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Few interacting particles in a random potential

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Abstract. – We study the localization length and the Breit-Wigner width of few interacting particles in a random potential. Concentrating on the case of three particles, we show that their localization length is strongly enhanced comparing to the enhancement for two interacting particles.

Recently it was shown that in a random potential two repulsing/attracting particles can propagate coherently on a distance l_c which is much larger than one-particle localization length l_1 in the absence of interaction [1]. In some sense interaction destroys quantum interference which leads to one-particle localization and creates an effective pair of two particles of size l_1 propagating on a large distance. For better understanding of this result Imry developed [2] a scaling block picture of localization for interacting particles which can be applied in principle for a larger number of particles and higher dimensions. Intensive numerical investigations by Pichard and coworkers [3] and von Oppen and coworkers [4] confirmed the existence of the two interacting particles (TIP) effect. While some additional checks are still required, the results [3], [4] definitely show that in the one-dimensional case the TIP length is $l_c \propto {l_1}^{\alpha}$, with α close to the theoretically predicted power [1] $\alpha = 2$. These results are also in agreement with previous studies by Dorokhov who analyzed the case of two particles confined by strong attraction in a well with size much smaller than l_1 [5]. Investigations of TIP effect in higher dimensions were done in [2], [6], [7] and they demonstrated that in dimension $d = 3$ the TIP pair can be delocalized below one-particle Anderson transition where all one-particle states are localized. It is interesting to note that possibilities of delocalization due to interaction via collective motion have been discussed long ago [8].

While now the properties of TIP propagation reached a level of qualitative understanding, the problem of a larger number of interacting particles is still not well understood. From the physical point of view the most interesting situation is the case of finite density of particles. However, the analysis in this case is quite complicated and at present only estimate [2] and

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numerical studies in [9] have addressed this problem. One of the ways to approach this problem is to analyze the case of a larger number of particles. The simplest case is three interacting particles where the situation is not so trivial since two-particle interaction leads to the Breit-Wigner structure of eigenstates [10]-[13]. In this paper we will concentrate on this three-particle model.

However, before the analysis of three-particle model, let us first discuss a generalized model of TIP (see also [14]). In this model two particles are moving separately in two independent parallel strips. Each strip has its own disordered potential. The particles interact with each other but each of them remains in its own strip. If the strips are identical then the model is equivalent to two interacting particles moving in the same strip. However, here we analyze the case of different strips. The first strip with the first particle is represented by the standard one-channel Anderson model with diagonal disorder which changes in the interval $\pm W_1$. The equidistant sites on this one-dimensional lattice are marked by the index n_1, the intersites hopping matrix element is V_1. For the first particle the localization length at the center of the band is $l_1 \approx 25(V_1/W_1)^2 \gg 1$. The second particle is moving in a quasi-one-dimensional Anderson model which forms the second strip with M transverse channels with sites marked by index n_2 along the strip and site index $\tilde{n}_2$ ($1 \leq \tilde{n}_2 \leq M$) in the transverse direction. The disorder in the second strip is independent of disorder in the first one, even when the number of channels in the second strip is the same as in the first one ($M = 1$). The amplitude of disorder in the second strip is $\pm W_2$. We note that the first strip has only one channel, while the second has M channels. The localization length for the second particle is $l_2 \propto M$. The hopping in the second strip is only between nearest sites and is $V_2 \approx V_1 \approx V$. We will assume that $l_2 > l_1$. The interaction between two particles in the strips is of the form $U\delta_{n_1,n_2}$. It is independent of the transverse index $\tilde{n}_2$ in the second strip. We put $\hbar$ and lattice constant a equal to unity. The eigenvalue equation for this model is

$$\begin{aligned}(E^{(1)}{}_{n_1} + E^{(2)}_{n_2,\tilde{n}_2} + U\delta_{n_1,n_2})\psi_{n_1,n_2,\tilde{n}_2} + V_1(\psi_{n_1+1,n_2,\tilde{n}_2} + \psi_{n_1-1,n_2,\tilde{n}_2}) + \\ + V_2(\psi_{n_1,n_2+1,\tilde{n}_2} + \psi_{n_1,n_2-1,\tilde{n}_2} + \psi_{n_1,n_2,\tilde{n}_2+1} + \psi_{n_1,n_2,\tilde{n}_2-1}) = E\psi_{n_1,n_2,\tilde{n}_2}\,,\end{aligned} \tag{1}$$

where $E^{(1,2)}$ are site energies in the strips with their own disorder.

To find the TIP localization length, one should first estimate, similarly to [1] (eqs. (1), (2) therein), the transition matrix elements U_s between eigenstates without interaction ($U = 0$). This gives

$$U_s = U \sum_{n_1,n_2,\tilde{n}_2} R^{+}_{n_1,m_1}\tilde{R}^{+}_{n_2,\tilde{n}_2,m_2,\tilde{m}_2} R_{n_1,m_1'}\tilde{R}_{n_2,\tilde{n}_2,m_2',\tilde{m'}_2}\delta_{n_1,n_2}\,, \tag{2}$$

where $R, \tilde{R}$ represent the transformation between the lattice basis and one-particle eigenstates so that $R_{n_1,m_1} \approx \exp[-\mid n_1 - m_1 \mid/l_1 - i\theta_{n_1,m_1}]/\sqrt{l_1}$ and $\tilde{R}_{n_2,\tilde{n}_2,m_2,\tilde{m}_2} \approx \exp[-\mid n_2 - m_2 \mid/l_2 - i\theta_{n_2,\tilde{n}_2,m_2,\tilde{m}_2}]/\sqrt{Ml_2}$ correspondingly for the first and second particle, indices m_1 and $m_2, \tilde{m}_2$ mark noninteracting eigenstates for the first and second particles, correspondingly. The phase θ randomly changes with indices. Due to the exponential decrease of R, one should take into account only the states with $\mid n_{1,2} - m_{1,2} \mid < l_{1,2}$. For the case $l_2 > l_1$ the sum in (2) contains approximately $l_1 M$ random terms so that $U_s \approx U/(l_2\sqrt{l_1 M})$. The interaction-induced transition rate is given by the Fermi golden rule $\Gamma \sim {U_s}^2\rho_{\mathrm{c}}$, where $\rho_{\mathrm{c}} \approx l_1 l_2 M/V$ is the density of states coupled by direct transitions. As a result, $\Gamma \sim U^2/(Vl_2)$ is independent of l_1 and M. As has been discussed in [10], it determines the Breit-Wigner width of the local density of states. The number of levels inside the Breit-Wigner peak is approximately $\Gamma\rho_{\mathrm{c}}$ and the peak should contain many levels ($\Gamma\rho_{\mathrm{c}}$) to have the above estimates valid (see also [10]-[12]).

With the rate Γ we can determine the interaction-induced diffusion rate for the first particle which is $D_1 \sim {l_1}^2\Gamma \sim U^2{l_1}^2/(Vl_2)$ and appears as a result of collisions of the first particle with the second one oscillating in the block of size l_2. Knowing the diffusion rate, it is possible to determine the localization length for a pair in a way similar to that used for the kicked rotator [15] and based on the uncertainty relation between the frequency and time (see also [14]). Indeed, the number of excited states in the first strip grows with time t as $\Delta n_1 \sim (D_1 t)^{1/2}$. Since two particles are propagating together so that $\mid n_1 - n_2 \mid < l_2$ the total number of excited states in both strips is $\Delta N \sim \Delta n_1 (M l_2)\delta E/V$, where δE takes into account the fact that the states are excited only in some energy interval inside the band width V. Generally, $\delta E < V$ and it is of the order of Breit-Wigner width Γ [10], but we will see that δE does not enter into the final expression for the localization length of the pair, and therefore actual value of δE is not very important (see also [15]). Indeed, all these ΔN levels are homogeneously distributed in the energy interval δE and the average splitting between them is $\Delta\nu \sim \delta E/\Delta N$. According to the uncertainty relation between frequency and time at the moment t, we can resolve discrete lines with the splitting $1/t$. Therefore, at the moment t^* defined by the equation $\Delta\nu \sim 1/t^*$ the discreteness of the spectrum is resolved and the diffusive propagation is stopped at $t^* \sim \Delta N(t^*)/\delta E$. This condition gives the localization time t^* for TIP pair and the localization length for the first particle l_{c1}:

$$t^* \sim U^2{l_1}^2 M^2 l_2/V; \quad l_{c1} \sim \Delta n_1 \sim (U/V)^2{l_1}^2 M\,. \tag{3}$$

The interesting feature of this result is that l_{c1} is not directly dependent on l_2. Indeed, for $M = 1$ the length l_{c1} is the same as in the case of TIP localization in one-dimensional Anderson model [1] and is independent of l_2 and the disorder in the second strip. However, the growth of the number of channels M in the second strip leads to the increase of l_{c1}. The localization length for the second particle is $l_{c2} \sim l_2$ if $l_2 \gg l_{c1}$ and $l_{c2} \sim l_{c1}$ if $l_2 \ll l_{c1}$. A similar approach can be used for analysis of TIP localization in higher dimensions [14].

Let us now consider three interacting particles in the same one-dimensional Anderson chain $E_n\phi_n + V(\phi_{n+1} + \phi_{n-1}) = E\phi_n$ with on site interaction $U_{12}\delta_{n_1,n_2}$, $U_{23}\delta_{n_2,n_3}$ and $U_{13}\delta_{n_1,n_3}$, where $n_{1,2,3}$ marks the site position of corresponding particle in the chain. As above, the one-particle localization length is l_1 and the band width is $4V$. For simplicity we will assume that $U_{13} = 0$ and $U_{23} > U_{12}$. However, this assumption is not of principal importance and all final expressions are correct also in the case of $U_{13} \sim U_{23} \sim U_{12}$. For $U_{13} = 0$ in first approximation the particles 2-3 form a pair of size l_1 which is localized on the length $l_{c2} \sim (U_{23}/V)^2{l_1}^2$. When this pair approaches the first particle at a distance l_1 the interaction between three particles in a block of size l_1 gives mixing between ${l_1}^3$ 3-particle states. An effective matrix element U_{s3} of interaction between 3-particle states in the block of size l_1 should be calculated in the second-order perturbation theory, since direct interaction couples only 2-particle states. Therefore, the matrix element between initial state $|123\rangle$ and final state $|1'2'3'\rangle$ is given by diagram presented in fig. 1 with intermediate state $|1'\bar{2}3\rangle$. It is of the form

$$U_{s3} = \sum_{\bar{2}} \frac{\langle 12|U_{12}|1'\bar{2}\rangle\langle\bar{2}3|U_{23}|2'3'\rangle}{(E_1 + E_2 + E_3 - E_{1'} - E_{\bar{2}} - E_3)} \sim \frac{U_{12}U_{23}}{{l_1}^3\Delta_1}\,. \tag{4}$$

It is important that the summation is carried out only over single-particle states $\bar{2}$, hence $\Delta_1 \sim V/l_1$ is a single-particle level spacing. Finally, this gives the mixing rate in a block of size l_1

$$\Gamma_3 \sim {U_{s3}}^2\rho_3 \sim (U_{12}U_{23}/V^2)^2 V/l_1\,, \tag{5}$$

where $\rho_3 \sim {l_1}^3/V$ is the density of 3-particle states in the block. This Γ_3 gives the mixing rate during the collision of the first particle with the pair 2-3 in the block l_1. Let us first

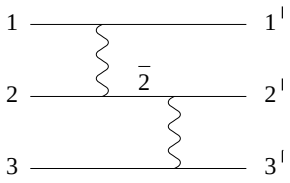

Fig. 1. – Diagram for the effective three-particle matrix element U_{s3} in (4).

discuss in more detail the case when all three particles are moving in a system of size l_1 and $U_{12} \sim U_{13} \sim U_{23} \sim U$. From (4) it follows that in such a block the number of 3-particle mixed levels is $\xi_3 \sim \Gamma_3\rho_3 \sim (U\sqrt{l_1}/V)^4$. This shows that many 3-particle states are mixed ($\xi_3 > 1$) only when there is a strong mixing of 2-particle states: $\xi_2 \sim \Gamma_2\rho_2 \sim (U\sqrt{l_1}/V)^2 > 1$ with $\Gamma_2 \sim (U/V)^2 V/l_1$ and $\rho_2 \sim l_1^2/V$. It is interesting to note that for $U < V$ always $\Gamma_3 < \Gamma_2$. Moreover, $\Gamma_3 < \Delta_2 = 1/\rho_2$ for $U/V < 1/{l_1}^{1/4}$. Similarly to the TIP case [13], the spectral rigidity characterized by $\Sigma_2(E)$ should have behavior as for random matrices for energy $E < \Gamma_3$, while for $E > \Gamma_3$ its behavior should be closer to the Poisson case of uncorrelated levels. For $U < V$ the width Γ_3 is smaller than Γ_2 and the eigenstates are not ergodic in the Hilbert space of eigenstates at zero interaction on the energy interval Γ_2. Let us mention that for k interacting particles in the block l_1 the mixing rate in the k-order of perturbation theory is $\Gamma_k \sim (U/V)^{2(k-1)}V/l_1$, the density of k-particle states is $\rho_k \sim {l_1}^k/V$ and the number of mixed levels $\xi_k \sim \Gamma_k\rho_k \sim (U\sqrt{l_1}/V)^{2k} > 1$. For $U < V$ we have $\Gamma_{k+1} < \Gamma_k$ so that total level mixing (ergodicity) in the energy interval Γ_2 is possible only for $U \sim V$ when for $k > 2$ the width $\Gamma_k \sim \Gamma_2$.

Let us now return back from one block l_1 to 3-particle dynamics on infinite chain. The frequency of collisions of the first particle with the pair 2-3 is of the order of $l_1/l_{c2} \sim 1/(U_{23}/V)^2 l_1$ since from ergodicity the ratio of time of the collision to the time between collisions is proportional to the ratio of volumes. Therefore, the average transition rate for 1-particle per unit time is $\tilde{\Gamma}_3 \sim \Gamma_3 l_1/l_{c2}$. Such transitions give the diffusion rate of the first particle $D_1 \sim \tilde{\Gamma}_3 {l_1}^2 \sim {U_{12}}^2/V$, since the size of transition is l_1. Similarly to the previous case with two chains, the total number of excited states after time t^* is $\Delta N \sim (D_1 t^*)^{1/2}(l_{c2}l_1)\delta E/V$, where δE is an energy width in which the levels are mixed. The localization time t^*, as previously, is determined from the condition $\Delta N \sim \delta E t^*$ which gives the localization length l_{3c1} for the first particle in the three-particle case:

$$t^* \sim D_1(l_{c2}l_1)^2/V^2; \qquad l_{3c1}/l_1 \sim D_1 l_{c2}/V \sim (U_{12}U_{23}/V^2)^2 {l_1}^2 . \tag{6}$$

For $U_{12} \sim U_{23} \sim U$ the localization length for the first particle is enhanced only if there is an enhancement for two-particle localization length, namely $(U/V)^2 l_1 > 1$. This result is quite natural since for $(U/V)^2 l_1 < 1$ the two-particle interaction is too weak and it is not able to mix three-particle levels. Another limiting case in (6) corresponds to $U \sim V$. For such interaction $l_{3c1} \sim {l_1}^3$ which is similar to the case of three particles trapped in a bag of size l_1. Indeed, one can consider 3-particle bag model like the TIP one [1] with effective number of transverse channels $M_{\rm ef} \sim l_1$, therefore for the 3-particle bag $l_{b3} \sim M_{\rm ef} {l_1}^2 \sim {l_1}^3$. The same estimate for l_{b3} was also obtained in [16] basing on the approaches developed in [5], [2]. In some sense the result (6) shows that similar to the TIP case the "size" and "form" of the bag is not important for the effect. Let us also mention that expression (6) is similar to previously analyzed model of TIP in two strips (3). Indeed, here the third particle gives the effective number of channels $M \sim (U/V)^2 l_1$ so that (6) becomes equivalent to (3). Generalization of the result (6) for k particles gives the enhancement $l_{kc}/l_1 \sim ((U/V)^2 l_1)^{k-1}$.

For the 3-dimensional case l_1 in the enhancement factor $(U/V)^2 l_1$ should be replaced by ${l_1}^3$ [2], [6], [14] so the delocalization takes place if $((U/V)^2 {l_1}^3)^{k-1} > 1$. This means that delocalization border for few particles coincides approximately with that for TIP and therefore it is not possible to have a propagating cluster with $k > 2$ repulsive particles. In some sense only TIP pairs are well defined. Finally, we note that the in this paper we discussed only the property of excited states which are relatively far from the Fermi energy. In the case when the TIP energy is close to the Fermi level one should take into account the reduction of the effective density of states. First estimates for such kind of situation were done by Imry [2]. However, more detailed further studies are required for a better understanding of the interaction effects for localization near the ground state.

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