N_{\phi} - 1/2$, $N_{\phi} + 1/2$. The dots (•) give the steady-state distribution, averaged from 135 < t < 150, obtained by iterating the quantum map (27). (a) $n_0 = 100$. (b) $n_0 = 200$. (c) $n_0 = 400$. For comparison, the classical distribution over unperturbed levels is also shown in (b) (dashed curve).



Fig. 5. Dependence of $\ln f_n$ versus rescaled number of photons $X = 2N_{\phi}/l_{\phi}$. Here l_{ϕ} is the experimental value obtained by least squares fits of 47 distributions as described in Fig. 4, for different values of ϵ_0 , ω_0 , n_0 . The constant parts in $\ln f_N$ have been substracted, so that perfect exponential localization would correspond to $\ln f_n = -X$ which is also drawn in the figure (full line).

10

20



Fig. 6. Plot of the logarithm of the rescaled experimental localization length log ($l_{\phi} \omega^{10/3}/3.33$) versus log ϵ^2 . The solid line gives the theoretical dependence (24). The points are obtained from numerical data of 43 different distributions, with $l_{\phi} > 1$.

Fig. 7. The ratio $R = l_{\phi}^{E}/l_{\phi}^{T}$ of the experimentally obtained localization length l_{ϕ}^{E} over the theoretical value $l^{T}\phi$ from (24) versus log l_{ϕ}^{T} for the same 43 cases of Fig. 6. The average value of R is $\langle R \rangle = 1.23 \pm 0.8$

mines the ionization threshold, above which the quantum process of ionization is close to the classical one.

In conclusion, a large amount of numerical data yield satisfactory agreement with the above developed theory in a broad parameter range. This theory can therefore be assumed to provide a fairly complete description of the process of excitation and ionization of 1-D hydrogen atoms in a monochromatic field.

In closing this section, we wish to add some comments of a more general character about the conditions that must be fulfilled in order that a deterministic quantum system, assumed to be classically integrable in the case of zero perturbation, may develop a mechanism of chaotic excitation (compare the discussion in the previous section). The parameters k and T in the Kepler map can be written as

$$k = 2\pi\mu\epsilon\rho$$
 $T = 2\pi\omega\partial\rho/\partial N = 2\pi\omega^2\partial\rho/\partial E$

so that the criterion for classical chaotic excitation takes the form

$$K = kT = 4\pi^2 \mu \epsilon \omega^2 \rho \partial \rho / \partial E > 1.$$

We already stressed that this criterion is, generally speaking, much more restrictive than the so-called quantum border of stability, that essentially requires the perturbation to be larger than the unperturbed level spacing. This is now clearly evident because this stability border is defined by

$$2\pi\mu\epsilon\rho = k \sim 1.$$

In order to have strong excitation, we have not only to satisfy k > 1, but also K > 1. In that case, the localization length is $l_{\phi} = 2\pi^2 \mu^2 \epsilon^2 \rho^2$ (7).

VI. THE KEPLER MAP FOR THE TWO-DIMENSIONAL Classical Hydrogen Atom

We turn now to the analysis of the two-dimensional case. The main results of this analysis were presented in [15]. We shall show that, insofar as the excitation in energy is examined, the more realistic 2-D model can be reduced to the 1-D one, even for not strictly 1-D initial states. We shall then conclude that the 1-D theory provides an understanding of the main essential features of the dynamics not just of the 2-D case, but even of the real 3-D case. Indeed, in that case, the component m of the angular momentum along the field direction is conserved. Thanks to this conservation law, the essential characters of the 2-D motion will not be essentially modified by the addition of the third spatial dimension.

In our study of the classical 2-D case, we will follow the same strategy successfully used in the 1-D case; namely, we shall seek for a simplified description of the 2-D dynamics by means of a map that describes the change of appropriate canonical variables between consecutive passages at the aphelion. Our 2-D case is obtained by setting m = 0, i.e., by assuming the orbital momentum to be normal to the direction of the field. The Hamiltonian in action-angle variables can then be written as [27], [37], [38]

$$H = -1/2n^{2} + \epsilon n^{2} \cos \omega t \left\{ (3/2)e \cos \psi - 2 \right.$$
$$\left. \cdot \sum_{s=1}^{\infty} (x_{s} \cos s\theta \cos \psi + -y_{s} \sin s\theta \sin \psi) \right\} + (3/2)\epsilon_{s}n^{2}e \cos \psi.$$
(32)

Here *e* is the eccentricity, connected to the orbital momentum *l* by $e = (1 - l^2/n^2)^{1/2}$, and

$$x_s = s^{-1} J'_s(se)$$
 $y_s = [(1 - e^2)^{1/2} / se] J_s(se).$ (32a)

The last term in (32) describes the effect of a static electric field, collinear with the microwave field. θ is the angle, conjugated to the action n, and ψ is the angle variable conjugate to l, i.e., it is the angle between the major axis of the ellipse and the direction of the external field.

The map we look for shall describe the change, during an orbital period, of N (energy divided by ω), of the conjugate phase φ , which is just the product of $-\omega$ and time, and of l, ψ . This map will therefore be a four-dimensional generalization of the 2-D Kepler map discussed in the previous section, and will be found much in the same way, i.e., by approximate integration of the Hamilton equations over one period of the unperturbed motion.

Moreover, we will be mainly concerned with the case $\omega n^3 >> 1$. Resonant terms in the Hamiltonian (32) will then correspond to s >> 1, so that we shall use asymptotic expansions of the Fourier amplitudes x_s , y_s for s >> 1. Since these expansions will turn out to be valid down to $s \sim 1$ within a 20 percent accuracy, the results thus obtained will have some validity throughout the region $\omega n^3 > 1$. In order to obtain such expansions, we start from the following formula

$$J_s(se) \approx (1/\sqrt{\pi})(2/s)^{1/3} \Phi[(s/2)^{2/3}(1-e^2)] \quad (33)$$

where Φ is the Airy function.

In our approximate integration of the equations of motion, the main contributions will be given by resonant terms, i.e., by terms with $s \approx \omega n^3$. For such terms,

$$J_s(se) \approx (1/\sqrt{\pi})(2/s)^{1/3} \Phi([(\omega/2)^{1/3}l]^2)$$

and, since $\Phi(x)$ decays exponentially as $x \to +\infty$, we shall consider the case in which $(\omega/2)^{1/3}l < 1$, i.e.,

$$l < (2/\omega)^{1/3}$$
. (34)

If (34) is satisfied and $\omega n^3 >> 1$, then $l \ll n$ so that the eccentricity *e* is close to 1. We can therefore write

$$y_s \approx (l/n)(2^{1/3}/\sqrt{\pi})s^{-4/3}\phi(0) \approx (l/n)0.447/s^{4/3}$$

having used the value $\phi(0) = 0.629$.

In order to estimate x_s , we use

$$J'_{s}(se) \approx J'_{s}(s) + s(e - 1) J''_{s}(s)$$

= $J'_{s}(s)(1 + l^{2}/2n^{2})$

together with

$$J'_{s}(s) = (1/s)\partial/\partial e J_{s}(se)\Big|_{e=1}$$
$$\approx -(2^{2/3}/\sqrt{\pi})\Phi'(0)/s^{2/3}$$

Thus, we finally get

$$x_s \approx (0.411/s^{5/3})(1 + l^2/2n^2)$$
 (35)

(we used the value $\Phi'(0) = -0.4587$). We are now ready to compute the map by a first-order perturbation theory. To this end, we substitute the unperturbed motion in the field dependent terms and we integrate the approximate equations thus obtained over one unperturbed period, keeping just the resonant term. We find

$$\Delta n = -\int_0^{2\pi n^3} dt \; \partial H / \partial \theta \approx k \omega n^3 [-(1 + l^2/2n^2) \sin \phi$$
$$\cdot \cos \psi - 1.09 \omega^{1/3} l \cos \phi \sin \psi]$$

where $\phi = -\omega t + s\theta$ is the value of $-\omega t$ at the perihelion $(\theta = 0)$ and $k = 0.822\pi\epsilon/\omega^{5/3}$ as in the 1-D case.

Analogously, we also find

$$\Delta l = -\int_0^{2\pi n^3} dt \,\partial H/\partial \psi \approx k \left[-(1 + l^2/2n^2) \cos \phi \right] \\ \cdot \sin \psi - 1.09 \omega^{1/3} l \sin \phi \cos \psi + 1 + 3\pi \epsilon_s n^3 (n^2 - l^2/2) \sin \psi \\ \Delta \psi \approx k \left[-(l/n^2) \cos \phi \cos \psi + 1.09 \omega^{1/3} \sin \phi \sin \psi + -3\pi \epsilon_s n^3 l \cos \psi \right] \\ \Delta \varphi \approx -2\pi \omega n^3.$$

By introducing a new variable $N = -1/(2n^2\omega)$, we can easily write a set of equations yielding ΔN , Δl , $\Delta \psi$, Δ_{ϕ} . These equations give a map from the values taken by N, l, ψ , ϕ , at a passage through the perihelion to their values \overline{N} , \overline{l} , $\overline{\psi}$, $\overline{\phi}$ at the next passage. However, as in the 1-D case, this map would not be a canonical one. In order to get a canonical map, we shall first write the generating function

$$G(\overline{N}, \overline{l}; \phi, \psi) = \overline{N}\phi + \overline{l}\psi - 2\pi (-2\omega\overline{N})^{-1/2} + k[(1 - \omega\overline{Nl}^2)\cos\phi\cos\psi - 1.09\omega^{1/3}\overline{l}\sin\phi\sin\psi] + 3\pi\epsilon_s(-2\omega\overline{N})^{-3/2}[(-2\omega\overline{N})^{-1} - \overline{l}^2/2]\cos\psi.$$
(36)

From this generating function we deduce, according to the usual procedure, the map

$$\overline{N} = N - k [(1 - \omega \overline{N} \overline{l}^2) \sin \phi \cos \psi + 1.09 \omega^{1/3} \overline{l} \cos \phi \sin \psi]$$

$$\overline{\phi} = \phi - 2\pi \omega (-2\omega \overline{N})^{-3/2} + k\omega \overline{l}^2 \cos \phi \cos \psi + 3\pi \epsilon_s \cos \psi [5\omega (-2\omega \overline{N})^{-7/2} - (3/2) \omega \overline{l}^2 (-2\omega \overline{N})^{-5/2}]$$

$$\overline{l} = l - k [(1 - \omega \overline{N} \overline{l}^2) \cos \phi \sin \psi + 1.09 \omega^{1/3} \overline{l} \sin \phi \cos \psi] +$$

+
$$3\pi\epsilon_s(-2\omega\overline{N})^{-3/2}[(-2\omega\overline{N})^{-1} - l^2/2]\sin\psi$$

$$\overline{\psi} = \psi + k [2\omega \overline{N}\overline{l}\cos\phi\cos\psi + 1.09\omega^{1/3}\sin\phi\sin\psi]$$

 $-3\pi\epsilon_s(-2\omega N)^{-3/2}l\cos\psi.$ (37)

This map is canonical, hence measure preserving, and for this reason it gives values of ΔN , Δl , $\Delta \psi$, $\Delta \phi$ slightly different from the previously computed ones.

A simpler version of this Kepler map can be obtained by going over to a different set (N, θ, J, χ) of canonical variables defined by a transformation that was used in [37], [38] in order to get a description of the motion near the main resonance s = 1. This transformation is defined by

$$tg\chi = (B/A)tg\psi \quad \theta = \phi + \chi$$
$$J + N = \int_0^l dl' AB/(A^2 \sin^2 \chi + B^2 \cos^2 \chi)$$
$$A = 1 - Nl^2\omega, \quad B = 1.09\omega^{1/3}l. \quad (38)$$

In these new variables, the generating function becomes

$$G(\overline{N}, \theta; \overline{J}, \chi) = \overline{J}\chi + \overline{N}\theta - 2\pi (-2\omega\overline{N})^{-1/2} - kH(\overline{J} + \overline{N}, \overline{N}, \chi) \cos \theta + 3\pi\epsilon_s (-2\omega\overline{N})^{-5/2} [1 + \omega\overline{N}\overline{l}^2] \cos \psi$$
(39)

where in the last term $\overline{l} = \overline{l}(\overline{N} + \overline{J}, \chi)$ and

$$H^{2}(N, J, \chi) = A^{2} \cos^{2} \psi + B^{2} \sin^{2} \psi.$$
 (40)

Let us consider the case with no static field $\epsilon_s = 0$. Then from the generating function (39), we obtain the map

$$N = N - kH \sin \theta$$

$$\overline{\theta} = \theta - 2\pi\omega (-2\omega\overline{N})^{-3/2} - k(\partial H/\partial\overline{N}) \cos \theta$$

$$\overline{J} = J + k \cos \theta \partial H/\partial \chi$$

$$\overline{\chi} = \chi - k \cos \theta \partial H/\partial \overline{J}.$$
(41)

As we will see below, under appropriate conditions, the dependence of H on N can be neglected and the changes ΔJ , $\Delta \chi$ after each iteration are so small that for J, χ we may use a continuous time approximation, i.e., we may assume the second couple of equations (41) to yield the time-derivative of J and χ .

A decisive simplification of the dynamics described by (41) is achieved under these assumptions—the validity of which, we repeat, we shall discuss below because they allow for a sort of decoupling of the (N, θ) motion from the (J, χ) motion. The latter, in the continuous time approximation, is described by the differential equations

$$dJ/dt = k \cos \theta \ \partial H/\partial \chi; \qquad d\chi/dt = -k \cos \theta \ \partial H/\partial J$$

where t is time measured in number of iterations. Defining a new time variable σ by

$$d\sigma/dt = k\cos\theta, \qquad (42)$$

these differential equations take the Hamiltonian form

$$dJ/d\sigma = \partial H/\partial \chi$$
 $d\chi/d\sigma = -\partial H/\partial J.$ (43)

Then *H* is an integral of the (J, χ) motion and can change in time only due to changes in *N*. Then, if the dependence of *H* on *N* can be neglected, the (N, θ) motion reduces to the map (16) with *kH* in place of *k*. The (J, χ) dynamics which, in the variable σ , do not depend on the (N, θ) motion, can also be described in the variables (l, ψ) . According to (38) and (40):

$$H = \left[\left(1 + l^2 / 2n^2 \right)^2 \cos^2 \psi + \left(1.09\omega^{1/3} l \right)^2 \sin^2 \psi \right]^{1/2}$$
(44)

where n, according to the made assumption, is constant.

Depending on the value of the Hamiltonian *H*, the (l, ψ) motion has different qualitative features. If H < 1, we shall have bounded oscillations of the phase ψ around $\psi = \pi/2$; instead, for H > 1, we will have phase rotations. These two types of motion are divided by the separatrix H = 1. We shall now analyze the (l, ψ) motion in two interesting limiting cases, corresponding to the regions around the unstable fixed point $(l = 0, \psi = 0)$ and the stable one $(l = 0, \psi = \pi/2)$.

The first of these limiting cases corresponds to very extended orbits along the direction of the field, i.e., to $l < n, \psi \ll 1$. In this case, (44) can be approximately written

$$H \approx l^2/2n^2 - (1/2)\psi^2 + 1.$$
 (45)

This Hamiltonian describes an unstable motion, with exponential growth of the variables and subsequent disruption of the original extended state

$$\psi = \psi_0 \operatorname{ch} \left(\sigma/\overline{\sigma} \right) + \left(l_0/n \right) \operatorname{sh} \left(\sigma/\overline{\sigma} \right)$$

$$l = n\psi_0 \operatorname{sh} \left(\sigma/\overline{\sigma} \right) + l_0 \operatorname{ch} \left(\sigma/\overline{\sigma} \right)$$
(46)

where l_0 , ψ_0 are the initial values and $\overline{\sigma} = n$ is the characteristic instability time. Of course, (46) is a linearized solution around the unstable fixed point $\psi = l = 0$. When l, ψ become large enough, this linear approximation becomes untenable and the original Hamiltonian (44) must be used, which describes a periodic motion of period $T \sim 4\overline{\sigma} \ln (1/\psi_0)$.

Introducing parabolic action variables n_1 , n_2 , we may also write $T \sim 2\overline{\sigma} \ln (n/n_2)$ (see below).

In the quantum case, extended states will be specified by a parabolic quantum number $n_2 \ll n$. In order to make contact with the quantum formulation, we must, therefore, go over to parabolic action angle variables $(n_1, n_2,$ $\eta_1, \eta_2)$. A connection between these variables and the polar variables used up to now is given by the relation

$$(n_1 - n_2)/n = (1 - l^2/n^2)^{1/2} \cos \psi.$$
 (47)

We also recall that $n = n_1 + n_2$ (since we have assumed m = 0). For $n_2 \ll n$, this relation gives approximately

$$n_2 \approx 1/4(l^2/n + n\psi^2)$$
 (48)

so that we can write

$$l_0 = 2(n_{20}n)^{1/2} \cos \eta_0$$

$$\psi_0 = 2(n_{20}/n)^{1/2} \sin \eta_0$$

and (46) and (48) yield

$$n_2 \approx n_{20} [\operatorname{ch} (2\sigma/n) + (\sin 2\eta_0) \operatorname{sh} (2\sigma/n)].$$
 (49)

When we shall discuss the numerical results of the computer simulation of the 2-D H-atom in which an initial state corresponding to assigned values of n_0 and n_{20} evolves under the action of the microwave field, we shall need, for comparison, certain statistical quantities related to the classical evolution. In order to define these quantities, we assume that the quantum evolution of an unperturbed eigenstate with given n, $n_2 = n_{20}$, corresponds to an ensemble of classical trajectories with a fixed value of n_{20} and phases η_{20} uniformly distributed in $[0, 2\pi)$. We can then use (49) to compute the dependence on time σ of the first two moments of the distribution in n_2 :

$$\mu_{1} = \langle n_{2} - n_{20} \rangle = n_{20} (\operatorname{ch} (2\sigma/n) - 1)$$

$$\mu_{2} = \langle (n_{2} - \langle n_{2} \rangle)^{2} \rangle = (n_{20}^{2}/2) \operatorname{sh}^{2} (2\sigma/n) \quad (50)$$

From these formulas we see that, for $2\sigma \ll n$,

$$\mu_2/\mu_1 \approx n_{20}.$$
 (51)

Instead, in the opposite case $2\sigma >> n$, we obtain

$$\langle n_2^2 \rangle / \langle n_2 \rangle^2 = 3/2.$$
 (52)

The results (49)-(52) have been deduced from the Hamiltonian formulation of the (χ, J) dynamics that was made possible by the introduction of the new "time" σ . In order to refer to the old time t (defined by the number of iterations of the Kepler map), (42) must be used. This equation shows that the connection between σ and t is determined by the time evolution of the phase θ which, in turn, is ruled by the first couple of equations (41). Under our assumptions, these equations describe a mapping in the (N, θ) variables, very similar to the map discussed in the previous section. As the value of the perturbation parameter kH is increased, the (N, θ) evolution undergoes a stochastic transition that will deeply modify the dependence of θ on time.

Let us first assume that the (N, θ) motion is regular. Then the average $\overline{\cos \theta} = \lambda \neq 0$ and therefore we may assume that, in the average, the connection between σ and t will be given by $\sigma = k\lambda t$. Then the instability in the (l, Ψ) motion near the unstable fixed point $l = \psi = 0$ will have a characteristic time

$$\sigma_R = \overline{\sigma}/k\lambda = n/k\lambda \approx \omega_0^{5/3}/(2.6\epsilon_0\lambda)$$
 (53)

which is in agreement with the instability time obtained in [38].

Instead, in the case of completely chaotic (N, θ) motion, θ depends on t in a random way; therefore, σ is a random function of t, with $\overline{\sigma}(t) = 0$ and $\overline{\sigma^2(t)} = (k^2/2)t$. Then the mean-square value of the argument of the hyperbolic functions in (46) is $(\overline{\sigma/n})^2 = k^2t/(2n^2)$ and the characteristic time of instability can now be estimated as

$$t_c = 2(n/k)^2 \approx \omega_0^{10/3}/(3.3\epsilon_0^2)$$
 (54)

so that $t_c \gg t_R$, and also $t_c \gg t_I$, $[t_I$ is the time of ionization given by (20)]:

$$t_c/t_I \approx 4\omega_0^2 \gg 1.$$
 (55)

Therefore, the instability in the (l, ψ) motion develops so slowly that the extended nature of the initial state cannot be seriously affected. This instability is not relevant for the (N, θ) motion.

The condition that allows us to use the continuous time approximation in the (χ, J) dynamics is satisfied in the regular region for $t_R >> 1$ which is always the case, because $\omega_0 > 1$ and ϵ_0 , $\lambda < 1$. In the chaotic region, t_c is even larger than t_R .

Instead, in order to check our assumption of negligible dependence of H on N, we notice that, for arbitrary states, the main change of H [see (40)] is due to the change of $Nl^2\omega$. Since $\omega\delta N \sim 1/n^2$, we have $\delta H \sim (l/n)^2 \ll 1$ for extended states. Notice, however, that numerical results of ours in cases when l/n is not very small indicate that the essential result of our analysis, i.e., the small effect of the (l, ψ) dynamics on the (N, θ) one, also holds for more general states.

Up to now, we discussed the (l, ψ) dynamics near the separatrix $H \approx 1$. Let us now turn to the case H < 1. From (38), we get the following approximate expressions for l, ψ :

$$l^{2} \approx 2L \sin^{2} \chi / (1.09\omega^{1/3})$$
$$tg\psi \approx 1 / [(2 \times 1.09\omega^{1/3}L)^{1/2} \cos \chi]$$
$$L = J + N$$

from which we obtain

$$H^2 \approx 2.18 \ \omega^{1/3} L.$$
 (56)

Now, since N is considered to be constant, L is conjugated to χ and plays the role of an action for the Hamiltonian H. Of course, (56) is just the leading term of the expansion in the small parameter L. The next higher order correction to (56) would be equal to $-2(1.09 \ \omega^{1/3}L)^2 \ \cos^2 \chi$.

According to (56), the period of the (l, ψ) motion in the time σ is

$$t_{\sigma} = 2\pi (\partial H/\partial L)^{-1} = 2\pi H/(1.09 \ \omega^{1/3}).$$

As before, in the case of regular (N, θ) motion, we get the period in the number of map iterations

$$t_R = t_\sigma / k\lambda \approx 2.2 (\omega_0^{4/3} / \epsilon_0 \lambda) H.$$
 (57)

If *H* is not too small, $t_R >> 1$ which again legitimates the use of the continuous time approximation. Instead, if the (N, θ) motion is chaotic—and this, for H < 1, will happen for $\epsilon_0 > 1/(49 \omega_0^{1/3} H)$ —proceeding as above, we get

$$t_c = 2t_{\sigma}^2/k^2 \approx 10(\omega_0^{8/3}/\epsilon_0^2)H^2.$$
 (58)

Also this value is much larger than 1 if H is not too small. τ_c is even larger than the ionization time t_l , which for H < 1 is given by $t_l = 2N_l^2/(k^2H^2)$:

$$t_c/t_l \approx 130 \; \omega_0^{4/3} H^4.$$
 (59)

In conclusion, for practically all values of H (except for very extended orbits in the direction perpendicular to the field, for which $l \ll n$, $\psi \approx \pi/2$, $H \ll 1$) the (l, ψ) motion is very slow and does not affect the excitation in energy. Therefore, the latter process can still be described with a good approximation by the one-dimensional theory discussed in previous sections. The case $m \neq 0$ (3-D atom) requires that the Hamiltonian (32) be multiplied by a factor $(1 - m^2/l^2)^{1/2}$ [27]. In (40), we must change H^2 into $H^2(1 - m^2/l^2)$. Insofar as m < l, this factor We conclude this section by briefly analyzing the effect of the static field ϵ_s that was hitherto assumed to be zero. Neglecting again the dependence of H and l on N, we can get again the map in the variables $(N, \theta; J, \chi)$ in a particularly transparent form. To this end, we use one more approximation; we substitute, in the (χ, J) equations, the average value λ of $\cos \theta$, which will always be legitimate near to the center of the resonance of the (N, θ) motion, $\theta = 0, \theta = \pi$.

Then from (39) we obtain the map

$$\overline{J} = J + k\lambda \,\partial H/\partial \chi - \partial F/\partial \chi$$
$$\overline{\chi} = \chi - k\lambda \,\partial H/\partial \overline{J} + \partial F/\partial \overline{J}$$

where F is the last term in (39). The (χ, J) dynamics are, therefore, ruled by the Hamiltonian $H_1 = F - k\lambda H$ which, for extended states and in (l, ψ) variables, has the form

$$H_{1} = 3\pi\epsilon_{s}n^{5}[1 - (l^{2}/n^{2} + \psi^{2})/2] - k\lambda[1 + (l^{2}/n^{2} - \psi^{2})/2].$$
(60)

If we use the value $\lambda = 1$ (corresponding to the center of the main resonance in the (N, θ) motion), we see immediately from (60) that the (l, ψ) motion becomes stable for

$$\epsilon_{so} \equiv |\epsilon_s n_0^4| > (2/3)(0.411/\omega_0^{5/3})\epsilon_0$$
 (61)

in agreement with [39].

If (61) is satisfied, *H* oscillates in the interval $\Delta H \sim \psi_0^2 \sim n_{20}/n$ around H = 1 with the frequency $\omega_s = 2[(3\pi\epsilon_{so})^2 - (\lambda k/n)]^{1/2}$. Then, if the static field is large enough, ω_s is essentially determined by ϵ_{so} alone; in that case, the influence of the microwave on the (l, ψ) motion can be neglected, the approximation $\cos \theta \sim \text{const is no}$ longer needed, and the (l, ψ) and (N, θ) motion are approximately decoupled. However, *H* becomes a periodic function of the iteration number *t*:

$$H = 1 + (2n_{20}/n) \cos (\omega_s t - 2\eta_0); \qquad \omega_s \approx 6\pi\epsilon_{so}$$
(62)

Then, above the chaotic threshold, we shall have a diffusive excitation in energy. The (l, ψ) motion will still have an essentially regular character; the chaotic part, which will be present in it, will be relatively small if ϵ_{so} $\gg \lambda k/(3\pi n)$. The time scale of the regular (l, ψ) motion, which is ω_s^{-1} , is nevertheless smaller than the characteristic time of the diffusion in *n*, which is of the order of t_l (20).

VII. Two-Dimensional Quantum Theory and Numerical Experiments

Whereas the 1-D Kepler map was easily quantized and thus opened the way to a straightforward analysis of the quantum 1-D problem, the quantization of the 2-D Kepler map is a much more difficult problem. However, the qualitative picture of the classical 2-D motion that was obtained in the previous section allows for the understanding of the essential features of the quantum 2-D problem.

Thanks to the appropriate decoupling of the two degrees of freedom, the (N, θ) dynamics should be very similar in the 2-D and in the 1-D case, provided that the perturbation parameter k is replaced by kH in the 2-D case. We are thus led to predict exponential photonic localization with a localization length

$$l_{\phi} \approx 3.33 \ \epsilon^2 H^2 / \omega^{10/3}.$$
 (63)

Since *H* is only approximately constant, we are faced with the important question of what the relevance of small changes in H would be on the (N, θ) quantum dynamics; in particular, we should ask whether these changes might lead to delocalization. In order to answer this question, we must remark that localization would persist unless the time scale t_c becomes comparable to the time $t_d \approx l_{\phi}$ required for the quantum suppression of chaotic diffusion. As a matter of fact, the effect of the (l, ψ) motion on the (N, θ) motion would be just a broadening of the lines in the discrete spectrum of the (N, θ) motion, up to a width $\sim 1/t_{c}$. Then, in order for delocalization to occur, it would be at least necessary-but perhaps not yet sufficient-that this broadening is larger than the average spacing of levels in the discrete spectrum, i.e., we should require that $t_c^{-1} > l_{\phi}^{-1}$. However, this condition cannot be met below the 1-D delocalization border, as can be seen from the following estimate of the ratio t_c/l_{ϕ} :

$$t_c/l_{\phi} \approx \frac{4\omega_0^2 (\epsilon_q \epsilon_0^{-1})^4}{100\omega_0^{4/3} (\epsilon_q \epsilon_0^{-1})^4} \quad \text{for } H \approx 1$$
(64)

where ϵ_q is the 1-D delocalization border (29). Equation (64) shows that, if $\epsilon_0 < \epsilon_q$, the ratio $t_c l_{\phi}^{-1}$ is much larger than 1.

In other words, the slow (l, ψ) motion acts as an adiabatic perturbation on the (N, θ) motion, and cannot therefore produce additional transitions in the latter. The ultimate reason of this adiabaticity is the Coulomb degeneracy. To summarize: the 2-D delocalization border co-incides with the 1-D one when $H \approx 1$, and is larger by a factor 1/H for H < 1.

It is interesting to remark that in principle quantum localization may be expected for the (l, ψ) motion, too. Indeed, far in the chaotic quasi-classical region (K >> 1) one has $\overline{\cos \theta} = 0$ and since $\sigma = k \Sigma \cos \theta_i$, a localization in σ may occur similar to the localization in the number of photons $N = -k \Sigma \sin \theta_i$ [map (41)].

According to the above qualitative picture, a possible way to destroy localization is the introduction of a strong static field ϵ_s , which would eliminate the Coulomb degeneracy. According to (41) and (62), the excitation would be approximately described by the map

$$\overline{N} = N - k (1 + (2n_{20}/n_0) \cos(\omega_s t)) \sin \theta$$

$$\overline{\theta} = \theta - 2\pi\omega (-2\omega\overline{N})^{-3/2}.$$
 (65)

The quantization of a similar map was investigated in [6] for $\omega_s \sim 1$. According to results obtained there, a necessary condition for delocalization is $k^2 n_{20}/n_0 \gtrsim 1$, which gives

Strictly speaking, due to the approximate character of (65), this delocalization should be understood as a sharp increase of the localization length, up to N_l , taking place above the border (66). Instead, if $\omega_s \sim 6\pi\epsilon_{0_s}$ is much smaller than 1, the critical field will be larger and further investigations are needed. As a preliminary comment, we notice that for delocalization it would be at least necessary that the time $t_D \sim l_{\phi}$ of 1-D localization be larger than the slow-frequency period $2\pi/\omega_s$. On the other hand, if the spreading of the shell is comparable with the shell separation ($\omega_s \sim 1$) then the 2-D border would still be given by (66). Notice that the static field term can stabilize the precession of stable orbits ($\theta = 0$) [39], but the (l, ψ) motion can still contribute in delocalizing the N-motion.

We shall now describe the numerical results we obtained by computer simulations of the classical and quantum 2-D models. Our algorithm for quantum computations made use of an expansion over a basis of unperturbed eigenstates. The (complex) amplitudes of the expansion of the state vector over a basis of unperturbed eigenstates labeled by the quantum numbers n, l obey the following set of ordinary differential equations:

$$ic_{nl} = (-1/2n^2) c_{nl} + \epsilon(t) \sum_{n'l'} z_{nl}^{n'l'} c_{n'l'}$$
(67)

where $\epsilon(t) = \epsilon \cos \omega t$ and $z_{nl}^{n'l'}$ are dipole matrix elements. Selection rules restrict the sum over l' to $l' = l \pm 1$.

In our numerical scheme, the sum over n' was truncated to n = 128, and the integration was carried out by the method described in (3). Most computations were performed by initially exciting levels with principal quantum number $n_0 = 66$ and with a prescribed value of the parabolic quantum number n_2 chosen in the range $0 \le n_2 \le$ 30; the field strength and frequency varied in the ranges $1 \le \omega_0 \le 2.5, 0.03 \le \epsilon_0 \le 0.06$. Since a spherical basis was used in the computation, a conversion from the spherical to the parabolic basis was required. For this conversion, we used the Clebsch coefficients given in [32]. In comparison to the 1-D case, 2-D computations are much more difficult, because the dimension of the basis is on the order of $n^2/2 \sim 10^4$. For this reason, we had to restrict the integration time to a relatively small number of field periods $\tau = 120$.

For each considered case, we computed the probability distribution over parabolic eigenstates $F(n, n_2)$, and the distribution over the principal quantum number $f(n) = \sum_{n_2} F(n, n_2)$. We also computed the excitation probability $W_{1.5}$ on states with $n > 1.5 n_0$, and we analyzed the dependence on time of the first two moments $\langle (n - \langle n \rangle)^2 \rangle$, $\langle n - n_0 \rangle$, $\mu_2 = \langle (n_2 - \langle n_2 \rangle)^2 \rangle$, $\mu_1 = \langle n_2 - n_{20} \rangle$. In order to suppress fluctuations, we averaged in

time the distribution f(n), typically over ten microwave periods.

In our numerical simulations of the classical model, the initial conditions corresponding to the quantum case were given by an ensemble of 900 trajectories with fixed n_0 , n_{20} values and homogeneously distributed phases in the interval $(0, 2\pi)$. A numerical integration procedure which was able to avoid the singularity arising near the nucleus was devised in analogy with the 1-D case [3], [23], [30].

The essential result of our quantum computations is that in all cases we observed localization in satisfactory agreement with the 1-D estimate (24). In Fig. 8 we show, for a typical case, the comparison of classical and quantum second moments of the distribution f(n). Here the parameter values are $\omega_0 = 2.5$, $\epsilon_0 = 0.04$, and $n_{20} = 15$ which lie above the classical chaos border (18). The localization of the quantum motion is clearly apparent; instead, the classical diffusion approximately agrees with the theoretical estimate, represented by the straight line.

Since in this case the stochasticity parameter is not very large ($K = \epsilon_{cr}/\epsilon_0 \approx 2.66$), formula (19) for the diffusion rate must be corrected by an appropriate factor. According to [10] and [19], this fact is $[0.6 (K - 0.97)^3/K^2] \approx 0.41$; the diffusion coefficient is then $D = (k^2/2) (0.41 \omega_0) = 1.12$. (The factor ω_0 was introduced, because (19) yields the diffusion in the number of photons per iteration and we now need instead the diffusion rate in the number of levels per microwave period.)

In Fig. 9, we compare the excitation probability $W_{1.5}$ in the classical and quantum case for the same parameter values as in Fig. 8. We see that the quantum probability is three orders of magnitude less than the classical. This is due to the localization phenomenon displayed in Fig. 10. Here we plot the distribution over unperturbed levels as a function of the photon number $N_{\phi} = (1/2n_0^2 - 1/2n^2)/\omega$. In the quantum case, the distribution appears to be exponentially localized, and the localization length agrees with the 1-D estimate (24). Instead the classical distribution over unperturbed levels diffuses according to the Fokker-Planck equation whose analytical solution is represented by the thin curve (see [3]).

In contrast to the localization in energy, we observed the theoretically predicted delocalization in n_2 thanks to which all levels inside the energy shell are excited. The moments of the quantum distribution in n_2 are close to their classical values. The time dependence of the quantum and classical second moment of the distribution is shown in Fig. 11. For this case, the chaotic time scale in number of microwave periods is $\tau_c = \omega_0 t_c \sim 10^4$, i.e., it is much larger than the time of numerical computation. Therefore, in spite of the chaotic (N, θ) motion, the main contribution in the second moment μ_2 appears to be due to the regular component in the (N, θ) motion whose existence is not surprising since the stochasticity parameter $K \approx 2.66$ is not very large.

In order to check the theoretical prediction (50), we plot in Fig. 11 the ratio $\mu_2/(\mu_1 n_{20})$ which, according to (51), should be roughly equal to one since in this case $2\sigma/n$



Fig. 8. Dependence of the classical (dashed curve) and quantum (full curve) second moments of the distribution f(n) on the number of microwave periods τ . Parameter values are $n_0 = 66$. $\omega_0 = 2.5$, $\epsilon_0 = 0.04$, and $n_{20} = 15$. The straight, dotted line gives the theoretical estimate for the classical diffusion rate (see the text).



Fig. 9. Classical (dashed curve) and quantum (full curve) excitation probability $W_{1.5}$ above the level $n = 1.5 n_0$ as a function of the number τ of microwave periods for the same parameter values of Fig. 8. In the figure, the quantum probability is multiplied by a factor 100.



Fig. 10. Classical (dashed curve) and quantum (full curve) distribution functions, averaged in the time interval $110 < \tau \le 120$, versus the number of absorbed photons $N = (1/2n_0^2 - 1/2n^2)/\omega$ for the same parameter values of Fig. 9. The straight, dotted line is the one-dimensional, quantum, theoretical exponential distribution; the dotted-dashed curve is the analytical solution of the Fokker-Planck equation.

<< 1. Numerical results satisfactorily confirm this theoretical prediction and allow us to extract the value of the characteristic regular time scale. Indeed, from (50) we have $\mu_2 \approx 2 n_{20}^2 (\tau/\tau_r)^2$ and from Fig. 11 we obtain $\tau_r \approx$



Fig. 11. Time dependence of second moments μ_2 of n_2 for the classical (dashed line) and quantum (full curve) distributions for the same parameter values of Fig. 10. Here we also show the ratio of n_2/n_{20} to the first moment μ_1 of n_2 in the classical case (open circles) and in the quantum case (full circles). It is seen that this ratio is close to one (right-hand scale) in agreement with the theoretical expression (51).

750. Then we can also find the corresponding value of the factor $\lambda = \overline{\cos \theta}$ which according to (53) is given by $\lambda = n \omega_0 / (k \tau_r) \approx 0.14$.

The relatively small value of λ may be related to the fact that the value $\omega_0 = 2.5$ lies just between two main resonances, so that the stable region is relatively small for the chosen initial conditions. It follows that the regular time scale $t_r \approx 300$ is much larger than the localization time $t_D \approx l_{\phi} \approx 1$ and even larger than the ionization time $t_I \approx 160$.

The distribution function $F(n, n_2)$ is shown in Fig. 12. In Fig. 13 we show the level curves $F(n, n_2) = \text{const}$ for the quantum distribution $F(n, n_0)$, for four different cases with the same $\epsilon_0 = 0.04$, $\omega_0 = 2.5$, $n_0 = 66$, and different values of n_2 . It is interesting to observe that the stronger excitation in energy occurs for extended states. The reason is that for such states the value of the integral H is \approx 1 [see (40) and (41)]; instead, for states with $n_2 \sim n$, the value of H becomes smaller than one and this decreases the localization length [see (63)].

All the obtained results show a localization phenomenon close to the 1-D case, and it is not affected by the strong excitation taking place inside the energy shell. The reasons of this behavior are the existence of the approximate integral of motion H and the long time scale associated with the (l, ψ) motion.

As previously noted, the reason why 1-D localization persists in the 2-D case is the Coulomb degeneracy. Therefore, in order to get 2-D delocalization one needs to destroy this degeneracy. One way is the introduction of a static field as described above. Another way would be to consider the ionization of Rydberg alkali atoms where the quantum defects eliminate the degeneracy of unperturbed levels. However, since for $l \ge 3$, quantum defects are very small, even for such atoms 1-D localization should be expected in practice. We performed several numerical simulations of situations in which a quantum defect was introduced. We still observed localization in number of photons close to the 1-D value (24). At the same time, all levels within one shell were strongly excited. We excited



Fig. 12. An example of a full quantum probability distribution $F(n, n_1)$ for the same case of Fig. 10.

different initial conditions with a given value of l or with given parabolic quantum numbers. In both cases, practically all the shell was excited; however, the excitation probability in energy $W_{1.5}$ and the moment of distributions over n were the same as without quantum defects.

VIII. LOCALIZATION IN ELLIPTICALLY POLARIZED FIELDS

In the foregoing sections, we discussed the localization phenomenon for H atoms in a linearly polarized monochromatic field. We will presently show that the same phenomenon occurs in the case of elliptically polarized fields, too.

Let the field vector, in (xyz) coordinates, be

 $\boldsymbol{\varepsilon} = (\epsilon \cos \alpha \cos \omega t, \epsilon \sin \alpha \sin \omega t, 0)$

where α is the polarization angle. It is convenient to use Euler angles φ , ψ , θ related to the usual Euler angles φ' , ψ' , θ' introduced, e.g., in [42] by $\varphi = \varphi' + \pi/2$, $\theta = \theta'$, $\psi = \psi' - \pi/2$. Then, taking into account that $\cos \theta = m/l$, we get the following Hamiltonian

$$H = -1/2n^{2} + \epsilon n^{2} \Big\{ \cos \alpha \cos \omega t \Big[(\cos \varphi \cos \psi - (m/l) \sin \varphi \sin \psi) x' + \\ - (\cos \varphi \sin \psi + (m/l) \sin \varphi \sin \psi) y' \Big] + \\ + \sin \alpha \sin \omega t \Big[(\sin \varphi \cos \psi + (m/l) \\ \cdot \cos \varphi \sin \psi) x' + \\ - (\sin \varphi \sin \psi - (m/l) \cos \varphi \cos \psi) y' \Big] \Big\}$$

where

$$x' = (3/2) e - 2 \sum_{s=1}^{\infty} x_s \cos s$$
$$y' = -2 \sum_{s=1}^{\infty} y_s \sin s\theta$$

θ



$$G(\overline{N}, \phi; \overline{l}, \psi; \overline{m}, \varphi)$$

= $\overline{N}\phi + \overline{l}\psi - 2\pi (-2\omega\overline{N})^{-1/2} - k \Big\{ \cos \phi \Big[\cos \varphi \cos \psi (\cos \alpha (1 - \omega\overline{N}\overline{l}^2)) \Big] \Big\}$

 $-\sin\varphi\sin\psi(\cos\alpha(1-\omega\overline{N}\overline{l}^{2}))$ $\cdot\overline{m}/\overline{l} + \sin\alpha(1.09)\omega^{1/3}\overline{l}) -$ $-\sin\phi[\cos\varphi\sin\psi(\cos\alpha(1.09))$ $\cdot\omega^{1/3}\overline{l} + \sin\alpha(1-\omega\overline{N}\overline{l}^{2}))\overline{m}/\overline{l}) +$ $+\sin\varphi\cos\psi(\cos\alpha(1.09)\omega^{1/3}\overline{m}) +$ $+\sin\alpha(1-\omega\overline{N}\overline{l}^{2}))] \}.$ (68)

N, ϕ , and k have here the same meaning as in Section VI. Again following the method of Section VI, we introduce a new phase θ conjugated to N, and we finally get the map

$$\overline{N} = N - kH \sin \theta$$

$$\overline{\theta} = \theta - 2\pi (-2\omega \overline{N})^{-3/2}$$
(69)

where

$$H^{2} = \left[\cos \varphi \cos \psi (\cos \alpha (1 + l^{2}/2n^{2}) + \sin \alpha 1.09 \omega^{1/3}m) - - \sin \varphi \sin \psi (\cos \alpha (1 + l^{2}/2n^{2}) + m/l + \sin \alpha 1.09 \omega^{1/3}l)\right]^{2} + \left[\cos \varphi \sin \psi (\cos \alpha 1.09 \omega^{1/3}l) + \sin \alpha (1 + l^{2}/2n^{2}) m/l) + \sin \varphi \cos \psi (\cos \alpha 1.09 \omega^{1/3}m) + \sin \alpha (1 + l^{2}/2n^{2}))\right]^{2}.$$
 (70)

Equation (69) was obtained from the much more complicated map described by the generating function (68), under the assumption of negligible dependence of H on N. In this approximation, the (N, θ) motion is decoupled; instead, the $(l, \psi; m, \varphi)$ motion in the new time $\sigma = k \int \cos \theta \, dt$ is described by the Hamiltonian H. These dynamics, in principle, can be chaotic. (Notice that the 1-D case corresponds to $\varphi = 0$, $\alpha = 0$, m = l.) If initially the plane of the motion coincides with the plane of the field $(m = l, \varphi = 0)$, then the microwave will not change this situation. Moreover, if the field is circularly polarized ($\alpha = \pi/4$), then H (70) simplifies to

$$H^{2} = (1/\sqrt{2}) \left[(1 + l^{2}/2n^{2}) + 1.09 \,\omega^{1/3} l \right]^{2}.$$
 (71)

Since (71) is independent of phase, the orbital momentum is an approximate integral of the motion.

Therefore, we can conclude that the localization picture that was discussed for the 1-D model in Section III still applies, even in the general case of elliptically polarized fields, with the localization length in number of photons given by (63).

It is interesting to remark that the classical model for the hydrogen atom in a circularly polarized field is formally very similar to a model for the motion of a comet in the solar system in which only the influence of the Sun and Jupiter is considered. Indeed, a map of the type (69) for the comet motion, in the case that the comet perihelion lies outside Jupiter's orbit, was obtained in (43). The Halley comet itself is described by a map very similar to (69), as discussed in [44] (in that case, however, sin θ is replaced by a sawtooth function).

IX. CLASSICAL PICTURE OF ABOVE-THRESHOLD IONIZATION

The distribution in energy of ionized electrons is now attracting much attention [45]. As we discussed in Section IV, our map description allows for some predictions about this distribution. On the other hand, since the Kepler map was derived under the assumption of not too intense a field, that picture of the distribution of electrons cannot be expected to be valid for strong fields. We shall presently analyze the conditions of applicability of the map and we will obtain a description of the distribution of ionized electrons in the complementary regime of a very strong field. It will appear that in this regime the distribution is sharply-peaked around some energy value which lies significantly above the ionization border.

In the one-dimensional case, the map description, which reduces the effect of the field to "kicks" near the perihelion, can only be valid when the individual kicks are large in comparison to the energy of the motion in the free field. A single kick corresponds to the energy $k\omega = 2.58$ $\epsilon\omega^{-2/3}$ [see (13)]. Therefore, since the energy of the freefield induced oscillations is ~0.5 (ϵ/ω)², the quantitative condition for the validity of the map is

$$0.5 (\epsilon/\omega)^2 \ll 2.58 \ \epsilon \omega^{-2/3}, \text{ i.e., } \epsilon \ll \epsilon_{\text{ATI}} \approx 5\omega^{4/3}.$$
(72)

In the opposite case, when $\epsilon \gg \epsilon_{ATI}$, a different picture is valid, that can be obtained by estimating the distance r from the nucleus at which the Coulomb interaction significantly influences the free-field motion: $1/r \sim (\epsilon/\omega)^2$. Then the Coulomb interaction will be effective during a time $t \sim r/v$, where the velocity v is $\sim (\epsilon/\omega)$, so that $t \sim (\omega/\epsilon)^3$. Therefore the change in field phase during the interaction is $\omega t \sim (\omega^4/\epsilon^3) \ll 1$. Then the interaction with the nucleus looks like a collision with an elastic wall. This suggests that the ionization process can be studied on a model in which the electron interacts with the free field in the presence of an elastic wall. If the amplitude $x \sim (\epsilon/\omega^2)$ of the oscillations in free field is much larger than the size of the unperturbed orbit, i.e., if $(\epsilon/\omega^2) > n_0^2$, then the appropriate initial conditions for the wall model will be given by a point very close to the wall with a very small initial velocity. The average velocity of an electron escaping to infinity will be determined by the initial phase of the field, which is supposed to be switched on suddenly. Moreover, we shall assume that the field is adiabatically switched off at infinity, because this corresponds to the physical situation in which the electron slowly escapes from the interaction region. Then the energy distribution of ionized electrons will correspond to the energy distribution of escaping electrons that is obtained by varying the initial phases of the field ϵ $\cos(\omega t + \varphi).$

We obtained the dependence of the energy on the initial field phase by numerical simulation of the above described wall model (Fig. 14). This figure can be used to determine the distribution of final energies in an ensemble of homogeneously distributed initial phases, which corresponds to the physical situation. The approximately 50 percent initial phases that in Fig. 14 lead to the same final energy $0.5 (\epsilon/\omega)^2$ corresponds to a δ -function in the distribution, centered on that energy. In the region $0 < \varphi$



Fig. 14. Dependence of the ionization energy E on the field phase φ for the wall model.

 $< \pi/2$, the particles undergo a large number of collisions, increasing to infinity as $\varphi \rightarrow 0$. For $\pi/2 < \varphi < \pi$, there are no collisions at all. For $\pi < \varphi < 3\pi/2$, the number of collisions ranges from 1 to ∞ each number of collisions corresponding to a local maximum in the curve. From $3\pi/2 < \varphi < 2\pi$, the number of collisions is inversely proportional to the initial velocity, which we assume to be very small. To check the predictions of this simple wall model, we carried out numerical experiments on the exact 1-D model. For each fixed phase of the field, we integrated a number ~ 500 of trajectories with the same initial energy and homogeneously distributed field and orbit phases. Fig. 15 shows how the distribution changes as the field strength is increased.

In Fig. 15(a), $\epsilon \ll \epsilon_{ATI}$ and we have a smooth distribution over a range determined by the strength of one kick. Fig. 15(c) corresponds to $\epsilon \gg \epsilon_{ATI}$ and the distribution is sharply peaked around the energy $E = 0.5 (\epsilon/\omega)^2$, as in the wall model. The intermediate region is illustrated in Fig. 15(b), with $\epsilon \approx \epsilon_{ATI}$. The transition between the two opposite regimes is not a very sharp one; therefore we determined the interval of values of ϵ in which it took place and compared its dependence on ω to check the theoretical estimate (72). The result is shown in Fig. 16 which demonstrates a satisfactory agreement.

In the quantum case, this classical picture of ionization will be valid when k >> 1 (if $\epsilon \ll \epsilon_{AT1}$), or, if $\epsilon >> \epsilon_{AT1}$, when 0.5 $(\epsilon/\omega)^2 >> \omega$ (one quantum). Indeed, in the latter case the quantum distribution would display several peaks spaced by one quantum, but the distance between such peaks will be much less than the typical energy 0.5 $(\epsilon/\omega)^2$ of ionized electrons; therefore a sharp peak will still be observed at an energy significantly higher than one quantum.

X. COMPARISON WITH LABORATORY EXPERIMENTS

The present-day experimental technique allows for preparation of atoms in states with principal quantum number $n \sim 100$. One procedure (Bayfield and Koch [24], [40], [46]-[49]) is to inject a 1 KeV proton beam through a noble gas; in this way, due to charge exchange, a beam of atoms with $n \sim 10$ is produced. Then high levels $n \sim 10$



Fig. 15. Distribution in energy of ionized electrons, for an ensemble of 500 classical orbits with homogeneously distributed field-and orbit phaseand scaled frequency $\omega_0 = 2$. (a) $\epsilon_0 = 1 \ll \epsilon_{0_{ATI}} \approx 13$; (b) $\epsilon_0 = 20$. (c) $\epsilon_0 = 80$. Arrows show the amplitude of one kick.



Fig. 16. Field intensity intervals (vertical bars) in which the transition from the Kepler-map regime to the wall-model regime was observed in numerical simulations of the exact, classical model of the hydrogen atom. The straight line is the theoretical estimate (72).

100 can be excited by means of a CO_2 laser. The beam then passes through a microwave cavity. Static field ionization is finally used to analzye the quantum numbers in the beam emerging from the interaction region; the direction of the static field is the same as that of the microwave field. Another procedure exploits two- or three-step laser excitation of alkali atoms in a thermal beam [50].

To the best of our knowledge, until now laboratory experiments (Bayfield, Koch) have been carried out inside or close to the classical chaotic region. An essential feature of these experiments was the relatively low frequency $\omega/2\pi \approx 9.9$ GHz (or even less); this means that for the practical totality of excited levels, the frequency ω_0 is less than one, and indeed in most cases $\omega_0 \sim 0.5$. For such frequencies either one falls in the classically stable region ($\epsilon_0 < \epsilon_c$) or, if the classical chaotic border is exceeded, the quantum motion is delocalized according to the analysis given here (see also Fig. 17). Therefore, in both cases the quantum behavior is expected to agree with classical predictions. For example, in a typical experimental condition, where $\omega_0 \approx 0.5$, $\epsilon_0 \approx 0.04$, $n_0 = 66$, the localization length (24) is $l_{\phi} \approx 230 >> N_I \approx 66$.

In Fig. 17 we plot recent experimental data [47], [48] for the field ϵ_0 at which 10 percent ionization is obtained during the interaction time ($\tau \approx 300$ field periods), with a fixed frequency $\omega/2\pi = 9.9$ GHz. The change in ω_0 was obtained by varying n_0 . These values of ϵ_0 can be identified with the ionization border because, for $\omega_0 < 1$, the interaction time is long enough. In the same Fig. 17 we compare the experimental results with two theoretical borders: the classical chaotic border ϵ_c and the quantum delocalization border ϵ_q . The value ϵ_c is given by formula (18a) which can be assumed to hold down to $\omega_0 \approx 0.5$. For smaller values of ω_0 , we smoothly extrapolated to the critical value for ionization border for fixed ω is given by

$$\epsilon_q \approx \left(\omega^{1/6}/\sqrt{66}\right) \omega_0 \approx 0.0417 \ \omega_0. \tag{73}$$

The actual ionization border for a given ω_0 is determined by the highest of these two values. Fig. 14 indicates a satisfactory agreement of the existing data with the theory; unfortunately, there is no available data above $\omega_0 =$ 1, where some deviation from the classical behavior begins to show up. Our prediction is that, by further increasing the initially excited level n_0 , the localization phenomenon should become more and more evident and the experimental data should deviate from classical predictions. However, actual experimental data should not be expected to faithfully reproduce the theoretical border (73) which gives the condition for strong ionization. Indeed, whereas (73) defines the threshold for strong excitation into the continuum, in actual experiments ionization is identified with excitation beyond some level \bar{n} so that the experimental threshold may turn out to be even appreciably lower than (73), depending on the actual value of \overline{n} . Nevertheless, we predict that they should be significantly higher than the classical threshold. Of course, our theory allows for estimates of experimental thresholds, once the



Fig. 17. The classical chaotic border equation 18(a) (full curve) as a function of a rescaled frequency ω_0 . Since (18a) can be assumed to hold only down to $\omega_0 \approx 0.5$, the dotted line is an extrapolation to the critical ionization value in the static field. The points are experimental results and give the threshold field values for 10 percent ionization probability which is the total probability above a given unperturbed level $n = \overline{n}$. The dashed curve gives the theoretical field value computed according to our quantum theory for 10 percent probability above level $\overline{n} = 180$.

actual values of \bar{n} and of the percent excitation beyond \bar{n} which defines the empirical threshold are known. We should also remark that the experiments of Fig. 17 used a microcanonical distribution inside the initially excited shell; nevertheless, we have shown in this paper that the 1-D approximation describes the excitation process sufficiently well. A numerical check of the quantum threshold for a wide range of frequencies ω_0 is given in [51].

We also wish to mention that a discrepancy between quantum and classical predictions can be already observed for not very large values of ω_0 . As an example we plotted in Fig. 18 the dependence of the excitation probability on the number of microwave periods for $\omega_0 = 1$ and $n_0 = 63$, $n_0 = 66$ at $\epsilon_0 = 0.03$ in the 1-D case. According to Fig. 18(a), for $n_0 = 63$ the quantum excitation probability is higher than for $n_0 = 66$ but is still smaller than in the classical case. For $n_0 = 66$ the excitation is much smaller in the quantum than in the classical case.

The quantum excitation for $n_0 = 63$ was also computed by Bardsley [52] who obtained a slightly higher value. In that paper the critical field for 10 percent ionization was shown to be approximately the same in the classical and in the quantum case. There is no contradiction here with our Fig. 18; indeed, the ionization probability, close to the chaos border, is a steep function of field strength, so that a small change in the later may yield large discrepancies between classical and quantum probabilities [see Fig. 18(a)]. Fig. 18(b) shows that also in the 2-D case the quantum excitation is smaller than the classical. Moreover, the 1-D and 2-D excitation probabilities are close to each other, both in the quantum and in the classical case. We remark that, even though initially $n_2 = 0$, after $\tau =$ 40 almost the whole shell $n_0 = 66$ was excited ($\mu_2 \approx$ 100) so that we are facing a truly 2-D case. As a matter of fact, we had here a short regular time scale $t_r \approx 13$, and this led to fast mixing inside the shell. However, this mixing took place essentially in the regular component of motion $(t_{ch} \sim 300 \gg t_r)$; therefore, the 2-D and 1-D



Fig. 18. Excitation probability $W_{1.5}$ as a function of the number of microwave periods τ . (a) One-dimensional case with $\epsilon_0 = 0.03$, $\omega_0 = 1$. The dashed curves give the quantum results for $n_0 = 63$ and $n_0 = 66$ ($W_{1.5}$ is averaged here over five microwave periods). The solid line represents the classical case. (b) Comparison between 1-D and 2-D classical and quantum computations for $\epsilon_0 = 0.03$, $\omega_0 = 1$, $n_0 = 66$. In the 2-D computations, n_2 was set initially equal to zero. (\Box) Classical case 1-D; full line: classical case 2-D; dashed line: quantum case, 1-D; (O) quantum case, 2-D.

probabilities were close, because the regular component does not contribute to excitation.

XI. CONCLUSIONS

In this paper we have attempted a general formulation of the theoretical reasons why we hold the quantum localization phenomenon to play a central role in the microwave excitation of Rydberg atoms.

Thanks to the "Kepler map" formulation of the dynamics of highly excited H-atom in a microwave field, we have been able to give a simple description of this phenomenon for the simplified one-dimensional model, and then to demonstrate that the picture of the excitation process obtained in this way is essentially unaltered on going over to a more realistic 2-D model. The assumption that the main contribution to ionization is given by extended states thus receives theoretical grounds, but the relevance of our results goes beyond that. Showing that the heavy impact of localization on the excitation process originally predicted on the 1-D model is not just an artifact of the 1-D dynamics greatly substantiates the expectation that this phenomenon may be detected in laboratory experiments.

A number of questions that are still open that should be analyzed in order that the physical relevance of localization may be exactly appreciated. For example, the sensitivity of localization to certain modifications of our basic model, such as, e.g., the introduction of noise, should be investigated. Our research work is currently developing in this direction.

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