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Two interacting particles in the Harper model

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The dynamics of two particles with a short-range repulsive or attractive interaction is studied numerically in the Harper model. It is shown that an interaction leads to the appearance of localized states and a pure-point spectrum component in the case when the noninteractive system is quasidiffusive or ballistic. In the localized phase an interaction gives only a stronger localization contrary to the case of two interacting particles in a random potential. [S0163-1829(96)06346-1]

The Harper model of electrons on a two-dimensional square lattice in the presence of a perpendicular magnetic field was intensively studied during the last decades (see, e.g., Refs. 1–6). After fixing the quasimomentum in one of the directions the eigenvalue equation is reduced to a very simple form of a one-dimensional quasiperiodic discrete chain:

$$2\lambda\cos(\hbar n+\beta)\phi_n+\phi_{n+1}+\phi_{n-1}=E\phi_n, \tag{1}$$

where the effective Plank's constant $\hbar/2\pi$ gives the ratio of magnetic flux through the lattice cell to one flux quantum and β is a constant related to the quasimomentum. For the original problem of electrons in a magnetic field the parameter λ should be fixed at $\lambda=1$ but generally one can consider the model (1) at different values of λ. Intensive analytical and numerical studies[2,6,3] for typical irrational values $\hbar/2\pi$ showed that for $\lambda>1$ the spectrum is pure-point like with gaps and all eigenstates are exponentially localized. For $\lambda<1$ the spectrum becomes continuous with extended eigenstates corresponding to ballistic classical motion. For $\lambda=1$ the situation is critical with a singular-continuous multifractal spectrum and power law localized eigenstates.

While the properties of the one-particle Harper model are now well understood in many respects the question about the effects of the particle interaction has been not much investigated up to now. Indeed, the main physical problem of interacting particles at a finite particle density is very complicated for both analytical and numerical investigations. One of the approaches to understanding this problem is to analyze the effect of an interaction of only two particles in the Harper model. Recently such an approach has given a number of interesting results for two interacting particles (TIP's) in a random potential showing that even repulsive particles can form a pair which can propagate on a large distance.[7,8] In this paper I address the problem of TIP's in a quasiperiodic potential showing that here interaction effects can be quite different from the case of random potential. The investigation of such a type of model will allow us to analyze the stability of the multifractal spectrum with respect to interactions.

For two particles on a square lattice (x,y) with magnetic flux and on-site interparticle interaction the eigenvalue equation has the form

$$\begin{aligned}&e^{i\hbar y_1}\psi_{x_1+1,y_1,x_2,y_2}+e^{-i\hbar y_1}\psi_{x_1-1,y_1,x_2,y_2}+\psi_{x_1,y_1+1,x_2,y_2}+\psi_{x_1,y_1-1,x_2,y_2}+e^{i\hbar y_2}\psi_{x_1,y_1,x_2+1,y_2}+e^{-i\hbar y_2}\psi_{x_1,y_1,x_2-1,y_2}\\&\quad+\psi_{x_1,y_1,x_2,y_2+1}+\psi_{x_1,y_1,x_2,y_2-1}+\widetilde{U}\delta_{x_1,x_2}\delta_{y_1,y_2}\psi_{x_1,y_1,x_2,y_2}=E\psi_{x_1,y_1,x_2,y_2}.\end{aligned} \tag{2}$$

Here (x,y) are integers marking the sites of the square lattice, the indices 1,2 note two particles, $\widetilde{U}$ is the on-site interaction, and $\hbar=2\pi\phi/\phi_0$ is determined by the ratio of magnetic flux ϕ through the unit cell to the quantum of flux ϕ_0. The direct investigation of Eq. (2) is a quite complicated problem. Therefore, I reduce it to a simpler one with Bloch waves propagating in the x direction $\psi_{x_1,y_1,x_2,y_2}=\varphi_{y_1,y_2}\int dk_1dk_2A_{k_1,k_2}\exp[i(k_1x_1+k_2x_2)]$, which leads to TIP's in the Harper model with an effectively renormalized interparticle interaction:

$$[2\lambda\cos(\hbar n_1+\beta_1)+2\lambda\cos(\hbar n_2+\beta_2)+U\delta_{n_1,n_2}]\varphi_{n_1,n_2}+\varphi_{n_1+1,n_2}+\varphi_{n_1-1,n_2}+\varphi_{n_1,n_2+1}+\varphi_{n_1,n_2-1}=E\varphi_{n_1,n_2}. \tag{3}$$

Here for generality the parameter λ is introduced which should be taken equal to 1 for model (2) and the coordinates $y_{1,2}$ are replaced by $n_{1,2}$. The parameters $\beta_{1,2}$ are $\beta_{1,2}=k_{1,2}$ and the strength of the renormalized interaction is $U=\widetilde{U}\int dkA_{k_1+k_2-k,k}/A_{k_1,k_2}$. The investigation of the properties of model (3) should allow one to understand better the properties of the original TIP problem on two-dimensional (2D) lattice (2).

The time evolution of models (2) and (3) is governed by Eqs. (2) and (3) with E on the right-hand side replaced by

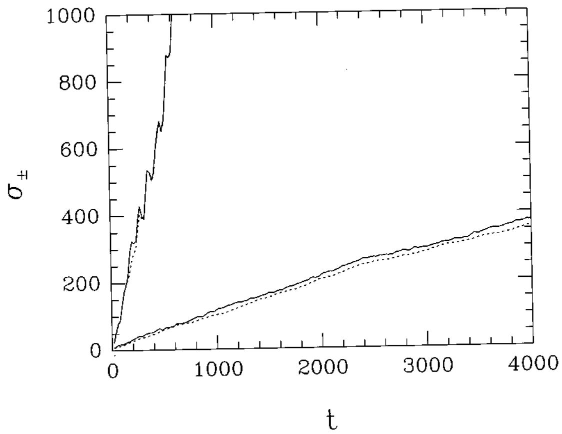

FIG. 1. Dependence of second moments $\sigma_{\pm}=\langle(n_1\pm n_2)^2\rangle$ on time t for TIP's in the Harper model (3) (σ_+ is the solid curve, σ_- is the dashed curve). The parameters are $\lambda=1$, $\hbar=\pi(5^{1/2}-1)$, $\beta_{1,2}=2^{1/2}$; $U=0$ for upper curves, $U=1$ for lower curves. The system size is $N\times N=301\times 301$ sites; initially particles are at the same $n_{1,2}=0$.

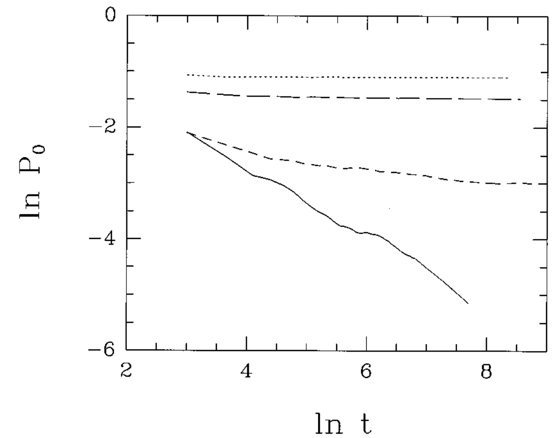

FIG. 3. Dependence of probability to stay at the initial state P_0 on time t for $\lambda=1$, $U=1$; $\lambda=5/6$, $U=5/6$; $\lambda=1$, $U=1/4$; $\lambda=1$, $U=0$ (curves from up to down), and $\hbar,\beta_{1,2}$ are as in Fig. 1.

$i\partial/\partial t$. This evolution for the model (3) was studied numerically on effective two-dimensional lattice with size $N\times N\leqslant 301\times 301$. The flux ratio was fixed at the golden mean value $\hbar/2\pi=(\sqrt{5}-1)/2$. For $\lambda=1$ the spectrum of a noninteracting problem is singular continuous with multifractal properties.[3–5] Due to this fact the spreading over the lattice is similar to the diffusive case with the second moments of probability distribution $\sigma_{\pm}=\langle(n_1\pm n_2)^2\rangle$ growing approximately linearly with time (see Fig. 1). The switched-on interaction leads to a significant decrease in the rate of this growth, namely, approximately 10 times for the case in Fig. 1. Here, initially at time $t=0$ two particles are located at the same site so that all probability is concentrated at $n_1=n_2=0$ (the same initial conditions were used in Figs. 2 and 3). The analysis of the probability distribution $P_{\pm}=\Sigma_{n_{\pm}=\mathrm{const}}P(n_1,n_2)$ dependence on $n_{\pm}=|n_1\pm n_2|/\sqrt{2}$ shows that its tail has Gaussian shape. However, while in the noninteractive case all the probabilities spread diffusively over the lattice in the case with an interaction a large part of probability (around $W_{\mathrm{loc}}\approx 0.9$) remains localized in the vicinity of the initial position of particles within the interval $-5\leqslant n_{1,2}\leqslant 5$ (Fig. 2). The numerical simulations show that in the interacting case the distribution mainly consists of two parts. One of them represents localized states and is frozen near the initial position of particles; another continues to diffuse as in the noninteractive case and corresponds to the Gaussian tail of distribution evolving in a diffusive way. Figure 2 represents a typical distribution shape at an instant moment of time. These numerical data clearly demonstrate the qualitative change induced by interaction: the appearence of localized component.

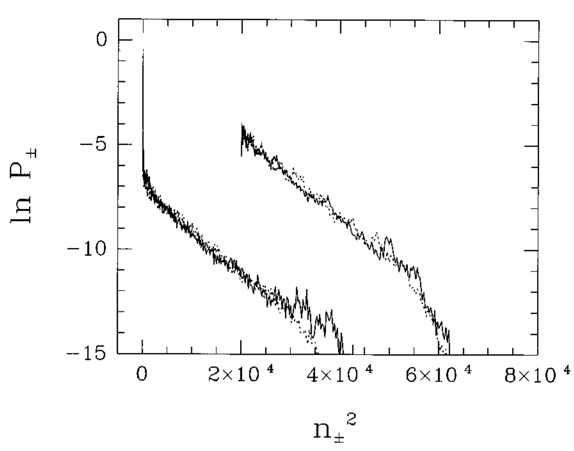

FIG. 2. Dependence of integrated probability distributions P_+ (solid curve) and P_- (dashed curve) on $n_{\pm}^2=(n_1\pm n_2)^2/2$ for the case of Fig. 1 and $t=4000$. The left curves are for $U=1$; right are for $U=0$ (shifted for clarity).

If initially the particles are located on different sites, then the value of W_{loc} decreases with the growth of the initial distance Δr between them but its value still remains quite large if the initial distance is about few sites. For fixed Δr the value of W_{loc} is not sensitive to the initial choice of $n_{1,2}$ and $\beta_{1,2}$. The dependence on time of the probability to stay at the origin P_0 averaged over the time interval $[0,t]$ is shown in Fig. 3. Only in the noninteractive case does P_0 go to zero with time while for nonzero U its value approaches some constant. It is interesting to note that asymptotically P_0 is larger than zero not only for $\lambda=1$ but also for $\lambda<1$ when the noninteracting case has a continuous spectrum with waves ballistically propagating along the lattice. With the interaction decreasing the value of P_0 decreases also but not very sharply (Fig. 3), also it is not sensitive to the sign of U. In the localized phase $\lambda>1$ the interaction gives only a decrease of spreading over the lattice sites similarly to the case with $\lambda\leqslant 1$. For example, in the case of Fig. 1 but with $\lambda=1.05$ and $U=1$ the probability P_0 is approximately 9 times larger than for $\lambda=1.05$ and $U=0$.

The numerical results discussed above demonstrate that the TIP behavior in a quasiperiodic potential is quite different from the case of random potential[7,8] where the interaction produces mainly delocalizing effects. The main reasons for this difference are probably the following. The delocalization in the Harper model (1) at $\lambda=1$ appears as the result of quantum tunneling between the sites with closes energies

$E_n = 2\lambda\cos(\hbar n+\beta)$ which are exponentially far from each other but are close in energy. Apparently, the interaction destroys these tiny resonance conditions, which leads to the appearance of localized states. These states are localized in all directions on the 2D lattice (n_1, n_2) (see Fig. 2). Therefore, they do not correspond to a situation in which the interaction creates a coupled state which can propagate along the lattice. According to the numerical data these localized states are centered on the plane (n_1, n_2) mainly along the diagonal $n_1 = n_2$ (two particles are close to each other) and their structure is approximately the same and independent of the position along the diagonal. This means that these states form a pure-point component in the energy spectrum. The question of how this component is placed with respect to the spectrum of noninteracting particles remains open. One possibility is that this pure-point spectrum is located completely outside of the noninteractive band $[-8,8]$. Such a case in some sense would be similar to an impurity state in a usual ballistic continuous band. Another possibility is that the pure-point spectrum is also partially located inside the band $[-8,8]$ in the gaps which exist for the noninteractive problem. The second possibility looks to be more probable and more interesting. One of the indications in this direction is that the strong decrease of U (Fig. 3) does not lead to the disappearance of the localized component. It is natural to assume that the structure of the pure-point spectrum remains approximately the same for $\lambda < 1$ (at least for not very small values of λ). According to numerical data the singular-continuous part of the spectrum at $\lambda = 1$ is not completely destroyed by the interaction, so that the quasidiffusive spreading still takes place (Fig. 1), but it would be quite desirable to have rigorous mathematical results about the structure of the spectrum in the presence of interaction.

The results discussed above were obtained for the TIP's in the one-dimensional Harper model (3). They indicate that the interaction induced localized states also should exist in the original problem (2) of TIP's on the two-dimensional lattice with magnetic flux. Indeed, here the interaction again should give a destruction of tiny resonance conditions for tunneling. However direct detailed investigations of the model (2) are required to make definite conclusions, but numerical studies of the model (2) are much more complicated than for the model (3). Finally, let us note that in the classical noninteractive limit the dynamics of models (2) and (3) is integrable. Therefore, it would be interesting to study the effect of TIP's in the kicked Harper model[9] where the one-particle classical dynamics is chaotic.

In conclusion, the numerical investigations of TIP's in the Harper model (3) show that the attractive-repulsive interaction leads to appearance of localized states and pure-point spectrum. This happens in the case when noninteractive system $(U=0)$ has quasidiffusive wave packet spreading with a singular-continuous spectrum $(\lambda=1)$ or even for $\lambda<1$ when it has the ballistic wave packet propagation and continuous spectrum. Such an effect of the interaction in quasiperiodic systems is attributed to the interaction induced destruction of tiny resonance conditions which in the noninteractive system allowed one to tunnel between quasiresonant states, leading to infinite wave packet spreading and decay of the probability to stay at the origin to zero. In the localized phase the interaction gives only a decrease of the localization length. Therefore, the situation in the quasiperiodic systems is quite different from the case of random potential where the interaction between two particles gives an increase of the localization length.

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